IOP Conf. Series: Journal of Physics: Conf. Series **804** (2017) 012015 doi:10.1088/1742-6596/804/1/012015

Positive Energy Unitary Irreducible Representations of the Superalgebras $osp(1|2n, I\!R)$ and Character Formulae for $n = 3$

V.K. Dobrev

Institute of Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences, 72 Tsarigradsko Chaussee, 1784 Sofia, Bulgaria

I. Salom

Institute of Physics, University of Belgrade, Pregrevica 118, 11080 Zemun, Belgrade, Serbia

Abstract. We overview our study of the positive energy (lowest weight) unitary irreducible representations of the superalgebras $osp(1|2n, I\!R)$. We give more explicitly character formulae for these representations in the case $n = 3$.

1. Introduction

Recently, superconformal field theories in various dimensions are attracting more interest, in particular, due to their duality to AdS supergravities. Until recently only those for $D \leqslant 6$ were studied since in these cases the relevant superconformal algebras satisfy [1] the Haag-Lopuszanski-Sohnius theorem [2]. Thus, such classification was known only for the $D = 4$ superconformal algebras $su(2, 2/N)$ [3] (for $N = 1$), [4, 5, 6, 7] (for arbitrary N). More recently, the classification for $D = 3$ (for even N), $D = 5$, and $D = 6$ (for $N = 1, 2$) was given in [8] (some results are conjectural), and then the $D = 6$ case (for arbitrary N) was finalized in [9].

On the other hand the applications in string theory require the knowledge of the UIRs of the conformal superalgebras for $D > 6$. Most prominent role play the superalgebras $osp(1|2n)$. Initially, the superalgebra $osp(1|32)$ was put forward for $D = 10$ [10]. Later it was realized that $osp(1|2n)$ would fit any dimension, though they are minimal only for $D = 3, 9, 10, 11$ (for $n = 2, 16, 16, 32$, resp.) [11]. In all cases we need to find first the UIRs of $osp(1|2n, I\!R)$ which study was started in [12] and [13]. Later, in [14] we finalized the UIR classification of [12] as Dobrev-Zhang-Salom (DZS) Theorem. There we also proved the DZS Theorem for $osp(1|6)$, while the case $osp(1|8)$ was proved in [15].

In the present paper we present more explicitly the character formulae for $osp(1|6)$. For the lack of space we refer for extensive literature on the subject in [12, 14].

2. Preliminaries on representations

Our basic references for Lie superalgebras are [16, 17], although in this exposition we follow [12].

The even subalgebra of $\mathcal{G} = osp(1|2n, I\!R)$ is the algebra $sp(2n, I\!R)$ with maximal compact subalgebra $\mathcal{K} = u(n) \cong su(n) \oplus u(1)$.

We label the relevant representations of $\mathcal{G}$ by the signature:

$$\chi \;=\; [\, d\, ;\, a_1\, , ..., a_{n-1}\,] \tag{1}$$

where d is the conformal weight, and $a_1, ..., a_{n-1}$ are non-negative integers which are Dynkin labels of the finite-dimensional UIRs of the subalgebra $su(n)$ (the simple part of $\mathcal{K}$).

We present the classification of the positive energy (lowest weight) UIRs of $\mathcal{G}$ following [12, 14] where were used the methods used for the $D = 4, 6$ conformal superalgebras, cf. [4, 5, 6, 7, 9]. The main tool is an adaptation of the Shapovalov form [18] on the Verma modules V^χ over the complexification $\mathcal{G}^{\mathbb{C}} \;=\; osp(1|2n)$ of $\mathcal{G}$.

The root system of $\mathcal{G}^{\mathbb{C}}$ are given in terms of $\delta_1 \ldots, \delta_n$, $(\delta_i, \delta_j) \;=\; \delta_{ij}$, $i, j = 1, ..., n$. The even and odd roots systems are [16]:

$$\begin{aligned} \Delta_{\bar 0} &= \{\pm\delta_i \pm \delta_j\ ,\ 1 \leqslant i < j \leqslant n\ ,\quad \pm 2\delta_i\ ,\ 1 \leqslant i \leqslant n\}\ , \\ \Delta_{\bar 1} &= \{\pm\delta_i\ ,\ 1 \leqslant i \leqslant n\} \end{aligned} \tag{2}$$

(we remind that the signs $\pm$ are not correlated). We shall use the following distinguished simple root system [16]:

$$\Pi \;=\; \{\, \delta_1 - \delta_2\, ,\, , \ldots , \delta_{n-1} - \delta_n\, ,\, \delta_n\, \}\ , \tag{3}$$

or introducing standard notation for the simple roots:

$$\begin{aligned} \Pi &= \{\, \alpha_1\,, ..., \, \alpha_n \,\}\ , \\ &\alpha_j \ = \ \delta_j - \delta_{j+1}\ , \quad j = 1, ..., n-1\ , \quad \alpha_n \ = \ \delta_n\ . \end{aligned} \tag{4}$$

The root $\alpha_n = \delta_n$ is odd, the other simple roots are even. The Dynkin diagram is:

$$\underset{1}{\circ} \text{——} \cdots \text{——} \underset{n-1}{\circ} \Longrightarrow \underset{n}{\bullet} \tag{5}$$

The black dot is used to signify that the simple odd root is not nilpotent. In fact, the superalgebras $B(0,n) = osp(1|\,2n)$ have no nilpotent generators unlike all other types of basic classical Lie superalgebras [16].

The corresponding to Π positive root system is:

$$\Delta^+_{\bar 0} \ = \{\delta_i \pm \delta_j\,,\ 1 \leqslant i < j \leqslant n,\ 2\delta_i\,,\ 1 \leqslant i \leqslant n\}, \qquad \Delta^+_{\bar 1} \ = \{\delta_i\,,\ 1 \leqslant i \leqslant n\} \tag{6}$$

Conversely, we give the elementary functionals through the simple roots:

$$\delta_k \ = \ \alpha_k + \cdots + \alpha_n\ . \tag{7}$$

From the point of view of representation theory more relevant is the restricted root system, such that:

$$\begin{aligned} \bar\Delta^+ \ &= \ \bar\Delta^+_{\bar 0} \cup \Delta^+_{\bar 1}\ , \\ \bar\Delta^+_{\bar 0} \ &\equiv \ \{\alpha \in \Delta^+_{\bar 0} \mid \tfrac{1}{2}\alpha \notin \Delta^+_{\bar 1}\} \ = \ \{\delta_i \pm \delta_j\ ,\ 1 \leqslant i < j \leqslant n\} \end{aligned} \tag{8}$$

The superalgebra $\mathcal{G} \ = \ osp(1|\,2n, I\!R)$ is a split real form of $osp(1|\,2n)$ and has the same root system.

