$SL(2, C)$ Instanton Sheaves

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Abstract. We constructed $SL(2, C)$ Yang-Mills instanton solutions which satisfy the complex ADHM equations and the monad construction by using the biquaternion method. And we also found that the $SL(2, C)$ instanton solutions can be used to explicitly construct instanton sheaves on CP^3 which similar to the case holomorphic vector bundles on CP^3 of $SU(2)$ ADHM construction. And the existence of these instanton sheaves is related to singularities on S^4 which do not exist in $SU(2)$ instanton case.

1. Introduction

In 1970s, searching classical exact solutions of Yang-Mills (SDYM) equation was an important issue. The instanton solutions are classical solutions of Euclidean $SU(2)$ (anti)self-dual Yang-Mills (SDYM). Yang-Mills instantons are not only important in quantum field theory for physicists but also influential on algebraic geometry for mathematicians. In nonperturbative quantum field theory, quantum instanton tunnelling has resolved the QCD $U(1)_A$ problem [1] and created the strong CP problem with associated QCD θ-vacua [2] structure. Mathematically, instanton can be used as a tool in algebraic geometry classify four-manifolds [3] is the key idea.

The first instanton solution [4] BPST 1-instanton was found in 1975. The $5k$ moduli parameters CFTW k-instanton solutions [5] were constructed later. And then JNR extended the number of moduli parameters of the k-instanton solutions from $5k$ to $5k + 4$ (5,13 for k = 1,2) [6] based on the $4D$ conformal symmetry group. Finally, mathematicians ADHM used method in algebraic geometry and worked out the complete $8k - 3$ moduli parameters instanton solutions for each k-th homotopy in 1978[7]. ADHM used the monad construction combining with the Penrose-Ward transform to construct the most general instanton solutions. One to one correspondence between anti-self-dual $SU(2)$-connections on S^4 and global holomorphic vector bundles of rank two on CP^3 is the key idea in ADHM construction. The explicit complete $SU(2)$ instanton solutions for $k \leq 3$ had been worked out in [8].

ADHM construction has been generalized to the cases of many other compact Lie groups[8, 9] SDYM theories. Recently, the present authors wrote a paper[10] and generalized the quaternion calculation in $SU(2)$ ADHM construction to the biquaternion calculation with biconjugation operation, and built a class of non-compact $SL(2, C)$ Yang-Mills instanton solutions with $16k-6$ parameters for each k-th homotopy class. The number of moduli parameters $16k-6$ is consistent with the conjecture made by Frenkel and Jardim in [11] and proved recently in this paper[12] from the mathematical sight. From these new $SL(2, C)$ instanton solutions, we can easily get $SL(2, C)$ (M, N) instanton solutions constructed in 1984 [13].

The discovery of instanton sheaf structure on the projective space in $SL(2, C)$ case is different from the holomorphic vector bundles on CP^3 in $SU(2)$ case. Comparing with the well know regular $SU(2)$ ADHM instanton solutions without any singularities on S^4 spacetime,there are singularities on S^4 for $SL(2,C)$ instanton solutions which can not be gauged away. Recalling that there is a fibration from CP^3 to S^4 with fibers being CP^1. A bundle E on CP^3 can descend down to a bundle over S^4 if and only if no fiber of the twistor fibration is a jumping line for E. This is precisely the case for the $SU(2)$ ADHM construction.

In $SL(2,C)$ instanton solutions, some twistor lines called jumping lines. After Penrose-Ward transform, we can expect $SL(2,C)$ instanton sheaf structure on CP^3. We will show the two examples of instanton sheaf in this paper. We discover some points on CP^3 for the 2-instanton case, the vector bundle description of $SL(2,C)$ 2-instanton on CP^3 breaks down, and we have to use the sheaves description or "sheaf instanton" on CP^3[11]

In section II and III of this paper, we will review the biquaternion ADHM construction of $SL(2,C)$ Yang-Mills instantons with $16k-6$ moduli parameters [10]. In section IV, we will show how to express the biquaternion instanton solution constructed in section III to complex ADHM data $(B_{lm}, I_{m,}J_m)$ with $l,m=1,2.$ which are solutions of the complex version of the ADHM equations [14],and we will also show the equivalence between biquaternion ADHM construction and complex ADHM equations. In section V, we will identify the relationship between α and β matrices in the monad construction and Kerβ/Imα the holomorphic vector bundles on CP^3.Then, we will extend the vector bundles case to the sheaves case which are the break down of vector bundle at some points of CP^3 and give two examples of sheaves.

