Supergroups in Critical Dimensions and Division Algebras

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Abstract.

We establish a link between classical heterotic strings and the groups of the magic square associated with Jordan algebras, allowing for a uniform treatment of the bosonic and superstring sectors of the heterotic string.

1. Division and Jordan Algebras, Magic Squares

This will be an introduction to the construction of the super-Poincaré group in special dimensions $d = 3, 4, 6, 10$, called critical dimensions in which it is possible to write supersymmetric classical Lagrangians for Yang-Mills theories and superstrings. It is known that the conventional quantization of superstrings[1] is only possible in $d = 10$. However, parastatistical quantization appears to be possible in the other critical dimensions[2]. In critical dimensions special algebraic structures occur. They are associated with the four division algebras of the Hurwitz's theorem[3], i.e. $\mathcal{R}$ (the real numbers), $\mathcal{C}$ (complex numbers), $\mathcal{H}$ (quaternions) and $\mathcal{O}$ (octonions or Cayley numbers). Their dimensions are respectively 1, 2, 4 and 8 or $d-2$ if d is the critical dimension. They have positive semi-definite quadratic multiplicative norms. The first two are commutative and associative. Quaternions are associative but not commutative and hence can be represented by 2×2 complex or 4×4 real matrices. Octonions are neither commutative nor associative and as a result have no direct matrix representations. However, their associator is totally antisymmetric, so that the octonion algebra is "alternative".

We shall make use of the norm groups and the automorphism groups of the Hurwitz algebras. The norm groups are formed by the linear transformations of the components of an element of the algebra that preserve the quadratic norm. For real numbers it it the discrete group Z_2. For complex numbers it is $O(2) \sim U(1)$. For quaternions it is $O(4) \sim SU(2) \times SU(2)$ and for octonions it is $O(8)$. The automorphism groups are the groups that leave invariant the multiplication table of the imaginary units of the algebra. It is Z_2 $(i \to -i)$ for complex numbers, $SU(2)$ for quaternions and the exceptional group G_2 for octonions. The automorphism groups, unlike the norm groups, leave the real (or scalar) part of an algebra element invariant. All Hurwitz algebras are power associative, so that if $N = n_1 + n_2 + \ldots + n_m$, we can write

$$x^N = x^{n_1}.\, x^{n_2} \ \ldots x^{n_m} \tag{1}$$

unambiguously and independently of the brackets in the nonassociative case. Besides alternative algebras Jordan algebras are also power associative. We now give a brief description of Jordan algebras.

Jordan algebras have elements that have a commutative product

$$a.b = b.a \tag{2}$$

which is not associative but is subject to the Jordan identity

$$[a\; b\; a^2] = 0\ , \tag{3}$$

where the square bracket denotes the associator defined by

$$[a\; b\; c] = (a.b).c \ - \ a.(b.c)\ . \tag{4}$$

An example of a Jordan algebra is the algebra of observables with elements that are $n \times n$ hermitian complex matrices with the composition law

$$a.b = \frac{1}{2}\{a\; b\} = \frac{1}{2}(a\, b + b\, a)\ . \tag{5}$$

The identity (3) is satisfied. There is also the exceptional Jordan algebra J_3^8 of 3×3 octonionic matrices that are hermitian with respect to octonionic conjugation. Note that in a conjugation the sign of imaginary units is reversed. The commutative Jordan product in this case is still defined by (5) and the Jordan identity still holds.