The above simple root system is also the simple root system of the complex simple Lie algebra B_n (dropping the distinction between even and odd roots) with Dynkin diagram:

$$\underset{1}{\circ} \text{——} \cdots \text{——} \underset{n-1}{\circ} \Longrightarrow \underset{n}{\circ} \tag{9}$$

and root system:

$$\Delta^+_{\mathrm{B_n}} \ = \ \{\delta_i \pm \delta_j\ ,\ 1 \leqslant i < j \leqslant n\ ,\ \ \delta_i\ ,\ 1 \leqslant i \leqslant n\} \ \cong \ \bar\Delta^+ \tag{10}$$

This shall be used essentially below.

We need explicitly the lowest weight $\Lambda \in \mathcal{H}^*$ (where $\mathcal{H}$ is the Cartan subalgebra of $\mathcal{G}^{\mathbb{C}}$) the values of which should be related to the signature (1):

$$(\Lambda, \alpha_k^\vee) \ = - \, a_k \ , \quad 1 \leqslant k \leqslant n \ , \tag{11}$$

where $\alpha_k^\vee \equiv 2\alpha_k/(\alpha_k, \alpha_k)$, and the minus signs anticipate the fact that we shall use lowest weight Verma modules (instead of the highest weight modules used in [17]) and to Verma module reducibility w.r.t. the roots α_k (this is explained in detail in [6, 12]).

Obviously, a_n must be related to the conformal weight d which is a matter of normalization so as to correspond to some known cases. Thus, our choice is:

$$a_n \ = \ - \, 2d - a_1 - \cdots - a_{n-1} \ . \tag{12}$$

The actual Dynkin labelling is given by:

$$m_k \ = \ (\rho - \Lambda, \alpha_k^\vee) \tag{13}$$

where $\rho \in \mathcal{H}^*$ is given by the difference of the half-sums $\rho_{\bar{0}}$, $\rho_{\bar{1}}$ of the even, odd, resp., positive roots (cf. (6):

$$\begin{aligned} \rho \ \ \doteq \ \ \rho_{\bar{0}} - \rho_{\bar{1}} \ &= \ (n - \tfrac{1}{2})\delta_1 + (n - \tfrac{3}{2})\delta_2 + \cdots + \tfrac{3}{2}\delta_{n-1} + \tfrac{1}{2}\delta_n \ , \qquad (14) \\ \rho_{\bar{0}} \ &= \ n\delta_1 + (n-1)\delta_2 + \cdots + 2\delta_{n-1} + \delta_n \ , \\ \rho_{\bar{1}} \ &= \ \tfrac{1}{2}(\delta_1 + \cdots + \delta_n) \ . \end{aligned}$$

Naturally, the value of ρ on the simple roots is 1: $(\rho, \alpha_i^\vee) = 1, i = 1, ..., n$.

Unlike $a_k \in \mathbb{Z}_+$ for $k < n$ the value of a_n is arbitrary. In the cases when a_n is also a non-negative integer, and then $m_k \in \mathbb{N}$ $(\forall k)$ the corresponding irreps are the finite-dimensional irreps of $\mathcal{G}$.

To introduce Verma modules we use the standard decomposition:

$$\mathcal{G}^{\mathbb{C}} \ = \ \mathcal{G}^+ \oplus \mathcal{H} \oplus \mathcal{G}^- \tag{15}$$

where $\mathcal{G}^+$, $\mathcal{G}^-$, resp., are the subalgebras corresponding to the positive, negative, roots, resp., and $\mathcal{H}$ denotes the Cartan subalgebra.

We consider lowest weight Verma modules, so that $V^\Lambda \cong U(\mathcal{G}^+) \otimes v_0$, where $U(\mathcal{G}^+)$ is the universal enveloping algebra of $\mathcal{G}^+$, and v_0 is a lowest weight vector v_0 such that:

$$\begin{aligned} Z \, v_0 \ &= \ 0 \ , \quad Z \in \mathcal{G}^- \\ H \, v_0 \ &= \ \Lambda(H) \, v_0 \ , \quad H \in \mathcal{H} \ . \end{aligned} \tag{16}$$

Further, for simplicity we omit the sign $\otimes$, i.e., we write $p\,v_0 \in V^\Lambda$ with $p \in U(\mathcal{G}^+)$.

Adapting the criterion of [17] to lowest weight modules, one finds that a Verma module V^Λ is reducible w.r.t. the positive root β iff the following holds [12]:

$$(\rho - \Lambda, \beta^\vee) = m_\beta\ , \qquad \beta \in \Delta^+\ , \quad m_\beta \in \mathbb{N}\ . \tag{17}$$

If a condition from (17) is fulfilled then V^Λ contains a submodule which is a Verma module $V^{\Lambda'}$ with shifted weight given by the pair m,β : $\Lambda' = \Lambda + m\beta$. The embedding of $V^{\Lambda'}$ in V^Λ is provided by mapping the lowest weight vector v_0' of $V^{\Lambda'}$ to the **singular vector** $v_s^{m,\beta}$ in V^Λ which is completely determined by the conditions [19]:

$$\begin{aligned} X\ v_s^{m,\beta} &= 0\ , \quad X \in \mathcal{G}^-\ , \\ H\ v_s^{m,\beta} &= \Lambda'(H)\ v_0\ , \quad H \in \mathcal{H}\ , \quad \Lambda' = \Lambda + m\beta\ . \end{aligned} \tag{18}$$

Explicitly, $v_s^{m,\beta}$ is given by a polynomial in the positive root generators [20, 21]:

$$v_s^{m,\beta} = P^{m,\beta}\, v_0\ , \quad P^{m,\beta} \in U(\mathcal{G}^+)\ . \tag{19}$$

Thus, the submodule I^β of V^Λ which is isomorphic to $V^{\Lambda'}$ is given by $U(\mathcal{G}^+)\, P^{m,\beta}\, v_0$.

Certainly, (17) may be fulfilled for several positive roots (even for all of them). Let Δ_Λ denote the set of all positive roots for which (17) is fulfilled, and let us denote: $\tilde{I}^\Lambda \equiv \cup_{\beta\in\Delta_\Lambda} I^\beta$. Clearly, $\tilde{I}^\Lambda$ is a proper submodule of V^Λ. Let us also denote $F^\Lambda \equiv V^\Lambda/\tilde{I}^\Lambda$.

The Verma module V^Λ contains a unique proper maximal submodule I^Λ ($\supseteq \tilde{I}^\Lambda$) [17, 22]. Among the lowest weight modules with lowest weight Λ there is a unique irreducible one, denoted by L_Λ, i.e., $L_\Lambda = V^\Lambda/I^\Lambda$.