2. Biquaternions

2.1. Quaternion

Now, we are going to review the construction of $SL(2,C)$ YM instantons [13, 10]. We will use the convention $\mu = 0,1,2,3$ and $\epsilon_{0123} = 1$ for $4D$ Euclidean space. In contrast to the quaternion in the $Sp(1)$ $(= SU(2))$ ADHM construction, the authors of [10] used *biquaternion* to construct $SL(2,C)$ Yang-Mills instantons. A quaternion x can be written as

$$x = x_\mu e_\mu, \;\; x_\mu \in R, \;\; e_0 = 1, e_1 = i, e_2 = j, e_3 = k \tag{1}$$

where e_1, e_2 and e_3 anticommute and obey

$$e_i \cdot e_j = -e_j \cdot e_i = \epsilon_{ijk} e_k; \quad i,j,k = 1,2,3, \tag{2}$$

$$e_1^2 = -1, e_2^2 = -1, e_3^2 = -1. \tag{3}$$

The conjugate quarternion is defined to be

$$x^\dagger = x_0 e_0 - x_1 e_1 - x_2 e_2 - x_3 e_3 \tag{4}$$

so that the norm square of a quarternion is

$$|x|^2 = x^\dagger x = x_0^2 + x_1^2 + x_2^2 + x_3^2. \tag{5}$$

Generally the unit quarternions can be expressed as Pauli matrices

$$e_0 \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, e_i \to -i\sigma_i \; ; \; i = 1,2,3. \tag{6}$$

2.2. Biquaternions

A biquaternion (or complex-quaternion) z can be written as

$$z = z_\mu e_\mu, \;\; z_\mu \in C, \tag{7}$$

which occasionally can be written as

$$z = x + yi \tag{8}$$

where x and y are quaternions and $i = \sqrt{-1}$, not to be confused with e_1 in Eq.(1). The biconjugation [16] of z is defined to be

$$z^{\circledast} = z_\mu e_\mu^\dagger = z_0 e_0 - z_1 e_1 - z_2 e_2 - z_3 e_3 = x^\dagger + y^\dagger i, \tag{9}$$

which was heavily used in the construction of $SL(2, C)$ instantons [10] in contrast to the complex conjugation

$$z^* = z_\mu^* e_\mu = z_0^* e_0 + z_1^* e_1 + z_2^* e_2 + z_3^* e_3 = x - yi. \tag{10}$$

The norm square of a biquarternion is defined to be

$$|z|_c^2 = z^{\circledast} z = (z_0)^2 + (z_1)^2 + (z_2)^2 + (z_3)^2, \tag{11}$$

which is a *complex* number in general so we use a subscript c in the norm.

3. Biquaternion ADHM

Now, we're going to review the biquaternion construction of $SL(2, C)$ instantons. The first step is to introduce the $(k + 1) \times k$ biquarternion matrix $\Delta(x) = a + bx$

$$\Delta(x)_{ab} = a_{ab} + b_{ab} x, \;\; a_{ab} = a^\mu_{ab} e_\mu, b_{ab} = b^\mu_{ab} e_\mu, x = x^\mu e_\mu \tag{12}$$

where a^μ_{ab} and b^μ_{ab} are complex numbers, and a_{ab} and b_{ab} are biquarternions. x^μ is the position in 4D Euclidean space. The biconjugation of the $\Delta(x)$ matrix is defined to be

$$\Delta(x)^{\circledast}_{ab} = \Delta(x)^\mu_{ba} e^\dagger_\mu = \Delta(x)^0_{ba} e_0 - \Delta(x)^1_{ba} e_1 - \Delta(x)^2_{ba} e_2 - \Delta(x)^3_{ba} e_3. \tag{13}$$

The quadratic condition of $SL(2, C)$ instantons reads

$$\Delta(x)^{\circledast} \Delta(x) = f^{-1} = \text{symmetric, non-singular } k \times k \text{ matrix for } x \notin J, \tag{14}$$

from which we can deduce that $a^{\circledast} a, b^{\circledast} a, a^{\circledast} b$ and $b^{\circledast} b$ are all symmetric matrices. The choice of *biconjugation* operation was crucial for the construction of the $SL(2, C)$ instantons. On the other hand, for $x \in J$, $\det \Delta(x)^{\circledast} \Delta(x) = 0$. The set J is called singular locus or "jumping lines". There are no jumping lines for the case of $SU(2)$ instantons on S^4. In the $Sp(1)$ quaternion case, the symmetric condition on f^{-1} implies f^{-1} is real; while for the $SL(2, C)$ biquaternion case, it implies f^{-1} is *complex* which means $[\Delta(x)^{\circledast} \Delta(x)]^\mu_{ij} = 0$ for $\mu = 1, 2, 3$.

