Representations of the Bondi−Metzner−Sachs group in three space−time dimensions

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Abstract. The original Bondi−Metzner−Sachs group B is the common asymptotic symmetry group of all asymptotically flat Lorentzian 4−dim space−times. As such, B is the best candidate for the universal symmetry group of General Relativity (G.R.). In 1973, with this motivation, P. J. McCarthy classified all relativistic B−invariant systems in terms of strongly continuous irreducible unitary representations (IRS) of B. Here, we construct the IRS of $B(2,1)$, the analogue of B, in 3 space−time dimensions. The IRS are induced from 'little groups' which are compact. The finite 'little groups' are cyclic groups of even order. The inducing construction is exhaustive notwithstanding the fact that $B(2,1)$ is not locally compact in the employed Hilbert topology.

1. Introduction

The Bondi−Metzner−Sachs (BMS) group B is the common asymptotic group of all curved real Lorentzian space−times which are asymptotically flat in future null directions [1, 2], and is the best candidate for the universal symmetry group of G.R..

In 1939 Wigner laid the foundations of special relativistic quantum mechanics [3] and relativistic quantum field theory by constructing the Hilbert space IRS of the (universal cover) of the Poincare group P.

The universal property of B for G.R. makes it reasonable to attempt to lay a similarly firm foundation for quantum gravity by following through the analogue of Wigner's programme with B replacing P. Some years ago McCarthy constructed explicitly [4, 5, 6, 7, 8, 9, 10, 11] the IRS of B for exactly this purpose. This work was based on G.W.Mackey's pioneering work on group representations [3, 12, 13, 14, 15]; in particular McCarthy's work extended G.W.Mackey's work to the relevant infinite−dimensional case.

It is difficult to overemphasize the importance of Piard's results [16, 17] who soon afterwards proved that *all* the IRS of B, when this is equipped with the Hibert topology, are derivable by the inducing construction. This proves the exhaustivity of McCarthy's list of representations and renders his results even more important.

However, in quantum gravity, complexified or euclidean versions of G.R. are frequently considered and the question arises: Are there similar symmetry groups for these versions of the theory? McCarthy constructed [18], in abstract form, all possible analogues of B, both real and in any signature, or complex, with all possible notions of asymptotic flatness 'near infinity'. There are, in fact, forty−one such groups. These abstract constructions were given

in a quantum setting; the paper was concerned with finding the IRS of these groups in Hilbert spaces (especially for the complexification $\mathcal{CB}$ of B itself). It has been argued [19, 20, 21, 22] that these Hilbert space representations are related to elementary particles and quantum gravity (via gravitational instantons).

Here, we follow this programme for 3−dim G.R. and construct in the Hilbert topology the IRS of $B(2,1)$, the analogue of B in three space−time dimensions. It is proved that all IRS of $B(2,1)$ are induced from IRS of *compact* 'little groups'. It follows that some IRS of $B(2,1)$ are controlled by IRS of the finite symmetry groups of regular polygons in ordinary euclidean 2−space. It is proved that *all* IRS of $B(2,1)$ are induced by the IRS of its little groups notwithstanding the fact that $B(2,1)$ is *not* locally compact in the employed Hilbert topology. The paper closes with the explicit construction of the IRS of $B(2,1)$.

We turn now to the study of to $B(2,1)$, the analogue of B in three space−time dimensions.

1.1. The group $B^{2,1}(N^+)$

Recall that the $2+1$ Minkowski space is the vector space R^3 of row vectors with 3 real components, with the inner product defined as follows. Let $x, y \in R^3$ have components x^μ and y^μ respectively, where $\mu = 0, 1, 2$. Define the inner product $x.y$ between x and y by

$$x.y = x^0y^0 - x^1y^1 - x^2y^2. \tag{1}$$

Then the $2+1$ Minkowski space, sometimes written $R^{2,1}$, is just R^3 with this inner product. The "2,1" refers to the one plus and two minus signs in the inner product. Let $SO(2,1)$ be the (connected component of the identity element of the) group of linear transformations preserving the inner product. Matrices $\Lambda \in SO(2,1)$ are taken as acting by matrix multiplication from the right, $x \longmapsto x\Lambda$, on row vectors $x \in R^{2,1}$.