It may happen that the maximal submodule I^Λ coincides with the submodule $\tilde{I}^\Lambda$ generated by all singular vectors. This is, e.g., the case for all Verma modules if $\mathrm{rank}\,\mathcal{G} \leqslant 2$, or when (17) is fulfilled for all simple roots (and, as a consequence for all positive roots). Here we are interested in the cases when $\tilde{I}^\Lambda$ is a proper submodule of I^Λ. We need the following notion.

Definition: [22, 23, 24] *Let* V^Λ *be a reducible Verma module. A vector* $v_{\rm ssv} \in V^\Lambda$ *is called a* **subsingular vector** *if* $v_{\rm su} \notin \tilde{I}^\Lambda$ *and the following holds:*

$$X\ v_{\rm su} \in \tilde{I}^\Lambda\ , \quad \forall X \in \mathcal{G}^- \tag{20}$$

Going from the above more general definitions to $\mathcal{G}$ we recall that in [12] it was established that from (17) follows that the Verma module $V^{\Lambda(\chi)}$ is

IOP Conf. Series: Journal of Physics: Conf. Series **804** (2017) 012015 doi:10.1088/1742-6596/804/1/012015

reducible if one of the following relations holds:

$$\mathbb{N} \ni m^-_{ij} = j - i + a_i + \cdots + a_{j-1} \tag{21a}$$
$$\mathbb{N} \ni m^+_{ij} = 2n - i - j + 1 + a_j + \cdots + a_{n-1} - a_1 - \cdots - a_{i-1} - 2d \tag{21b}$$
$$\mathbb{N} \ni m_i = 2n - 2i + 1 + a_i + \cdots + a_{n-1} - a_1 + \cdots - a_{i-1} - 2d \tag{21c}$$
$$\mathbb{N} \ni m_{ii} = n - i + \tfrac{1}{2}(1 + a_i + \cdots + a_{n-1} - a_1 + \cdots - a_{i-1}) - d \tag{21d}$$

corresponding to the roots $\delta_i - \delta_j$, $\delta_i + \delta_j$, $(i < j)$, δ_i, $2\delta_i$, resp. Further we shall use the fact from [12] that we may eliminate the reducibilities and embeddings related to the roots $2\delta_i$. Indeed, since $m_i = 2m_{ii}$, whenever (21d) is fulfilled also (21c) is fulfilled.

For further use we introduce notation for the root vector $X^+_j \in \mathcal{G}^+$, $j = 1, \ldots, n$, corresponding to the simple root α_j.

Further, we notice that all reducibility conditions in (21a) are fulfilled. In particular, for the simple roots from those condition (21a) is fulfilled with $\beta \to \alpha_i = \delta_i - \delta_{i+1}$, $i = 1, ..., n-1$ and $m^-_i \equiv m^-_{i,i+1} = 1 + a_i$. The corresponding submodules $I^{\alpha_i} = U(\mathcal{G}^+)\, v^i_s$, where $\Lambda_i = \Lambda + m^-_i \alpha_i$ and $v^i_s = (X^+_i)^{1+a_i}\, v_0$. These submodules generate an invariant submodule which we denote by $I^\Lambda_c \subset \tilde{I}^\Lambda$. Since these submodules are nontrivial for all our signatures in the question of unitarity instead of V^Λ we shall consider also the factor-modules:

$$F^\Lambda_c = V^\Lambda / I^\Lambda_c \supset F^\Lambda . \tag{22}$$

We shall denote the lowest weight vector of F^Λ_c by $|\Lambda_c\rangle$ and the singular vectors above become null conditions in F^Λ_c :

$$(X^+_i)^{1+a_i}\, |\Lambda_c\rangle = 0 , \quad i = 1, ..., n-1. \tag{23}$$

If the Verma module V^Λ is not reducible w.r.t. the other roots, i.e., (21b,c,d) are not fulfilled, then $F^\Lambda_c = F^\Lambda$ is irreducible and is isomorphic to the irrep L_Λ with this weight.

In fact, for the factor-modules reducibility is controlled by the value of d, or in more detail:

The maximal d coming from the different possibilities in (21b) are obtained for $m^+_{ij} = 1$ and they are:

$$d_{ij} \equiv n + \tfrac{1}{2}(a_j + \cdots + a_{n-1} - a_1 - \cdots - a_{i-1} - i - j) , \quad i < j, \tag{24}$$

the corresponding root being $\delta_i + \delta_j$.

IOP Conf. Series: Journal of Physics: Conf. Series **804** (2017) 012015 doi:10.1088/1742-6596/804/1/012015

The maximal d coming from the different possibilities in (21c) are obtained for $m_i = 1$ and they are:

$$d_i \equiv n - i + \tfrac{1}{2}(a_i + \cdots + a_{n-1} - a_1 - \cdots - a_{i-1}) \ , \tag{25}$$

the corresponding roots being δ_i .

There are some orderings between these maximal reduction points [12]:

$$\begin{aligned} d_1 &> d_2 > \cdots > d_n \ , \\ d_{i,i+1} &> d_{i,i+2} > \cdots > d_{in} \ , \\ d_{1,j} &> d_{2,j} > \cdots > d_{j-1,j} \ , \\ d_i &> d_{jk} > d_\ell \ , \qquad i \leqslant j < k \leqslant \ell \ . \end{aligned} \tag{26}$$

Obviously the first reduction point is:

$$d_1 = n - 1 + \tfrac{1}{2}(a_1 + \cdots + a_{n-1}) \ . \tag{27}$$

Below we shall use the following notion. The singular vector v_1 is called **descendant** of the singular vector $v_2 \notin \mathbb{C}v_1$ if there exists a homogeneous polynomial P_{12} in $U(\mathcal{G}^+)$ so that $v_1 = P_{12}\, v_2$. Clearly, in this case we have: $I^1 \subset I^2$, where I^k is the submodule generated by v_k . Thus, when we factor the submodule I_2 this means factoring also the submodule I_1.