The classical Lie groups can be regarded as the automorphism groups of Jordan algebras. If J_n^1, J_n^2 and J_n^4 denote the Jordan algebras of $n \times n$ hermitian matrices over $\mathcal{R}$, $\mathcal{C}$ and $\mathcal{H}$ respectively they have the corresponding automorphism groups $O(n)$, $SU(n)$ and $Sp(n)$. The automorphism group of the exceptional Jordan algebra is the exceptional group F_4. The automorphism groups leave the trace of the Jordan matrices invariant. Hence traceless hermitian matrices are modules for their automorphism groups. For 3×3 matrices we find the following groups and dimensions:

Division algebras	$\mathcal{R}$	$\mathcal{C}$	$\mathcal{H}$	$\mathcal{O}$
Jordan algebras	J_3^1	J_3^2	J_3^4	J_3^8
Dim. of traceless elements	5	8	14	26
Automorphism groups	$O(3)$	$SU(3)$	$Sp(3)$	F_4
Dim. of automorphism groups	3	8	21	52

These groups form the first line of the celebrated magic square of Rozenfeld, Freudenthal and Tits[5]. To proceed further we must introduce the Freudenthal algebra[6]. Consider a 3×3 complex octonionic matrix J that is hermitian with respect to octonionic conjugation. It is a 27-dimensional module (27) for the exceptional group E_6, which is a complex group like $SU(3)$. Since $27 \times \bar{27}$ contains the identity representation and 27×27 contains the $(\bar{27})$ representation of E_6 we can define a scalar product and a symmetric product associated with the mappings

$$J \times J^* \to \mathcal{C} \quad \text{and} \quad J \times J \to J^*, \qquad J^* \times J^* \to J \, . \tag{6}$$

The scalar product is defined by

$$(J \, , K^*) = \mathrm{Tr} \, (J \, .K^*) \tag{7}$$

while the "Freudenthal product" of J and K, both belonging to the (27) representation of E_6 is obtained by first introducing

$$J \times J = J^{-1} \, \mathrm{Det} \, J = J^2 - J \, \mathrm{Tr} \, J - \frac{1}{2} \, I \, \mathrm{Tr} \, (J^2 - J \, \mathrm{Tr} \, J) \, , \tag{8}$$

and then defining

$$J \times K = \frac{1}{2} \, (J + K) \times (J + K) \; - \; \frac{1}{2} \, J \times J \; - \; \frac{1}{2} \, K \times K \, , \tag{9}$$

we find

$$J \times K = J.K - \frac{1}{2} \, J \, \mathrm{Tr} \, K - \frac{1}{2} \, K \, \mathrm{Tr} \, J - \frac{1}{2} \, I \, (\mathrm{Tr} \, J.K - \mathrm{Tr} \, J \, \mathrm{Tr} \, K) \, . \tag{10}$$

If J and K transform like (27), then $J \times K$ transforms like $\bar{27}$ of E_6. Similarly $J^* \times K^*$ transforms like a (27). The group E_6 is an automorphism group for the combination of the scalar product (7) and the Freudenthal product (10).

If we consider automorphism groups of 3×3 matrices with entries that are combinations of complex numbers with the four division algebras, requiring hermiticity with respect to the latter only, and then define scalar and Freudenthal products as for the octonionic matrices, we find

	$\mathcal{R}$	$\mathcal{C}$	$\mathcal{H}$	$\mathcal{O}$
Automorphism group $\mathcal{C}$:	$SU(3)$	$SU(3) \times SU(3)$	$SU(6)$	E_6
Dimension of group	8	16	35	78
Complex Dim. of module	6	9	15	27

This is the second line of the magic square. Note that the traces of the 3×3 matrices are not invariant under the automorphism groups of the Freudenthal algebra, but Det J, Tr $(J.K^*)$ and Tr $(J^*.K)$ are invariant.