To construct the self-dual gauge field, we introduce a $(k + 1) \times 1$ dimensional biquaternion vector $v(x)$ satisfying the following two conditions

$$v^{\circledast}(x) \Delta(x) = 0, \tag{15a}$$
$$v^{\circledast}(x) v(x) = 1 \tag{15b}$$

where $v(x)$ is fixed up to a $SL(2, C)$ gauge transformation

$$v(x) \longrightarrow v(x) g(x), \quad g(x) \in \; 1 \times 1 \text{ Biquaternion}. \tag{16}$$

Note that in general a $SL(2,C)$ matrix can be written in terms of a 1×1 biquaternion as

$$g=\frac{q_\mu e_\mu}{\sqrt{q^\circledast q}}=\frac{q_\mu e_\mu}{|q|_c}. \tag{17}$$

The next step is to define the gauge field

$$G_\mu(x)=v^\circledast(x)\partial_\mu v(x), \tag{18}$$

which is a 1×1 biquaternion. The $SL(2,C)$ gauge transformation of the gauge field is

$$\begin{aligned} G_\mu(x)->G'(x)&=(g^\circledast(x)v^\circledast(x))\partial_\mu(v(x)g(x))\\ &=g^\circledast(x)G_\mu(x)g(x)+g^\circledast(x)\partial_\mu g(x) \end{aligned} \tag{19}$$

where in the calculation Eq.(15b) has been used. Note that, unlike the case for $Sp(1)$, $G_\mu(x)$ needs not to be anti-Hermitian.

One can then define the $SL(2,C)$ field strength

$$F_{\mu\nu}=\partial_\mu G_\nu(x)+G_\mu(x)G_\nu(x)-[\mu\longleftrightarrow\nu], \tag{20}$$

and prove the self-duality of $F_{\mu\nu}$. To count the number of moduli parameters for the $SL(2,C)$ k-instantons, one can use transformations which preserve conditions Eq.(14), Eq.(15a) and Eq.(15b), and the definition of G_μ in Eq.(18) to bring a and b in Eq.(12) into the following simple canonical form

$$b=\begin{bmatrix}0_{1\times k}\\ I_{k\times k}\end{bmatrix}, a=\begin{bmatrix}\lambda_{1\times k}\\ -y_{k\times k}\end{bmatrix} \tag{21}$$

where λ and y are biquaternion matrices with orders $1\times k$ and $k\times k$ respectively, and y is symmetric

$$y=y^T. \tag{22}$$

Thus the constraints for the moduli parameters are

$$a^\circledast_{ci}a_{cj}=0, i\neq j,\text{ and } y_{ij}=y_{ji}. \tag{23}$$

The total number of moduli parameters for k-instanton can be calculated through Eq.(23) to be

$$\#\text{ of moduli for } SL(2,C)\ k\text{-instantons}=16k-6, \tag{24}$$

which is twice of that of the case of $Sp(1)$. Roughly speaking, there are $8k$ parameters for instanton "biquaternion positions" and $8k$ parameters for instanton "sizes". Finally one has to subtract an overall $SL(2,C)$ gauge group degree of freedom 6.

We provide two explicit examples of $SL(2,C)$ instantons here. These will be used in section V for the discussion of instanton sheaves.

4. Complex ADHM Equations

In this section, we're going to show the equivalence between biquaternion ADHM construction complex ADHM equations.

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4.1. From Biquaternion to ADHM Data

We begin with the canonical form (21) and using the explicit matrix representation(6) to express the biquaternions as Pauli matrices.

$$a = \begin{bmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_k \\ y_{11} & y_{12} & \cdots & y_{1k} \\ y_{21} & y_{22} & \cdots & y_{2k} \\ .. & \cdots & \cdots & \cdots \\ y_{k1} & y_{k2} & \cdots & y_{kk} \end{bmatrix} \tag{25}$$