3. Unitarity

The first results on the unitarity were given in [12], and then improved in [14]. Thus, the statement below should be called *Dobrev-Zhang-Salom Theorem*:

Theorem DZS: All positive energy unitary irreducible representations of the superalgebras $osp(1|\,2n, I\!\!R)$ characterized by the signature χ in (1) are obtained for real d and are given as follows:

$$\begin{aligned}
d &\geqslant n-1+\tfrac{1}{2}(a_1+\cdots+a_{n-1}) = d_1 , \quad a_1 \neq 0 , && (28)\\
d &\geqslant n-\tfrac{3}{2}+\tfrac{1}{2}(a_2+\cdots+a_{n-1}) = d_{12} , \quad a_1=0,\ a_2\neq 0 ,\\
d &= n-2+\tfrac{1}{2}(a_2+\cdots+a_{n-1}) = d_2 > d_{13} , \quad a_1=0,\ a_2\neq 0 ,\\
d &\geqslant n-2+\tfrac{1}{2}(a_3+\cdots+a_{n-1}) = d_2 = d_{13} , \quad a_1=a_2=0,\ a_3\neq 0 ,\\
d &= n-\tfrac{5}{2}+\tfrac{1}{2}(a_3+\cdots+a_{n-1}) = d_{23} > d_{14} , \quad a_1=a_2=0,\ a_3\neq 0 ,\\
d &= n-3+\tfrac{1}{2}(a_3+\cdots+a_{n-1}) = d_3 = d_{24} > d_{15} , \quad a_1=a_2=0,\ a_3\neq 0 , && (29)\\
&\ldots\\
d &\geqslant n-1-\kappa+\tfrac{1}{2}(a_{2\kappa+1}+\cdots+a_{n-1}) , \quad a_1=\ldots=a_{2\kappa}=0,\ a_{2\kappa+1}\neq 0 ,\\
&\qquad \kappa=\tfrac{1}{2},1,\ldots,\tfrac{1}{2}(n-1) ,\\
d &= n-\tfrac{3}{2}-\kappa+\tfrac{1}{2}(a_{2\kappa+1}+\cdots+a_{n-1}) , \quad a_1=\ldots=a_{2\kappa}=0,\ a_{2\kappa+1}\neq 0 ,\\
&\ldots\\
d &= n-1-2\kappa+\tfrac{1}{2}(a_{2\kappa+1}+\cdots+a_{n-1}) , \quad a_1=\ldots=a_{2\kappa}=0,\ a_{2\kappa+1}\neq 0 ,\\
&\ldots\\
d &\geqslant \tfrac{1}{2}(n-1) , \quad a_1 = \ldots = a_{n-1} = 0\\
d &= \tfrac{1}{2}(n-2) , \quad a_1 = \ldots = a_{n-1} = 0\\
&\ldots\\
d &= \tfrac{1}{2} , \quad a_1 = \ldots = a_{n-1} = 0\\
d &= 0 , \quad a_1 = \ldots = a_{n-1} = 0
\end{aligned}$$

Parts of the *Proof* were given in [12], while in [14] was given a detailed sketch of the Proof. In [14] was given also the Proof for the case $n=3$, while the proof for $n=4$ was given in [15].

4. Character formulae

Let $\hat{\mathcal{G}}$ be a simple Lie algebra of rank ℓ with Cartan subalgebra $\hat{\mathcal{H}}$, root system $\hat{\Delta}$, simple root system $\hat{\pi}$. Let Γ, (resp. Γ_+), be the set of all integral, (resp. integral dominant), elements of $\hat{\mathcal{H}}^*$, i.e., $\lambda \in \hat{\mathcal{H}}^*$ such that $(\lambda, \alpha_i^\vee) \in \mathbb{Z}$, (resp. $\mathbb{Z}_+$), for all simple roots α_i, ($\alpha_i^\vee \equiv 2\alpha_i/(\alpha_i,\alpha_i)$). Let V be a lowest weight module with lowest weight Λ and lowest weight vector v_0. It has the following decomposition:

$$\begin{aligned}
V &= \mathop{\oplus}_{\mu\in\Gamma_+} V_\mu , && (30)\\
V_\mu &= \{u \in V \mid Hu = (\Lambda+\mu)(H)u,\ \forall\ H \in \mathcal{H}\}
\end{aligned}$$

(Note that $V_0 = \mathbb{C}v_0$.) Let $E(\mathcal{H}^*)$ be the associative abelian algebra consisting of the series $\sum_{\mu\in\mathcal{H}^*} c_\mu e(\mu)$, where $c_\mu \in \mathbb{C},\ c_\mu = 0$ for μ outside the union of a finite number of sets of the form $D(\lambda) = \{\mu \in \mathcal{H}^* | \mu \geqslant \lambda\}$, using some ordering of $\mathcal{H}^*$, e.g., the lexicographic one; the formal exponents $e(\mu)$ have the properties: $e(0) = 1,\ e(\mu)e(\nu) = e(\mu+\nu)$.

Then the (formal) character of V is defined by:

$$\begin{aligned} ch_0\ V &= \sum_{\mu\in\Gamma_+} (\dim\ V_\mu)\ e(\Lambda+\mu) = \\ &= e(\Lambda) \sum_{\mu\in\Gamma_+} (\dim\ V_\mu)\ e(\mu) \end{aligned} \tag{31}$$

(We shall use subscript '0' for the even case.)

For a Verma module, i.e., $V = V^\Lambda$ one has dim $V_\mu = P(\mu)$, where $P(\mu)$ is a generalized partition function, $P(\mu) = \#$ of ways μ can be presented as a sum of positive roots β, each root taken with its multiplicity $\dim \mathcal{G}_\beta$ $(= 1$ here), $P(0) \equiv 1$. Thus, the character formula for Verma modules is:

$$\begin{aligned} ch_0\ V^\Lambda &= e(\Lambda) \sum_{\mu\in\Gamma_+} P(\mu)e(\mu) = \\ &= e(\Lambda) \prod_{\alpha\in\Delta^+} (1-e(\alpha))^{-1} \end{aligned} \tag{32}$$

Further we recall the standard reflections in $\hat{\mathcal{H}}^*$:

$$s_\alpha(\lambda) = \lambda - (\lambda, \alpha^\vee)\alpha\,, \quad \lambda \in \hat{\mathcal{H}}^*\,, \quad \alpha \in \hat{\Delta}. \tag{33}$$

The Weyl group W is generated by the simple reflections $s_i \equiv s_{\alpha_i}$, $\alpha_i \in \hat{\pi}$. Thus every element $w \in W$ can be written as the product of simple reflections. It is said that w is written in a reduced form if it is written with the minimal possible number of simple reflections; the number of reflections of a reduced form of w is called the length of w, denoted by $\ell(w)$.