The combination of the scalar product and the Freudenthal product also allows for the definition of a triple product that is E_6 covariant. Given F and K that belong to the (27) representation we define

$$M = \{F \; K^* \; F\} = F \; \mathrm{Tr} \; (K^*.F) - 2 \; (F \times F) \times K^* \; . \tag{11}$$

Then M also transforms like a (27). In the case of ordinary matrices it can be expressed in terms of the usual matrix product as

$$M = F K^* F \, , \tag{12}$$

so that

$$\mathrm{Det} \; M = (\mathrm{Det} F)^2 \; \mathrm{Det} \; K^* \, , \tag{13}$$

a property that also holds for the exceptional Jordan algebra. Given three elements of the Freudenthal algebra F, K, L, we have the triple product:

$$N = \{F\ K^*\ L\} = \frac{1}{2}\left\{(F+L)\ K^*\ (F+L)\right\} - \frac{1}{2}\left\{F\ K^*\ F\right\} - \frac{1}{2}\left\{L\ K^*\ L\right\}. \tag{14}$$

In terms of the Jordan product it can be written in the form[4]

$$N = \{F\ K^*\ L\} = (F.K^*).L + F.(K^*.L) - K^*.(F.L)\ . \tag{15}$$

E_6 is also the automorphism group of this ternary algebra.

In the case of the exceptional group E_7 there is a 56-dimensional fundamental representation X consisting of two complex scalars α and β^* and two Jordan matrices J and K^* belonging respectively to the (27) and ($\bar{27}$) representations of the E_6 subgroup. Now the direct product of three 56 dimensional representations also contains a (56). Hence there is a ternary algebra of the 56-dimensional modules of E_7. There is also a scalar product of a (56) and its complex conjugate. E_7 can be regarded as the automorphism group of the combined algebra of ternary products and scalar products.

Ranging over the automorphism groups of the ternary algebra when the module is constructed out of the four division algebras we find the third line of the magic square

	$\mathcal{R}$	$\mathcal{C}$	$\mathcal{H}$	$\mathcal{O}$
Aut. group of the ternary prod.	$Sp(3)$	$SU(6)$	$SO(12)$	E_7
Dimension of group	21	35	66	133
Dimension of the module X	14	20	32	56

Finally we can generate groups by considering fractional linear representations of these modules[7], giving nonlinear (coset space) realizations of the exceptional groups F_4, E_6, E_7 and E_8 that form the last line of the magic square. They have the respective dimensions 52, 78, 133 and 248.

The Magic square groups also have many non compact forms. A typical example is the $SL(3,R)$ real form associated with $SU(3)$ and the non compact groups $SL(3,R)\times SL(3,R)$ and $SL(3,C)$ associated with the compact group $SU(3)\times SU(3)$. In general they are subgroups of the complexified forms of the compact groups. They appear in the couplings of scalar fields in supergravity[8],[9] and also in the algebras associated with non simply laced lattices[10]. We shall see that they are also connected with the representations of the supersymmetric kinematical groups in critical dimensions.

Here is one set of a magic square of non compact groups that are subgroups of the non compact group $E_{8(-24)}$. Note that the number in brackets represents the number of non compact generators minus that of compact generators.

$\mathcal{R}$	$\mathcal{C}$	$\mathcal{H}$	$\mathcal{O}$
$O(2,1)$	$SU(2,1)$	$Sp(2,1)$	$F_{4(-20)}$
$SL(3,R)$	$SL(3,C)$	$SU^*(6)$	$E_{6(-26)}$
$Sp(6,R)$	$SU(3,3)$	$SO^*(12)$	$E_{7(-25)}$
$F_{4(+4)}$	$E_{6(+2)}$	$E_{7(-5)}$	$E_{8(-24)}$

The first line can be replaced by the compact groups $O(3)$, $SU(3)$, $Sp(3)$ and F_4 which are also subgroups of the non-compact groups of the second line. The starred groups are quaternionic groups. $SU^*(6)$ is isomorphic to the linear group over quaternions $SL(3,H)$. All of these non compact groups occur in supergravity theories.

Another remarkable series of critical dimensions is given by J_2^D, the Jordan algebra of 2×2 hermitian matrices over the division algebras, with D= 1, 2, 4 and 8. They form subalgebras of J_3^8. Here are the dimensions of the algebras and their automorphism groups.