$$a = \begin{bmatrix} \lambda_1^0 - i\lambda_1^3 & -\left(\lambda_1^2 + i\lambda_1^1\right) & \lambda_2^0 - i\lambda_2^3 & -\left(\lambda_2^2 + i\lambda_2^1\right) & \cdots & \lambda_k^0 - i\lambda_k^3 & -\left(\lambda_k^2 + i\lambda_k^1\right) \\ \lambda_1^2 - i\lambda_1^1 & \lambda_1^0 + i\lambda_1^3 & \lambda_2^2 - i\lambda_2^1 & \lambda_2^0 + i\lambda_2^3 & .. & \lambda_k^2 - i\lambda_k^1 & \lambda_k^0 + i\lambda_k^3 \\ y_{11}^0 - iy_{11}^3 & -\left(y_{11}^2 + iy_{11}^1\right) & y_{12}^0 - iy_{12}^3 & -\left(y_{12}^2 + iy_{12}^1\right) & \cdots & y_{1k}^0 - iy_{1k}^3 & -\left(y_{1k}^2 + iy_{1k}^1\right) \\ y_{11}^2 - iy_{11}^1 & y_{11}^0 + iy_{11}^3 & y_{12}^2 - iy_{12}^1 & y_{12}^0 + iy_{12}^3 & \cdots & y_{1k}^2 - iy_{1k}^1 & y_{1k}^0 + iy_{1k}^3 \\ y_{12}^0 - iy_{12}^3 & -\left(y_{12}^2 + iy_{12}^1\right) & y_{22}^0 - iy_{22}^3 & -\left(y_{22}^2 + iy_{22}^1\right) & \cdots & y_{2k}^0 - iy_{2k}^3 & -\left(y_{2k}^2 + iy_{2k}^1\right) \\ y_{12}^2 - iy_{12}^1 & y_{12}^0 + iy_{12}^3 & y_{22}^2 - iy_{22}^1 & y_{22}^0 + iy_{22}^3 & \cdots & y_{2k}^2 - iy_{2k}^1 & y_{2k}^2 - iy_{2k}^1 \\ .. & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ y_{1k}^0 - iy_{1k}^3 & -\left(y_{1k}^2 + iy_{1k}^1\right) & y_{2k}^0 - iy_{2k}^3 & -\left(y_{2k}^2 + iy_{2k}^1\right) & \cdots & y_{kk}^0 - iy_{2k}^3 & -\left(y_{kk}^2 + iy_{kk}^1\right) \\ y_{1k}^2 - iy_{1k}^1 & y_{1k}^0 + iy_{1k}^3 & y_{2k}^2 - iy_{2k}^1 & y_{2k}^0 + iy_{2k}^3 & \cdots & y_{kk}^2 - iy_{kk}^1 & y_{kk}^0 + iy_{kk}^3 \end{bmatrix}. \tag{26}$$

After rearrangement, we can reorganize this ADHM data into the block matrices.

$$a \rightarrow \begin{bmatrix} \lambda_1^0 - i\lambda_1^3 & \lambda_2^0 - i\lambda_2^3 & .. & \lambda_k^0 - i\lambda_k^3 & -\left(\lambda_1^2 + i\lambda_1^1\right) & -\left(\lambda_2^2 + i\lambda_2^1\right) & \cdots & -\left(\lambda_k^2 + i\lambda_k^1\right) \\ \lambda_1^2 - i\lambda_1^1 & \lambda_2^2 - i\lambda_2^1 & .. & \lambda_k^2 - i\lambda_k^1 & \lambda_1^0 + i\lambda_1^3 & \lambda_2^0 + i\lambda_2^3 & \cdots & \lambda_k^0 + i\lambda_k^3 \\ y_{11}^0 - iy_{11}^3 & y_{12}^0 - iy_{12}^3 & \cdots & y_{1k}^0 - iy_{1k}^3 & -\left(y_{11}^2 + iy_{11}^1\right) & -\left(y_{12}^2 + iy_{12}^1\right) & \cdots & -\left(y_{1k}^2 + iy_{1k}^1\right) \\ y_{12}^0 - iy_{12}^3 & y_{22}^0 - iy_{22}^3 & \cdots & y_{2k}^0 - iy_{2k}^3 & -\left(y_{12}^2 + iy_{12}^1\right) & -\left(y_{22}^2 + iy_{22}^1\right) & \cdots & -\left(y_{2k}^2 + iy_{2k}^1\right) \\ .. & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ y_{1k}^0 - iy_{1k}^3 & y_{2k}^0 - iy_{2k}^3 & \cdots & y_{kk}^0 - iy_{kk}^3 & -\left(y_{1k}^2 + iy_{1k}^1\right) & -\left(y_{2k}^2 + iy_{2k}^1\right) & \cdots & -\left(y_{kk}^2 + iy_{kk}^1\right) \\ y_{11}^2 - iy_{11}^1 & y_{12}^2 - iy_{12}^1 & \cdots & y_{1k}^2 - iy_{1k}^1 & y_{11}^0 + iy_{11}^3 & y_{12}^0 + iy_{12}^3 & \cdots & y_{1k}^0 + iy_{1k}^3 \\ y_{12}^2 - iy_{12}^1 & y_{22}^2 - iy_{22}^1 & \cdots & y_{2k}^2 - iy_{2k}^1 & y_{12}^0 + iy_{12}^3 & y_{22}^0 + iy_{22}^3 & \cdots & y_{2k}^0 + iy_{2k}^3 \\ .. & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ y_{1k}^2 - iy_{1k}^1 & y_{2k}^2 - iy_{2k}^1 & \cdots & y_{kk}^2 - iy_{kk}^1 & y_{1k}^0 + iy_{1k}^3 & y_{2k}^0 + iy_{2k}^3 & \cdots & y_{kk}^0 + iy_{kk}^3 \end{bmatrix} \tag{27}$$

$$= \begin{bmatrix} J_1 & J_2 \\ B_{11} & B_{21} \\ B_{12} & B_{22} \end{bmatrix} \tag{28}$$