The Weyl character formula for the finite-dimensional irreducible LWM L_Λ over $\hat{\mathcal{G}}$, i.e., when $\Lambda \in -\Gamma_+$, has the form:

$$ch_0\ L_\Lambda = \sum_{w\in W} (-1)^{\ell(w)}\ ch_0\ V^{w\cdot\Lambda}\,, \quad \Lambda \in -\Gamma_+ \tag{34}$$

where the dot $\cdot$ action is defined by $w \cdot \lambda = w(\lambda-\rho)+\rho$. For future reference we note:

$$s_\alpha \cdot \Lambda = \Lambda + n_\alpha \alpha \tag{35}$$

where

$$n_\alpha \ = \ n_\alpha(\Lambda) \ \doteq \ (\rho - \Lambda, \alpha^\vee) \ = \ (\rho - \Lambda)(H_\alpha)\,, \qquad \alpha \in \Delta^+. \tag{36}$$

In the case of basic classical Lie superalgebras the first character formulae were given by Kac. They are more complicated than the bosonic case, except for the algebras we consider. Actually, for $osp(1/2n)$ the Verma module character formula is the same as (32):

$$ch\ V^\Lambda \ = \ e(\Lambda) \prod_{\alpha\in\bar\Delta^+} \frac{1}{1-e(\alpha)} \tag{37}$$

using the restricted root system $\bar\Delta^+$. Naturally, the character formula for the finite-dimensional irreducible LWM L_Λ is again (34) using the Weyl group W_n of B_n.

Multiplets

A Verma module V^Λ may be reducible w.r.t. to many positive roots, and thus there maybe many Verma modules isomorphic to its submodules. They themselves may be reducible, and so on.

One main ingredient of the approach of [20] is as follows. We group the (reducible) Verma modules with the same Casimirs in sets called *multiplets* [20]. The multiplet corresponding to fixed values of the Casimirs may be depicted as a connected graph, the vertices of which correspond to the reducible Verma modules and the lines between the vertices correspond to embeddings between them. The explicit parametrization of the multiplets and of their Verma modules is important for understanding of the situation.

If a Verma module V^Λ is reducible w.r.t. to all simple roots (and thus w.r.t. all positive roots), i.e., $m_k \in \mathbb{N}$ for all k, then the irreducible submodules are isomorphic to the finite-dimensional irreps of $\mathcal{G}^\mathbb{C}$ [20]. (Actually, this is a condition only for m_n since $m_k \in \mathbb{N}$ for $k = 1,\ldots,n-1$.) In these cases we have the *main multiplets* which are isomorphic to the Weyl group of $\mathcal{G}^\mathbb{C}$ [20].

In the cases of non-dominant weight Λ the character formula for the irreducible LWM is [25] :

$$ch\ L_\Lambda \ = \ \sum_{\substack{w\in W\\ w\leqslant w_\Lambda}} \ (-1)^{\ell(w_\Lambda w)}\ P_{w,w_\Lambda}(1)\ ch\ V^{w\cdot(w_\Lambda^{-1}\cdot\Lambda)}\,, \quad \Lambda\in\Gamma \tag{38}$$

where $P_{y,w}(u)$ are the Kazhdan–Lusztig polynomials $y,w \in W$ [25] (for an easier exposition see [24]), w_Λ is a unique element of W with minimal length such that the signature of $\Lambda_0 = w_\Lambda^{-1} \cdot \Lambda$ is anti-dominant or semi-anti-dominant:

$$\chi_0 = (m'_1, \ldots, m'_n), \qquad m'_k = 1 - \Lambda_0(H_k) \in \mathbb{Z}_- \ . \tag{39}$$

Note that $P_{y,w}(1) \in \mathbb{N}$ for $y \leqslant w$.

When Λ_0 is semi-anti-dominant, i.e., at least one $m'_k = 0$, then in fact W is replaced by a reduced Weyl group W_R.

Most often the value of $P_{y,w}(1)$ is equal to 1 (as in the character formula for the finite-dimensional irreps), while the cases $P_{y,w}(1) > 1$ are related to the appearance of subsingular vectors, though the situation is more subtle, see [24].

It is interesting to see how the reducible points relevant for unitarity fit in the multiplets. In the case of d_{ij} using (24) we have:

$$m_n(d_{ij}) = 1 - 2m_j - \cdots - 2m_{n-1} - m_i - \cdots - m_{j-1} \ . \tag{40}$$

In the case of d_i (25) we have:

$$m_n(d_i) = 1 - 2m_i - \cdots - 2m_{n-1} \ . \tag{41}$$

As expected the weights related to positive energy d are not dominant $(m_n(d_{ij}) \in \mathbb{Z}_-,\ m_n(d_i) \in -\mathbb{N},\ (i<n))$, since the positive energy UIRs are infinite-dimensional. (Naturally, $m_n(d_n) = 1$ falls out of the picture since $d_n < 0$.)

Thus, the Verma modules with weights related to positive energy would be somewhere in the main multiplet (or in a reduction of the main multiplet), and the first task for calculating the character is to find the w_Λ in the character formula (38). This we do in the next subsection in the case $n = 3$.

5. The case of osp(1|6)

For $n = 3$ formula (26) simplifies to:

$$d_1 \ > \ d_{12} \ > \ d_2 \ > \ d_{23} \ > \ d_3$$

$$\searrow \ > \ d_{13} \ > \ \nearrow$$

The Theorem now reads:

$$\begin{aligned}
d &\geqslant 2+\tfrac{1}{2}(a_1+a_2) \ = \ d_1 \ , \quad a_1 \neq 0 \ , \\
d &\geqslant \tfrac{3}{2}+\tfrac{1}{2}a_2 \ = \ d_{12} \ , \quad a_1=0, \ a_2\neq 0 \ , \\
d &= 1+\tfrac{1}{2}a_2 \ = \ d_2 > d_{13} \ , \quad a_1=0, \ a_2 \neq 0 \ , \\
d &\geqslant 1 \ = \ d_2 = d_{13} \ , \quad a_1=a_2=0 \ , \\
d &= \tfrac{1}{2} \ = \ d_{23} \ , \quad a_1=a_2=0 \ , \\
d &= 0 \ = \ d_3 \ , \quad a_1=a_2=0 \ .
\end{aligned} \tag{42}$$