The future null cone $N^+ \subset R^{2,1}$ is just the set of nonzero vectors with zero length and $x^0 > 0$:

$$N^+ = \left\{x \in R^{2,1} | x.x = 0, x^0 > 0\right\}. \tag{2}$$

Let R^*_+ denote the multiplicative group of all *positive* real numbers. Obviously, if $x \in N^+$, then $tx \in N^+$ for any $t \in R^*_+$. Let $F_1(N^+)$ denote the vector space (under pointwise addition) of all functions $f : N^+ \to R$ satisfying the homogeneity condition

$$f(tx) = tf(x) \tag{3}$$

for all $x \in N^+$ and $t \in R^*_+$. Define a representation T of $SO(2,1)$ on $F_1(N^+)$ by setting, for each $x \in N^+$ and $\Lambda \in SO(2,1)$,

$$(T(\Lambda)f)(x) = f(x\Lambda). \tag{4}$$

Now let $B^{2,1}(N^+)$ be the semi−direct product

$$B^{2,1}(N^+) = F_1(N^+) \rtimes_T SO(2,1). \tag{5}$$

That is to say, $B^{2,1}(N^+)$ is, as a set, just the product $F_1(N^+) \times SO(2,1)$, and the group multiplication law for pairs is

$$(f_1, \Lambda_1)(f_2, \Lambda_2) = (f_1 + T(\Lambda_1)f_2, \Lambda_1\Lambda_2). \tag{6}$$

1.2. The double cover $B^{2,1}(N^+)_c$

Let $\mathrm{SL}(2,R)$ be the group of all real 2×2 matrices with determinant one. $\mathrm{SL}(2,R)$ is sometimes denoted by G below. Let $\mathrm{M}_s(2,R)$ be the set of all 2×2 symmetric real matrices. We define a right action of G on $\mathrm{M}_s(2,R)$ by $\mathrm{M}_s(2,R)\times G\to \mathrm{M}_s(2,R)$ with

$$(m,g)\mapsto g^\top m g, \tag{7}$$

where the superscript $\top$ means transpose. Clearly any element $\mu\in \mathrm{M}_s(2,R)$ can be parameterized as follows:

$$\mu=\begin{bmatrix} x^{\mathrm{o}}-x^1 & x^2\\ x^2 & x^o+x^1\end{bmatrix}$$

where $x^{\mathrm{o}},x^1,x^2\in R$. We now consider the map $b:R^3\to \mathrm{M}_s(2,R)$ defined by

$$b(x)=\begin{bmatrix} x^{\mathrm{o}}-x^1 & x^2\\ x^2 & x^o+x^1\end{bmatrix}, \tag{8}$$

where the x^μ are the components of $x\in R^3$. This map is a linear bijection, so the right action of G on $\mathrm{M}_s(2,R)$ induces a linear right action of G on R^3. Since

$$\det(b(x))=x\cdot x \tag{9}$$

and the G action preserves determinants (indeed $\det g = 1$) in $\mathrm{M}_s(2,R)$, G acts as transformations from $SO(2,1)$. In fact, this construction gives an homomorphism

$$\gamma: G\to SO(2,1) \tag{10}$$

which is onto, and has kernel $Z_2=\{Id,-Id\}$ in G, Id denoting the identity element of G. Thus γ identifies G as the double cover of $SO(2,1)$

$$G=SO(2,1)_c. \tag{11}$$

Therefore, the double cover of the group $B^{2,1}(N^+)$, given in (5), has the form

$$B^{2,1}(N^+)_c=F_1(N^+)\rtimes_T \mathrm{SL}(2,R). \tag{12}$$

Strictly speaking, "T" should read "$T\gamma$", but the notation is simpler as above.

1.3. The group $B(2,1)$

So far, the supertranslation space $F_1(N^+)$ has been defined as a space of truly *arbitrary* homogeneous functions of degree one. This has been merely for clarity; for physical applications, it is necessary to give this space additional structure. For reasons discussed in detail in McCarthy [18], we now give a new realization of $B^{2,1}(N^+)_c$ where the supertranslation space is restricted to be the separable Hilbert space $L^2(P_1(R),\lambda,R)$ of real−valued functions defined on $P_1(R)\simeq S^1$; functions square integrable with respect to the standard normalized (Lesbegue) measure λ on $P_1(R)\simeq S^1$; $P_1(R)\equiv S^1/Z_2$ is the one−dimensional real projective space (the circle quotient the antipodal map).