here we have the *rearrangement rule* for an element z_{ij} in a

$$\begin{aligned} z_{2n-1,2m-1} &\rightarrow z_{n,m} \ , \\ z_{2n-1,2m} &\rightarrow z_{n,k+m} \ , \\ z_{2n,2m-1} &\rightarrow z_{k+n,m} \ , \\ z_{2n,2m} &\rightarrow z_{k+n,k+m}. \end{aligned} \tag{29}$$

and we also do the same process on $a^{\circledast}$

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$$a^{\circledast} = \begin{bmatrix} \lambda_1^{\circledast} & y_{11}^{\circledast} & y_{12}^{\circledast} & .. & y_{1k}^{\circledast} \\ \lambda_2^{\circledast} & y_{12}^{\circledast} & y_{22}^{\circledast} & .. & y_{2k}^{\circledast} \\ .. & ... & ... & ... & ... \\ \lambda_k^{\circledast} & y_{1k}^{\circledast} & y_{2k}^{\circledast} & .. & y_{kk}^{\circledast} \end{bmatrix} \rightarrow \tag{30}$$

$$\tag{31}$$

$$= \begin{bmatrix} -I_2 & B_{22} & -B_{21} \\ I_1 & -B_{12} & B_{11} \end{bmatrix} \tag{32}$$

4.2. The Complex ADHM Equations
After the rearrangement and identification, we get symmetric condition as

$$a^{\circledast}a \rightarrow \begin{bmatrix} (a^{\circledast}a)_{11}^0 & 0 & ... & 0 & 0 & 0 & 0 & 0 \\ 0 & (a^{\circledast}a)_{22}^0 & ... & 0 & 0 & 0 & 0 & 0 \\ .. & ... & ... & ... & 0 & 0 & 0 & 0 \\ 0 & 0 & ... & (a^{\circledast}a)_{kk}^0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & (a^{\circledast}a)_{11}^0 & 0 & ... & 0 \\ 0 & 0 & 0 & 0 & 0 & (a^{\circledast}a)_{22}^0 & ... & 0 \\ 0 & 0 & 0 & 0 & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & 0 & 0 & ... & (a^{\circledast}a)_{kk}^0 \end{bmatrix} \tag{33}$$

$$= \begin{bmatrix} -I_2J_1 + B_{22}B_{11} - B_{21}B_{12} & -I_2J_2 + B_{22}B_{21} - B_{21}B_{22} \\ I_1J_1 + B_{11}B_{12} - B_{12}B_{11} & I_1J_2 + B_{11}B_{22} - B_{12}B_{21} \end{bmatrix}, \tag{34}$$

We can easily derive the complex ADHM equations

$$[B_{11}, B_{12}] + I_1J_1 = 0, \tag{35}$$
$$[B_{21}, B_{22}] + I_2J_2 = 0, \tag{36}$$
$$[B_{11}, B_{22}] + [B_{21}, B_{12}] + I_1J_2 + I_2J_1 = 0. \tag{37}$$

4.3. The Reality Conditions
We can impose the condition

$$\begin{aligned} I_1 &= J^{\dagger}, I_2 = -I, J_1 = I^{\dagger}, J_2 = J, \\ B_{11} &= B_2^{\dagger}, B_{12} = B_1^{\dagger}, B_{21} = -B_1, B_{22} = B_2 \end{aligned}$$

Then we can get the Real ADHM equation which are the constrain equations for $SU(2)$ case.

$$\begin{aligned} [B_1, B_2] + IJ &= 0, \\ \left[B_1, B_1^{\dagger}\right] + \left[B_2, B_2^{\dagger}\right] + II^{\dagger} - J^{\dagger}J &= 0. \end{aligned}$$

5. Monad Construction

In this section, we will talk about α and β matrices in the monad construction.

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5.1. *From Complex ADHM to Monad Construction*

First, we define α and β as functions of homogeneous coordinates x, y, z, w on CP^3

$$\alpha = \begin{bmatrix} zB_{11} + wB_{21} + x \\ zB_{12} + wB_{22} + y \\ zJ_1 + wJ_2 \end{bmatrix}, \tag{38}$$

$$\beta = \begin{bmatrix} -zB_{12} - wB_{22} - y & zB_{11} + wB_{21} + x & zI_1 + wI_2 \end{bmatrix}. \tag{39}$$

Just like the real ADHM equations, the following equation is set up if and only if the ADHM data is satisfied in complex ADHM equations(37).

$$\beta\alpha = 0 \tag{40}$$

In the monad construction, Eq.(40) tells us that Im α is a subspace of Ker β which will make us to consider the quotient space Kerβ/Imα. The quotient space Kerβ/Imα is a function of each point at CP^3 contains two parts of linear transformations. If the transformation α is injective and the transformation β is surjective, then the dimension of Ker β/ Im α will be $k + 2 - k = 2$ on all points of CP^3 ,so we can use holomorphic vector bundles to describe instantons. In $SU(2)$ case, we can always use holomorphic vector bundles to study instantons ,but for $SL(2, C)$ case, the situation is different. β and α may not be surjective and injective at some points of CP^3 for some ADHM data.The dimension of (Ker β/ Im α) may vary from point to point on CP^3, and we have to use sheaf description[11] instead of the holomorphic vector bundles in the non-compact $SL(2, C)$ instanton case.