The Weyl group W_n of B_n has $2^n n!$ elements, i.e., 48 for B_3. Let $S = (s_1,s_2,s_3)$, $s_i \equiv s_{\alpha_i}$, be the simple reflections. They fulfill the following relations:

$$s_1^2 = s_2^2 = s_3^2 = e, \quad (s_1s_2)^3 = e, \quad (s_2s_3)^4 = e, \quad s_1s_3 \ = \ s_3s_1 \, , \tag{43}$$

e being the identity of W_3. The 48 elements may be listed as:

$$\begin{aligned}
& e\,,\ s_1\,,\ s_2\,,\ s_3 \\
& s_1s_2\,,\ s_1s_3\,,\ s_2s_1\,,\ s_2s_3\,,\ s_3s_2\,, \\
& s_1s_2s_1\,,\ s_1s_2s_3\,,\ s_1s_3s_2\,,\ s_2s_1s_3\,,\ s_2s_3s_2\,, \\
& \ s_3s_2s_1\,,\ s_3s_2s_3\,, \\
& s_1s_2s_1s_3\,,\ s_1s_2s_3s_2\,,\ s_1s_3s_2s_1\,,\ s_1s_3s_2s_3\,, \\
& s_2s_3s_2s_1\,,\ s_2s_1s_3s_2\,,\ s_3s_2s_3s_1\,,\ s_3s_2s_3s_2\,, \\
& s_1s_2s_3s_2s_1\,,\ s_1s_3s_2s_1s_3\,,\ s_1s_2s_1s_3s_2\,, \\
& s_1s_3s_2s_3s_2\,, s_2s_1s_3s_2s_1\,,\ s_2s_1s_3s_2s_3\,, \\
& s_3s_2s_3s_1s_2\,,\ s_3s_2s_3s_2s_1\,, \\
& s_1s_3s_2s_3s_2s_1\,,\ s_1s_3s_2s_1s_3s_2\,,\ s_1s_2s_1s_3s_2s_1\,, \\
& s_2s_1s_3s_2s_1s_3\,, s_2s_1s_3s_2s_3s_2\,,\ s_3s_2s_3s_1s_2s_1\,, \\
& s_3s_2s_3s_1s_2s_3\,, s_2s_1s_3s_2s_3s_2s_1\,, \\
& s_2s_1s_3s_2s_3s_1s_2\,, s_3s_2s_1s_2s_3s_2s_1\,, \\
& s_3s_2s_3s_1s_2s_1s_3\,,\ s_3s_2s_3s_1s_2s_3s_2\,, \\
& s_2s_3s_2s_1s_2s_3s_2s_1\,,\ s_3s_2s_1s_3s_2s_3s_2s_1\,, \\
& s_3s_2s_1s_3s_2s_3s_1s_2\,, s_2s_3s_2s_1s_3s_2s_3s_2s_1\ .
\end{aligned} \tag{44}$$

The character formula for the Verma modules in our case is given explicitly by:

$$ch\ V^\Lambda = \frac{e(\Lambda)}{(1-t_1)(1-t_2)(1-t_1t_2)} \times$$
$$\times\frac{1}{(1-t_3)(1-t_2t_3)(1-t_1t_2t_3)} \times$$
$$\times\frac{1}{(1-t_2t_3^2)(1-t_1t_2t_3^2)(1-t_1t_2^2t_3^2)} \tag{45}$$

where $t_j \equiv e(\alpha_j)$.

Now we give the character formulae of the five boundary or isolated unitarity cases. Below we shall denote the signature of the dominant weight Λ_0 which determines the main multiplet by (m'_1, m'_2, m'_3), $m'_k \in \mathbb{N}$, using primes to distinguish from the signatures of the weights we are interested. We shall use also reductions of the main multiplet when the weights are semi-dominant, i.e., when some $m'_k = 0$.

• In the case of $d = d_1 = 2 + \frac{1}{2}(a_1 + a_2)$ there are twelve members of the multiplet which is a submiltiplet of a main multiplet. (Remember that that $m_1 > 1$ since $a_1 \neq 0$.) They are grouped into two standard $sl(3)$ submultiplets of six members. The first submultiplet starts from $V^{\Lambda_0^{d_1}}$, where $\Lambda_0^{d_1} = w \cdot \Lambda_0$, $w = w_{\Lambda_0^{d_1}} = s_2 s_1 s_3 s_2 s_3$, with signature:

$$\Lambda_0^{d_1} \quad : \quad (m_1, m_2, m'_3 = 1 - 2m_{12}) \ , \tag{46}$$
$$m_1, m_2 \in \mathbb{N} \ , \quad m_{12} \equiv m_1 + m_2 \ .$$

The other submultiplet starts from $V^{\Lambda'_0}$ with $\Lambda'_0 = \Lambda_0^{d_1} + \delta_1 = \Lambda_0^{d_1} + \alpha_1 + \alpha_2 + \alpha_3$, with signature: $\Lambda'_0 : (m_1 - 1, m_2, m'_3 = 1 - 2m_{12})$, $m_1 > 1$. The character formula is (38) with $w_\Lambda = w_{\Lambda_0^{d_1}}$:

$$\mathrm{ch}\,\Lambda_0^{d_1} = \frac{e(\Lambda_0^{d_1})}{(1-t_3)(1-t_2t_3)(1-t_1t_2t_3)} \times \tag{47}$$
$$\times \frac{1}{(1-t_2t_3^2)(1-t_1t_2t_3^2)(1-t_1t_2^2t_3^2)} \times$$
$$\times \{ \ \mathrm{ch}\,\Lambda_{m_1,m_2}(t_1,t_2) \ - \ t_1t_2t_3\,\mathrm{ch}\,\Lambda_{m_1-1,m_2}(t_1,t_2) \ \} \ , \ m_1 > 1$$

where $\mathrm{ch}\,\Lambda_{m_1,m_2}(t_1,t_2)$ is the normalized character of the finite-dimensional $sl(3)$ irrep with Dynkin labels (m_1, m_2) (and dimension $m_1m_2(m_1 + m_2)/2$):

$$\mathrm{ch}\ \Lambda_{m_1,m_2}(t_1,t_2) \ = \ \frac{1 - t_1^{m_1} - t_2^{m_2} + t_1^{m_1}t_2^{m_{12}} + t_1^{m_{12}}t_2^{m_2} - t_1^{m_{12}}t_2^{m_{12}}}{(1-t_1)(1-t_2)(1-t_1t_2)} \tag{48}$$

Naturally, the latter formula is a polynomial in t_1, t_2, e.g., $\mathrm{ch}\,\Lambda_{1,1}(t_1,t_2) = 1$, $\mathrm{ch}\,\Lambda_{2,1}(t_1,t_2) = 1 + t_1 + t_1 t_2$.