In particular in [23] the following Theorem is proved:

Theorem 1.1 *The group $B^{2,1}(N^+)_c$ can be realised as*

$$B(2,1)=L^2(P_1(R),\lambda,R)\rtimes_T G \tag{13}$$

with semi−direct product specified by

$$(T(g)\alpha)(x)=\kappa_g(x)\alpha(xg), \tag{14}$$

where $G = \mathrm{SL}(2,R)$, $g \in G$, $\alpha \in L^2(P_1(R), \lambda, R)$. Moreover, if

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \tag{15}$$

then the components x_1, x_2 of $\mathbf{x} \in R^2 - \{0\}$ transform linearly under g, so that the ratio $x = x_1/x_2,\ x_2 \neq 0,$ transforms fraction linearly under g. Writing xg for the transformed ratio,

$$xg = \frac{(\mathbf{x}g)_1}{(\mathbf{x}g)_2} = \frac{x_1 a + x_2 c}{x_1 b + x_2 d} = \frac{xa + c}{xb + d}. \tag{16}$$

The ratio $x = x_1/x_2,\ x_2 \neq 0$, is a local inhomogeneous coordinate of $P_1(R)$. We denote our final realization of our group by $B(2,1)$ to distinguish it from the previous realizations $B^{2,1}(N^+)$ and $B^{2,1}(N^+)_c$. In analogy to B, it is natural to choose a measure λ on $P_1(R)$ which is invariant under the maximal compact subgroup $SO(2)$ of G; we choose λ to be the standard normalized Lesbegue measure $\mathrm{d}\lambda = \frac{\mathrm{d}\theta}{2\pi}$.

The factor $\kappa_g(x)$ on the right hand side of (14) is defined by

$$\kappa_g(x) = \frac{(xb + d)^2 + (xa + c)^2}{1 + x^2}. \tag{17}$$

It is well known [24] that the topological dual of a Hilbert space can be identified with the Hilbert space itself, so that we have $L^{2'}(P_1(R), \lambda, R) \simeq L^2(P_1(R), \lambda, R)$. In fact, given a continuous linear functional $\phi \in L^{2'}(P_1(R), \lambda, R)$, we can write, for $\alpha \in L^2(P_1(R), \lambda, R)$

$$(\phi, \alpha) = < \phi, \alpha >, \tag{18}$$

where the function $\phi \in L^2(P_1(R), \lambda, R)$ on the right is uniquely determined by (and denoted by the same symbol as) the linear functional $\phi \in L^{2'}(P_1(R), \lambda, R)$ on the left. The representation theory of $B(2,1)$ is governed by the dual action T' of G on the topological dual $L^{2'}(P_1(R), \lambda, R)$ of $L^2(P_1(R), \lambda, R)$. The dual action T' is defined by:

$$< T'(g)\phi, \alpha > = < \phi, T(g^{-1})\alpha > \cdot \tag{19}$$

A short calculation gives

$$(T'(g)\phi)(x) = \kappa_g^{-2}(x)\phi(xg). \tag{20}$$

Now, this action T' of G on $L^{2'}(P_1(R), \lambda, R)$, given explicitly above is, like the action T of G on $L^2(P_1(R), \lambda, R)$, continuous. The 'little group' L_ϕ of any $\phi \in L^{2'}(P_1(R), \lambda, R)$ is the stabilizer

$$L_\phi = \{g \in G \mid T'(g)\phi = \phi\}. \tag{21}$$

By continuity, $L_\phi \subset G$ is a closed subgroup.

2. Representation theory

Let A and $\mathcal{G}$ be topological groups, and let T be a given homomorphism from $\mathcal{G}$ into the group of automorphisms $\mathrm{Aut}(A)$ of A. Suppose A is abelian and $\mathcal{H} = A \rtimes_T \mathcal{G}$ is the semi−direct product of A and $\mathcal{G}$, specified by the continuous action $T : \mathcal{G} \longrightarrow \mathrm{Aut}(A)$. In the product topology of $A \times \mathcal{G}$, $\mathcal{H}$ then becomes a topological group. It is assumed that it becomes a separable locally compact topological group. In order to give the operators of the induced representations explicitly it is necessary ([3], [12], [13], [14], [15] and references therein) to give the following information:

(i) An irreducible unitary representation U of L_ϕ on a Hilbert space D for each L_ϕ.

(ii) A $\mathcal{G}$ −quasi−invariant measure μ on each orbit $\mathcal{G}\phi \approx \mathcal{G}/L_\phi$, where L_ϕ denotes the little group of the base point $\phi \in A^{'}$ of the orbit $\mathcal{G}\phi$; $A^{'}$ is the topological dual of A.