5.2. *Example of Sheaves*

In this subsection, we will talk about the examples of $SL(2, C)$ Yang-Mills 2-instanton sheaves. In order to discuss explicitly, we define

$$l = \begin{vmatrix} \lambda_1^0 & \lambda_1^3 \\ \lambda_2^0 & \lambda_2^3 \end{vmatrix} - \begin{vmatrix} \lambda_1^1 & \lambda_1^2 \\ \lambda_2^1 & \lambda_2^2 \end{vmatrix}, \tag{41a}$$

$$n = \begin{vmatrix} \lambda_1^0 & \lambda_1^2 \\ \lambda_2^0 & \lambda_2^2 \end{vmatrix} - \begin{vmatrix} \lambda_1^0 & \lambda_1^1 \\ \lambda_2^0 & \lambda_2^1 \end{vmatrix}, \tag{41b}$$

$$m = \begin{vmatrix} \lambda_1^0 & \lambda_1^1 \\ \lambda_2^0 & \lambda_2^1 \end{vmatrix} - \begin{vmatrix} \lambda_1^2 & \lambda_1^3 \\ \lambda_2^2 & \lambda_2^3 \end{vmatrix}. \tag{41c}$$

And we know y_{12} can be express from other parameters λ_1 , λ_2 , y_{11} , and y_{22}

$$y_{12} = \frac{1}{2}\frac{(y_1 - y_2)}{|y_1 - y_2|_c^2}(\lambda_2^{\circledast}\lambda_1 - \lambda_1^{\circledast}\lambda_2). \tag{42}$$

5.2.1. *Example One*

For the first sample solution, we take the moduli parameters$\lambda_1^1 = \lambda_1^2 = \lambda_1^3 = \lambda_2^0 = \lambda_2^2 = \lambda_2^3 = 0$, then we get $l = 0$, $n = 0$ and $m = \lambda_1^0\lambda_2^1$. With these inputs, $w = \frac{1}{im}\left[0 \pm \sqrt{-m^2}\right] = \pm 1$ and the constraints from common eigenvector become

$$\begin{bmatrix} \pm\left(\frac{-i}{2d}\right)m \\ d - \sqrt{d^2 - \frac{m^2}{4d^2}} \end{bmatrix} \sim \begin{bmatrix} \left(\frac{-i}{2d}\right)m \\ \pm d - \sqrt{d^2 - \frac{m^2}{4d^2}} \end{bmatrix} \sim \begin{bmatrix} \mp i\lambda_2^1 \\ -\lambda_1^0 \end{bmatrix}. \tag{43}$$

If we choose $d^2 - \frac{m^2}{4d^2} = 0$, we have

$$m = 2d^2 = \lambda_1^0\lambda_2^1, \;\; \lambda_1^0 = -\lambda_2^1, \; x = y = 0. \tag{44}$$

Let's set $\lambda_1^0 = a, \lambda_2^1 = -a$ where a is a complex number and $a \neq 0$, then the corresponding solutions of moduli parameters are

$$\begin{bmatrix} \lambda_1 & \lambda_2 \\ y_{11} & y_{12} \\ y_{12} & y_{22} \end{bmatrix} \sim \begin{bmatrix} \lambda_1^0 - i\lambda_1^3 & -\left(\lambda_1^2 + i\lambda_1^1\right) & \lambda_2^0 - i\lambda_2^3 & -\left(\lambda_2^2 + i\lambda_2^1\right) \\ \lambda_1^2 - i\lambda_1^1 & \lambda_1^0 + i\lambda_1^3 & \lambda_2^2 - i\lambda_2^1 & \lambda_2^0 + i\lambda_2^3 \\ -d & 0 & \left(\frac{-i}{2d}\right) l & \left(\frac{-i}{2d}\right)(m - in) \\ 0 & -d & \left(\frac{-i}{2d}\right)(m + in) & -\left(\frac{-i}{2d}\right) l \\ \left(\frac{-i}{2d}\right) l & \left(\frac{-i}{2d}\right)(m - in) & d & 0 \\ \left(\frac{-i}{2d}\right)(m + in) & -\left(\frac{-i}{2d}\right) l & 0 & d \end{bmatrix}$$

$$= \begin{bmatrix} a & 0 & 0 & ia \\ 0 & a & ia & 0 \\ \frac{-i}{\sqrt{2}}a & 0 & 0 & \frac{a}{\sqrt{2}} \\ 0 & \frac{-i}{\sqrt{2}}a & \frac{a}{\sqrt{2}} & 0 \\ 0 & \frac{a}{\sqrt{2}} & \frac{i}{\sqrt{2}}a & 0 \\ \frac{a}{\sqrt{2}} & 0 & 0 & \frac{i}{\sqrt{2}}a \end{bmatrix}, a \neq 0. \tag{45}$$