In the case $m_1 = 2, m_2 = 1$ the character formula (47) simplifies to:

$$\mathrm{ch}\,\Lambda_0^{d_1} = \frac{e(\Lambda_0^{d_1})}{(1-t_3)(1-t_2t_3)(1-t_2t_3^2)(1-t_1t_2t_3^2)(1-t_1t_2^2t_3^2)} \times \times \Big(1 + \frac{t_1(1+t_2)}{1-t_1t_2t_3} \Big), \qquad m_1 = 2, m_2 = 1 \tag{49}$$

• In the case of $d = d_{12} = \frac{1}{2}(3+a_2)$ which is relevant for unitarity, i.e., $m_1 = 1$, there are again twelve members of the multiplet. Omitting the details [14] the character f-la is:

$$\mathrm{ch}\,\Lambda_0^{d_{12}} = \frac{e(\Lambda_0^{d_{12}})}{(1-t_3)(1-t_2t_3)(1-t_1t_2t_3)} \times \tag{50}$$

$$\times \frac{1}{(1-t_2t_3^2)(1-t_1t_2t_3^2)(1-t_1t_2^2t_3^2)}$$

$$\times\{ \mathrm{ch}\,\Lambda_{1,m_2}(t_1,t_2) - (t_1t_2^2t_3^2)^{m_2}\,\mathrm{ch}\,\Lambda_{1,m_2-1}(t_1,t_2) \}, \ m_2 > 1$$

In the case $m_2 = 2$ it simplifies to:

$$\mathrm{ch}\,\Lambda_0^{d_{12}} = \frac{e(\Lambda_0^{d_{12}})}{(1-t_3)(1-t_2t_3)(1-t_1t_2t_3)(1-t_2t_3^2)(1-t_1t_2t_3^2)} \times \times\{ 1 + t_1t_2^2t_3^2 + \frac{t_2(1+t_1)}{1-t_1t_2^2t_3^2} \} \tag{51}$$

• In the case $d = d_2 = d_{13} = 1$ and $a_1 = a_2 = 0$, $m_1 = m_2 = 1$, the signature is:

$$\Lambda_0^{d_2=d_{13}} \ : \ (1,1,-1) \ . \tag{52}$$

Again there are twelve members of the multiplet which has two $sl(3)$ submultiplets. First there is a $sl(3)$ sextet starting from $\Lambda_0^{d_2=d_{13}}$ with parameters $(1,1)$. Then there is a $sl(3)$ sextet starting from $\Lambda_0^{d_2=d_{13}} + \alpha_1 + 2\alpha_2 + 3\alpha_3$ with parameters $(1,1)$. Note that that $\alpha_1 + 2\alpha_2 + 3\alpha_3 = \delta_1 + \delta_2 + \delta_3$ is the weight of a subsingular vector [14], yet the corresponding KL polynomial $P_{y,w}(1)$ is equal to 1. Thus, the character formula is [14]:

$$\mathrm{ch}\,\Lambda_0^{d_2=d_{13}} = \tag{53}$$

$$= \frac{e(\Lambda_0^{d_2=d_{13}})\,(\,1 - t_1t_2^2t_3^3\,)}{(1-t_3)(1-t_2t_3)(1-t_1t_2t_3)(1-t_2t_3^2)(1-t_1t_2t_3^2)(1-t_1t_2^2t_3^2)}$$

Note that the above formula may be rewritten as:

$$\mathrm{ch}\,\Lambda_0^{d_2=d_{13}} = \frac{e(\Lambda_0^{d_2=d_{13}})}{(1-t_2t_3)(1-t_1t_2t_3)(1-t_2t_3^2)(1-t_1t_2t_3^2)} \times \times \left(\frac{1}{1-t_1t_2^2t_3^2} + \frac{t_3}{1-t_3} \right) \tag{54}$$

• In the case of $d = d_2 = 1+\frac{1}{2}a_2 > d_{13} = 1$, i.e., $m_1 = 1$, $m_2 = 1+a_2 > 1$. The multiplet has 24 members for $m_2 > 2$. Omitting the details [14] the character f-la is:

$$\begin{aligned}\mathrm{ch}\,\Lambda_0^{\prime d_2} &= \frac{e(\Lambda_0^{\prime d_2})}{(1-t_3)(1-t_2t_3)(1-t_1t_2t_3)} \times \\ &\times \frac{1}{(1-t_2t_3^2)(1-t_1t_2t_3^2)(1-t_1t_2^2t_3^2)} \times \\ &\times \{ \mathrm{ch}\,\Lambda_{1,m_2}(t_1,t_2) - t_2t_3\,\mathrm{ch}\,\Lambda_{2,m_2-1}(t_1,t_2) + \\ &+ t_1t_2^3t_3^3\,\mathrm{ch}\,\Lambda_{2,m_2-2}(t_1,t_2) - t_1^2t_2^4t_3^4\,\mathrm{ch}\,\Lambda_{1,m_2-2}(t_1,t_2) \}\end{aligned} \tag{55}$$

When $m_2 = 2$ $(a_2 = 1)$ the multiplet reduces to only 12 members, and the character formula simplifies to:

$$\begin{aligned}\mathrm{ch}\,\Lambda_0^{\prime d_2} &= \frac{e(\Lambda_0^{\prime d_2})}{(1-t_2t_3^2)(1-t_1t_2t_3^2)(1-t_1t_2^2t_3^2)} \times \\ &\times \{ \frac{1}{(1-t_3)(1-t_1t_2t_3)} + \frac{t_2}{(1-t_3)(1-t_2t_3)} \\ &\quad + \frac{t_1t_2}{(1-t_2t_3)(1-t_1t_2t_3)} \}\end{aligned} \tag{56}$$

• In the case of $d = d_{23} = \frac{1}{2}$, $a_1 = a_2 = 0$, i.e., $m_1 = m_2 = 1$, again we have a multiplet with 24 members. Omitting the details [14] the character formula is:

$$\begin{aligned}
\mathrm{ch}\,\Lambda_0^{d_{23}} \;&=\; \frac{e(\Lambda_0^{d_{23}})}{(1-t_3)(1-t_2t_3)(1-t_1t_2t_3)} \times \\
&\times \frac{1}{(1-t_2t_3^2)(1-t_1t_2t_3^2)(1-t_1t_2^2t_3^2)} \times \\
&\times \{\, 1 \;-\; t_2t_3^2\,\mathrm{ch}\,\Lambda_{2,1}(t_1,t_2) \;+\; t_1t_2^2t_3^4\,\mathrm{ch}\,\Lambda_{1,2}(t_1,t_2) \;-\; t_1^2t_2^4t_3^6 \,\} \;= \\
&= \frac{e(\Lambda_0^{d_{23}})}{(1-t_3)(1-t_2t_3)(1-t_1t_2t_3)\;(1-t_2t_3^2)(1-t_1t_2t_3^2)(1-t_1t_2^2t_3^2)} \times \\
&\times \{\, 1 \;-\; t_2t_3^2\,(1+t_1+t_1t_2) \;+\; t_1t_2^2t_3^4\,(1+t_2+t_1t_2) \;-\; t_1^2t_2^4t_3^6 \,\}
\end{aligned} \tag{57}$$

Note that the above formula may be rewritten as:

$$\mathrm{ch}\,\Lambda_0^{d_{23}} \;=\; \frac{e(\Lambda_0^{d_{23}})}{(1-t_3)(1-t_2t_3)(1-t_1t_2t_3)} \tag{58}$$

Note that formulae (49),(51),(54),(56),(58) are new w.r.t. [14].