Let D_μ be the space of functions $\psi : \mathcal{G} \to D$ which satisfy the conditions

$$(a) \qquad \psi(gl) = U(l^{-1})\psi(g), \qquad g \in \mathcal{G},\ l \in L_\phi,$$

$$(b) \quad \int_{\mathcal{G}\phi} < \psi(q), \psi(q) > \mathrm{d}\mu(q) < \infty,$$

where the scalar product under the integral sign is that of D. Note, that the constraint (a) implies that $< \psi(gl), \psi(gl) >=< \psi(g), \psi(g) >$, and therefore the inner product $< \psi(g), \psi(g) >$, $g \in \mathcal{G}$, is constant along every element q of the coset space $\mathcal{G}/L_\phi \approx \mathcal{G}\phi$. This allows to assign a meaning to $< \psi(q), \psi(q) >$, where $q = gL_\phi$, by defining $< \psi(q), \psi(q) >:=< \psi(g), \psi(g) >$. Thus the integrand in (b) becomes meaningful due to the condition (a). A pre−Hilbert space structure can now be given to D_μ by defining the scalar product

$$< \psi_1, \psi_2 >= \int_{\mathcal{G}\phi} < \psi_1(q), \psi_2(q) > \mathrm{d}\mu(q), \tag{22}$$

where $\psi_1, \psi_2 \in D_\mu$. It is convenient to complete the space D_μ with respect to the norm defined by the scalar product (22). In the resulting Hilbert space, functions are identified whenever they differ, at most, on a set of μ–measure zero. Thus our Hilbert space is

$$D_\mu = L^2(\mathcal{G}\phi, \mu, D). \tag{23}$$

Define now an action of $\mathcal{H} = A \rtimes_T \mathcal{G}$ on D_μ by

$$(g_\mathrm{o}\psi)(q) = \sqrt{\frac{\mathrm{d}\mu_{g_\mathrm{o}}}{\mathrm{d}\mu}(q)}\psi(g_\mathrm{o}^{-1}q), \tag{24}$$

$$\alpha\psi(q) = e^{i<g_\mathrm{o}\phi,\alpha>}\psi(q), \tag{25}$$

where $g_\mathrm{o} \in \mathcal{G}$, $q \in \mathcal{G}\phi$, and $\alpha \in A$. Eqs. (24) and (25) define the IRS of $B(2,1)$ induced from *each* $\phi \in A^{'}$ and *each* irreducible representation U of L_ϕ. The 'Jacobian' $\frac{\mathrm{d}\mu_{g_\mathrm{o}}}{\mathrm{d}\mu}$ of the group transformation is known as the *Radon−Nikodym* derivative of μ_{g_o} with respect to μ and ensures that the resulting IRS of $B(2,1)$ are unitary.

The central results of induced representation theory ([3], [12], [13], [14], [15] and references therein) are the following:

(i) Given the topological restrictions on $\mathcal{H} = A \rtimes_T \mathcal{G}$ (separability and local compactness), any representation of $\mathcal{H}$, constructed by the method above, is irreducible if the representation U of L_ϕ on D is irreducible. Thus an irreducible representation of $\mathcal{H}$ is obtained for *each* $\phi \in A^{'}$ and *each* irreducible representation U of L_ϕ.

(ii) If $\mathcal{H} = A \rtimes_T \mathcal{G}$ is a regular semi−direct product (i.e., $A^{'}$ contains a Borel subset which meets each orbit in $A^{'}$ under $\mathcal{H}$ in just one point) then *all* of its irreducible representations can be obtained in this way.

3. Obstructions and resolutions

Two remarks are in order regarding the representations of $B(2,1)$ obtained by the above construction:

(i) As it is explained in [23] the subgroup $L^2(P_1(R),\lambda,R)$ of $B(2,1)=L^2(P_1(R),\lambda,R)\rtimes_T G$ is topologised as a (pre) Hilbert space by using a natural measure on $P_1(R)$ and by introducing a scalar product into $L^2(P_1(R),\lambda,R)$. If R^4 is endowed with the natural metric topology then the group $G=SL(2,R)$, considered as a subset of R^4, inherits the induced topology on G. In the product topology of $L^2(P_1(R),\lambda,R)\times G$, $B(2,1)$ is a non−locally compact group (the proof follows, without substantial change, Cantoni's proof [25], see also [4]). (In fact the subgroup $L^2(P_1(R),\lambda,R)$, and therefore the group $B(2,1)$ can be employed with many different topologies. The Hilbert type topology employed here appears to describe quantum mechanical systems in asymptotically flat space−times [10]). Since in the Hilbert type topology $B(2,1)=L^2(P_1(R),\lambda,R)\rtimes_T G$ is not locally compact the theorems dealing with the irreducibility of the representations obtained by the above construction no longer apply (see e.g. [13]). However, it can be proved that the induced representations obtained above *are* irreducible. The proof follows very closely the one given in [7] for the case of the original BMS group B.

(ii) Here it is assumed that $B(2,1)$ is equipped with the Hilbert topology. It is of outmost significance that it can be proved [23] that in this topology $B(2,1)$ is a regular semi−direct−product. The proof follows the corresponding proof [16, 17] for the group B. Regularity amounts to the fact [12] that $L^{2'}(P_1(R),\lambda,R)$ can have no equivalent classes of quasi−invariant measures μ such that the action of $\mathcal{G}$ is strictly ergodic with respect to μ. When such measures μ do exist it can be proved [12] that an irreducible representation of the group, with the semi−direct−product structure at hand, may be associated with each such measure μ, that is not equivalent to any of the IRS constructed by the Wigner−Mackey's inducing method. In a different topology it is not known if $B(2,1)$ is a regular or irregular semi−direct−product. Irregularity of $B(2,1)$ in a topology different from the Hilbert topology would imply that there are IRS of $B(2,1)$ that are not unitary equivalent to any of the IRS obtained by the inducing construction. Strictly ergodic actions are notoriously hard to deal with even in the locally compact case. Indeed, for locally compact non−regular semi−direct products, there is no known example for which all inequivalent irreducibles arising from strictly ergodic actions have been found. For the other 41 groups defined in [18] regularity has only been proved for B [16, 17] when B is equipped with the Hilbert topology. Similar remarks apply to all of them regarding IRS arising from strictly ergodic actions in a given topology.

4. Little groups and inducing construction

In [23] it is proved that when $B(2,1)$ is employed with the Hilbert topology *all* little groups of $B(2,1)$ are compact. In particular the following Theorem is proved:

Theorem 4.1 *The little groups L_ϕ for $B(2,1)$ are precisely the closed subgroups of $K=SO(2)$ which contain the element $-I$, I being the identity element of G. These are (A) K itself, and (B) the cyclic groups C_n of even order n.*

For a given little group L_ϕ the elements $\phi\in A'$ which are invariant under L_ϕ, i.e., the elements $\phi\in A'$ which satisfy

$$(T'(g)\phi)(x)=\phi(x) \tag{26}$$

form a subspace of $L^2(P_1(R),\lambda,R)$. We denote this subspace by $L^2(L_\phi)$. Then we have the following Theorem:

Theorem 4.2 *The Hilbert space $L^2(SO(2))$ of invariant vectors $\phi \in A^{'}$ under $SO(2)$ is:*

$$L^2(SO(2)) = \left\{\phi \in L^2(P_1(R), \lambda, R) \mid \phi(x) = c,\ c \in R\right\}. \tag{27}$$

So $L^2(SO(2))$ is just the one−dimensional space of constant real−valued functions defined on $P_1(R)$. The Hilbert space $L^2(C_n)$ of invariant vectors $\phi \in A^{'}$ under C_n is:

$$L^2(C_n) = L^2(E_n), \tag{28}$$

where $L^2(E_n)$ is the Hilbert space of square integrable real−valued functions defined on

$$E_n = \left\{\theta \in P_1(R) \mid 0 < \theta < \frac{4\pi}{n}\right\}. \tag{29}$$

Moreover in [23] it is shown that the Wigner−Mackey's inducing construction is exhaustive despite the fact that $B(2,1)$ is not locally compact in the employed Hilbert topology. This result is rather important because other group theoretical approaches to quantum gravity which invoke Wigner−Mackey's inducing construction (see e.g. [26]) are typically plagued by the non−exhaustiveness of the inducing construction which results precisely from the fact that the group in question is not locally compact in the prescribed topology. Exhaustiveness is not just a mathematical nicety: If the inducing construction is not exhaustive one cannot know if the most interesting information or part of it is coded in the irreducibles which cannot be found by the Wigner−Mackey's inducing procedure. These results, i.e. compactness of the little groups and exhaustiveness of the inducing construction, not only are significant for the group theoretical approach to quantum gravity advocated here, but also they have repercussions [23] for other approaches to quantum gravity.