Note that since $\lambda_2^1 \neq 0$, this set of ADHM data is outside of CFTW case. Thus, we have discovered that, for points $[x : y : z : w] = [0 : 0 : 1 : \pm 1]$ on CP^3 and the ADHM data given in Eq.(45), the vector bundle description of $SL(2, C)$ 2-instanton on CP^3 breaks down, and which is led to use sheaf description for these non-compact Yang-Mills instantons or "sheaf instantons" on CP^3[11]. Note in addition that for the case of $SU(2)$ 2-instanton, λ_1^0 , λ_2^1 and d are real numbers inconsistent with Eq.(44). This is consistent with the known vector bundle description of $SU(2)$ 2-instanton on CP^3.

5.2.2. Example Two For the second sample solution, we can take the moduli parameters as $\lambda_1^1 = \lambda_1^2 = \lambda_1^3 = \lambda_2^0 = \lambda_2^1 = \lambda_2^3 = 0$, then we get $l = 0$, $n = \lambda_1^0\lambda_2^2$ and $m = 0$. With these inputs, $w = \pm i$ and the constraints from common eigenvector become

$$\begin{bmatrix} \pm\left(\frac{-i}{2d}\right) in \\ id - \sqrt{-d^2 + \frac{n^2}{4d^2}} \end{bmatrix} \sim \begin{bmatrix} \left(\frac{-i}{2d}\right) n \\ \pm d - \sqrt{d^2 - \frac{n^2}{4d^2}} \end{bmatrix} \sim \begin{bmatrix} \mp i\lambda_2^2 \\ -\lambda_1^0 \end{bmatrix}.$$

If we choose $d^2 - \frac{n^2}{4d^2} = 0$, we have

$$n = 2d^2 = \lambda_1^0\lambda_2^2\ ,\ \ \lambda_1^0 = -\lambda_2^2,\ \ x = y = 0. \tag{46}$$

Let's set $\lambda_1^0 = a, \lambda_2^2 = -a$ where a is a complex number and $a \neq 0$, then the corresponding solutions of moduli parameters are

$$\begin{bmatrix} \lambda_1 & \lambda_2 \\ y_{11} & y_{12} \\ y_{12} & y_{22} \end{bmatrix} \sim \begin{bmatrix} \lambda_1^0 - i\lambda_1^3 & -\left(\lambda_1^2 + i\lambda_1^1\right) & \lambda_2^0 - i\lambda_2^3 & -\left(\lambda_2^2 + i\lambda_2^1\right) \\ \lambda_1^2 - i\lambda_1^1 & \lambda_1^0 + i\lambda_1^3 & \lambda_2^2 - i\lambda_2^1 & \lambda_2^0 + i\lambda_2^3 \\ -d & 0 & \left(\frac{-i}{2d}\right) l & \left(\frac{-i}{2d}\right)(m - in) \\ 0 & -d & \left(\frac{-i}{2d}\right)(m + in) & -\left(\frac{-i}{2d}\right) l \\ \left(\frac{-i}{2d}\right) l & \left(\frac{-i}{2d}\right)(m - in) & d & 0 \\ \left(\frac{-i}{2d}\right)(m + in) & -\left(\frac{-i}{2d}\right) l & 0 & d \end{bmatrix}$$

$$= \begin{bmatrix} a & 0 & 0 & a \\ 0 & a & -a & 0 \\ \frac{-i}{\sqrt{2}}a & 0 & 0 & \frac{-ia}{\sqrt{2}} \\ 0 & \frac{-i}{\sqrt{2}}a & \frac{ia}{\sqrt{2}} & 0 \\ 0 & \frac{-ia}{\sqrt{2}} & \frac{i}{\sqrt{2}}a & 0 \\ \frac{ia}{\sqrt{2}} & 0 & 0 & \frac{i}{\sqrt{2}}a \end{bmatrix}, a \neq 0. \tag{47}$$

Notice that $\lambda_2^2 \neq 0$, this ADHM data is again outside of CFTW case. Eq.(47) gives the second example of the $SL(2, C)$ 2-instanton sheaf solution on CP^3. Note again that for the case of $SU(2)$ 2-instanton, λ_1^0 , λ_2^2 and d are real numbers which are inconsistent with Eq.(44). We have discovered that, for points $[x : y : z : w] = [0 : 0 : 1 : \pm i]$ on CP^3 and the ADHM data given in Eq.(47), the description of vector bundle of $SL(2, C)$ 2-instanton on CP^3 breaks down.