Acknowledgments
V.K.D. thanks the Organizers for the kind invitation to give a plenary talk at the 24-th International Conference on Integrable Systems and Quantum Symmetries, Prague, June 2016. V.K.D. was supported by COST Actions MP1210 and MP1405, and by Bulgarian NSF Grant DFNI T02/6.
I.S. was supported by COST Action MP1405 and Serbian Ministry of Science and Technological Development, grant OI 171031.

References

[1] W. Nahm, *Supersymmetries and their representations*, Nucl. Phys. **B135** (1978) 149-166.

[2] R. Haag, J.T. Lopuszanski and M. Sohnius, *All possible generators of supersymmetries of the S-matrix*, Nucl. Phys. **B88** (1975) 257-274.

[3] M. Flato and C. Fronsdal, *Representations of conformal supersymmetry*, Lett. Math. Phys. **8** (1984) 159-162.

[4] V.K. Dobrev and V.B. Petkova, *On the group-theoretical approach to extended conformal supersymmetry : classification of multiplets*, Lett. Math. Phys. **9** (1985) 287-298.

[5] V.K. Dobrev and V.B. Petkova, *All positive energy unitary irreducible representations of extended conformal supersymmetry*, Phys. Lett. **162B** (1985) 127-132.

[6] V.K. Dobrev and V.B. Petkova, *On the group-theoretical approach to extended conformal supersymmetry : function space realizations and invariant differential operators*, Fortschr. d. Phys. **35** (1987) 537-572.

[7] V.K. Dobrev and V.B. Petkova, *All positive energy unitary irreducible representations of the extended conformal superalgebra*, in: A.O. Barut and H.D. Doebner (eds.), *Conformal Groups and Structures*, Lecture Notes in Physics, Vol. 261 (Springer-Verlag, Berlin, 1986) pp. 300-308.

[8] S. Minwalla, *Restrictions imposed by superconformal invariance on quantum field theories*, Adv. Theor. Math. Phys. **2** (1998) 781-846.

[9] V.K. Dobrev, *Positive energy unitary irreducible representations of D=6 conformal supersymmetry*, J. Phys. **A35** (2002) 7079-7100.

[10] P.K. Townsend, *M(embrane) theory on T(9)*, Nucl. Phys. Proc. Suppl. **68** (1998) 11-16; J.P. Gauntlett, G.W. Gibbons, C.M. Hull and P.K. Townsend, *BPS states of D=4 N=1 supersymmetry*, Commun. Math. Phys. **216** (2001) 431-459.

[11] R. D'Auria, S. Ferrara, M.A. Lledo and V.S. Varadarajan, *Spinor algebras*, J. Geom. Phys. **40** , (2001) 101-128; R. D'Auria, S. Ferrara and M.A. Lledo, *On the embedding of space-time symmetries into simple superalgebras*, Lett. Math. Phys. **57** (2001) 123-133; S. Ferrara and M.A. Lledo, *Considerations on super Poincare algebras and their extensions to simple superalgebras*, Rev. Math. Phys. **14** (2002) 519-530.

[12] V.K. Dobrev and R.B. Zhang, *Positive Energy Unitary Irreducible Representations of the Superalgebras* $osp(1|2n,R)$, Phys. Atom. Nuclei, **68** (2005) 1660-1669.

[13] V.K. Dobrev, A.M. Miteva, R.B. Zhang and B.S. Zlatev, *On the unitarity of D=9,10,11 conformal supersymmetry*, Czech. J. Phys. **54** (2004) 1249-1256.

[14] V.K. Dobrev and I. Salom, *Positive Energy Unitary Irreducible Representations of the Superalgebras* $osp(1|2n,R)$ *and Character Formulae*, Proceedings of the VIII Mathematical Physics Meeting, (Belgrade, 24-31 August 2014) SFIN XXVIII (A1), eds. B. Dragovich et al, (Belgrade Inst. Phys. 2015) [ISBN 978-86-82441-43-4], pp. 59-81.

[15] V.K. Dobrev and I. Salom, *Positive Energy Unitary Irreducible Representations of the Superalgebras* $osp(1|8,R)$, Publications de l'Institut Mathematique, Belgrade, to appear (2016), (IF 0.195), arXiv:1607.03008.

[16] V.G. Kac, *Lie superalgebras*, Adv. Math. **26** (1977) 8-96; *A sketch of Lie superalgebra theory*, Commun. Math. Phys. **53** (1977) 31-64; the second paper is an adaptation for physicists of the first paper.

[17] V.G. Kac, *Representations of classical Lie superalgebras*, Lect. Notes in Math. **676** (Springer-Verlag, Berlin, 1978) pp. 597-626.

[18] N.N. Shapovalov, *On a bilinear form on the universal enveloping algebra of a complex semisimple Lie algebra*, Funkts. Anal. Prilozh. **6** (4) 65 (1972); English translation: Funkt. Anal. Appl. **6**, 307 (1972).

[19] J. Dixmier, *Enveloping Algebras*, (North Holland, New York, 1977).

[20] V.K. Dobrev, *Canonical construction of intertwining differential operators associated with representations of real semisimple Lie groups*, Rept. Math. Phys. **25** (1988) 159-181.

[21] V.K. Dobrev, *Singular vectors of quantum groups representations for straight Lie algebra roots*, Lett. Math. Phys. **22** (1991) 251-266.

[22] I.N. Bernstein, I.M. Gel'fand and S.I. Gel'fand, *Structure of representations generated by highest weight vectors*, Funkts. Anal. Prilozh. **5** (1) (1971) 1; English translation: Funct. Anal. Appl. **5** (1971) 1.

[23] V.K. Dobrev, *Subsingular vectors and conditionally invariant (q-deformed) equations*, J. Phys. **A28** (1995) 7135-7155.

[24] V.K. Dobrev, *Kazhdan-Lusztig polynomials, subsingular vectors, and conditionally invariant (q-deformed) equations*, Invited talk at the Symposium "Symmetries in Science IX", Bregenz, Austria, (August 1996), Proceedings, eds. B. Gruber and M. Ramek, (Plenum Press, New York and London, 1997) pp. 47-80.

[25] D. Kazhdan and G. Lusztig, *Representations of Coxeter Groups and Hecke Algebras*, Inv. Math. **53** (1979) 165-184.