5. Construction of the IRS of $B(2,1)$

To find explicitly the operators of the induced representations of $B(2,1)$, it suffices to provide the information cited in (i) and (ii) in section 2 for each of the orbit types. We note that all the little groups are abelian. All IRS of an abelian group are one−dimensional.

(i), ($\mathbf{i}_\alpha$) $L_\phi = K = SO(2)$.
The IRS U of K are parameterized by an integer ν which for distinct representations takes the values $\nu = ..., -2, -1, 0, 1, 2, ...$. Denoting these representations by $U^{(\nu)}$, they are given by multiplication in one complex dimension $D \approx C$ by

$$U^{(\nu)}(R(\theta)) = e^{i\nu\theta}, \tag{30}$$

where $R(\theta) = \begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix} \in SO(2)$.
($\mathbf{i}_\beta$) $L_\phi = C_n$, n is even.
The IRS U of C_n are parameterized by an integer λ which for distinct representations takes the values $\lambda = 0, 1, 2, ..., n-1$. Denoting these representations by $U^{(\lambda)}$, they are given by multiplication in one complex dimension $D \approx C$ by

$$U^{(\lambda)}\left(R\left(\frac{2\pi}{n}j\right)\right) = e^{i\frac{2\pi}{n}\lambda j}, \tag{31}$$

where j parameterizes the elements of the group C_n.

(ii) A $\mathcal{G}$ −quasi−invariant measure μ on each orbit $\mathcal{G}\phi \approx \mathcal{G}/L_\phi$ is required, however, a $\mathcal{G}$ −invariant measure μ on each orbit $\mathcal{G}\phi \approx \mathcal{G}/L_\phi$ will be given in all cases. We note that when μ is $\mathcal{G}$ −invariant then $\frac{\mathrm{d}\mu_{\mathrm{go}}}{\mathrm{d}\mu}(q) = 1$, and this is precisely what happens in the case of $B(2,1)$. Moreover, in the case of $B(2,1)$, $\mathcal{G} = G =$SL(2,R), and the little groups L_ϕ are given in Theorem 4.1.

(ii$_\alpha$) In [23] it is proved that the construction of a unique (up to a constant factor) G−invariant measure on the orbit 01 $\equiv G/L_\phi$, $L_\phi = K = SO(2)$, necessitates the construction of a G−invariant measure on G, and the construction of a K−invariant measure on K. A G−invariant measure on

$$G = \mathrm{SL}(2,R) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a,b,c,d \in R, \quad ad - bc = 1 \right\},$$

is given by

$$\mathrm{d}g = \frac{\mathrm{d}a \wedge \mathrm{d}b \wedge \mathrm{d}c}{a}. \tag{32}$$

G−invariant measure means $\mathrm{d}g = \mathrm{d}(gg_{\mathrm{o}}) = \mathrm{d}(g_{\mathrm{o}}g)$, $g_{\mathrm{o}} \in \mathrm{SL}(2,R)$. A K−invariant measure on K is given by the 1−form $\mathrm{d}\theta$ where θ is the usual angular coordinates which covers $P_1(R) \simeq S^1$.

(ii$_\beta$) The orbits 02 $\equiv G/L_\phi$, $L_\phi = C_n$, where n is even, can be endowed with the G−invariant measure on G given in case 01. Indeed, for a given little group L_ϕ, the orbit 02 $\equiv G/L_\phi$ is the space of orbits of the right action $T : G \times L_\phi \to G$ of the group L_ϕ on G given by

$$g \star c := g \cdot c, \tag{33}$$

where $g \in G$ and $c \in L_\phi$. Thus the action T denoted by $\star$ is identical to the group multiplication in G. Since the group L_ϕ is finite and since the action (33) is fixed point free the coset spaces G/L_ϕ inherit the measure on G.

This completes the necessary information in order to construct the induced representations of $B(2,1)$. As already stated, it is shown in [23] that when $B(2,1)$ is equipped with the Hilbert topology the inducing construction is exhaustive notwithstanding the fact that $B(2,1)$ is not locally compact. To conclude, in this paper *all* IRS of $B(2,1)$ have been constructed in the Hilbert topology.

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