To the further discussion of the geometric picture we have shown above points $[x : y : z : w] = [0 : 0 : 1 : \pm 1]$ or $[0 : 0 : 1 : \pm i]$ (or setting $w = 1$ instead of $z = 1$ with similar formulas) the vector bundle description is valid. In general, any other biquaternion ADHM data a set of points can be found as above. So the vector bundle description is necessarily valid outside these finitely many points. (this does not mean, however, that the vector bundle description has to be broken at these finitely many points)

We can make a conclusion that a certain proper subset in our biquaternion ADHM data (including the examples above), the description of vector bundle will break down only at some finitely many points (which depend on the ADHM data). Nevertheless, for ADHM data out of this proper subset the vector bundle description remains valid on the whole CP^3. In mathematics[11], the breakdown of the vector bundle description is related to the third Chern number c_3 of the obtained sheaf, which could be nonzero in the sheaf case in contrast to the vector bundle case in which c_3 is necessarily zero because the bundle is of rank two (two dimensional).

Secondly, the one to one correspondence between ASD connections on the one side and certain holomorphic objects on the other side-twistor space in ADHM construction will help us to to understand ASD connections by using the knowledge and information on the twistor side which is mainly accomplished by Penrose-Ward transform. While this correspondence works well for the vector bundle case to the $SU(2)$ instantons, $SL(2, C)$ case the holomorphic objects are no longer vector bundles on the twistor space as we have discussed in this paper. For example, a vector bundle on CP^3 can descend down to S^4 if and only if its set of jumping lines does not include any fiber of the fibration map $CP^3 \to S^4$. It is worth working on examining the singularities of the Penrose-Ward transformed object on S^4. When the preceding jumping-line condition is not met, or when the holomorphic object on CP^3 is a sheaf instead of a vector bundle, the singularities may appear. The cases vary a lot and the nature of the problem appears really different from one case to another case. So, studying these singularities on S^4 is an important future work to do.

Finally, it is natural to ask if our biquaternion ADHM solutions give all solutions to complex ADHM equations. In our biquaternion construction, we can easily find the number $16k - 6$ of parameters match the number expected by mathematicians[11]. An interested reader will easily find the solution for $k = 1$ with $I_1 = [1\ 0]$, $I_2 = [0\ 1]$, $J_1 = J_2 = 0$ and all $B_{lm} = 0$ [11] seem to be out of the biquaternion ADHM data. Now if you take $I_1 = [t\ 0]$, $I_2 = [0\ t]$, $J_1 = -I_1^\dagger$, $J_2 = I_2^\dagger$ and all $B_{lm} = 0$, which is seen to be a biquaternion ADHM solution and is equivalent to $(B_{lm}, gI_m, J_m g^{-1})$ in general for any nonzero complex number g, then you will see, by letting $g = t^{-1}$ and setting $t \to 0$ the above biquaternion solution under equivalence indeed reproduces the above (non-biquaternion) solution.

6. Conclusion

In $SU(2)$ Yang-Mills instanton ADHM construction, an one to one correspondence between anti-self-dual $SU(2)$ connections on S^4 and global holomorphic vector bundles of rank two on CP^3 satisfying certain reality conditions. We are going to extend this correspondence to the case of non-compact $SL(2, C)$ Yang-Mills instanton. First, we use biquaternion ADHM construction to build $SL(2, C)$ Yang-Mills instanton solutions which can be shown the complex ADHM data satisfy the complex version of the ADHM equations and the monad construction.

The next step is to calculate whether the ADHM data makes α injective and β surjective. In the $SL(2, C)$ CFTW k-instanton solutions case with $10k$ moduli parameters, although the jumping lines exist on S^4, the corresponding ADHM data are locally free which means α is injective and β is surjective and the vector bundle description of $SL(2,C)$ CFTW k-instanton on CP^3 remains valid as in the case of $SU(2)$ instantons. We then continue to calculate the second case of complete known $SL(2,C)$ 2-instanton solutions with 26 moduli parameters. We find that, for some subset of the complex ADHM data of $SL(2,C)$ 2-instanton solutions on some points on CP^3, the vector bundle description of $SL(2,C)$ 2-instanton on CP^3 breaks down, and this is led to use sheaf description for these non-compact Yang-Mills instantons or "sheaf instantons" on CP^3.

Although we have found the instanton sheaves by using quaternion methods in the previous discussion, we have not worked out the explicit constructions of instanton sheaves yet. We hope the explicit forms of the $SL(2,C)$ Yang-Mills sheaf instanton solutions constructed in this paper will help us to dig more materials in both physical and mathematical fields.

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