	$\mathcal{R}$	$\mathcal{C}$	$\mathcal{H}$	$\mathcal{O}$
Algebra	J_2^1	J_2^2	J_2^4	J_2^8
Real Dim. of alg.	3	4	6	10
Aut. group **[covering group]:**	$O(2)$ $[U(1)]$	$O(3)$ $[SU(2)]$	$O(5)$ $[Sp(2)]$	$O(9)$ $[spin9]$

These groups are seen to be subgroups of the first lines of the magic squares for compact and non compact groups given above. To look for the corresponding subgroups of the second line of the non-compact magic square, we start from the well known case of J_2^2, the algebra of 2×2 hermitian complex matrices. An example is the matrix associated with the momentum p that can be written

$$p = p_0 + \vec{\sigma}\cdot\vec{p} = \begin{pmatrix} p_o + p_3 & p_1 - ip_2 \\ p_1 + ip_2 & p_o - p_3 \end{pmatrix} \tag{16}$$

in terms of the Pauli matrices. The unitary representations of the Poincaré group are built through the use of the little groups that leave the standard form of p invariant. In the time-like and space-like cases p can be brought along its time component or one space-like components respectively, being left invariant by the corresponding groups $SO(3)$ and $SO(2,1)$. These are just the automorphism groups of the first lines of the magic squares given above. Similarly, the momentum vectors in critical dimensions are represented by hermitian 2×2 matrices over the division algebras. They are $(D+2)$ dimensional vectors in the Minkowski spaces $(D+1,1)$. The transverse part of the vector $(p_\perp)$, relevant in the massless, light-like case is represented by an element of the division algebra of dimension D[11]. The Lorentz group in critical dimensions is given by linear, homogeneous transformations that preserve the hermiticity of p and its Minkowski norm given by

$$\text{Det}\, p = p\,\bar{p}, \qquad \bar{p} = -p + I\ \text{Tr}p\ , \tag{17}$$

with I being 2×2 identity matrix.

In $d=4$ ($D=2$) dimensions it is given by

$$p' = L\,p\,L^\dagger, \qquad (\text{Det}L = 1)\ , \tag{18}$$

where L is a unimodular complex 2×2 matrix. Then we have

$$L\,\varepsilon\, SL(2,C) \sim SO(3,1)\ . \tag{19}$$

For $d=3$ it is a real unimodular matrix representing $SL(2,R) \sim SO(2,1)$, while for $d=6$ it is a quaternionic unimodular matrix $SL(2,H) \sim SO(5,1)$. For $d=8$ the matrix is octonionic and the linear action on p must also involve associators[12]. The action with associators is contained in the automorphism group G_2, the 14 parameters of which must be added to the $4\times 8-1=31$ parameters of a 2×2 octonionic matrix, giving a linear group of dimension 45, namely $O(9,1)$. Thus we can construct the following table:

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Algebras	J_2^1	J_2^2	J_2^4	J_2^8
Minkowski dim.	3	4	6	10
Norm preserving group	$SL(2,R)$	$SL(2,C)$	$SL(2,H)$	$Spin(9,1)$
[Lorentz group]	$[\sim SO(2,1)]$	$[\sim SO(3,1)]$	$[\sim SO(5,1)]$	$[\sim SO(9,1)]$
Little group for massless particle	T_1	$IO(2)$	$IO(4)$	$IO(8)$
Helicity group	$\cdots$	$O(2)$	$O(4)$	$O(8)$

In this table I means the inhomogeneous group, T_n is the translation group in n dimension and the helicity group is the homogeneous part of the little group for massless particles which preserves $p_0{+}p_{d-1}$. The helicity groups are seen to be the norm groups of the division algebras associated with the critical Minkowski dimensions. The Lorentz groups are seen to be subgroups of the second line of the magic square of the non compact groups, including $O(9,1)$ which is a subgroup of $E_{6(-26)}$. We have already seen that the little groups are subgroups of the first line of the same table.

2. Spinors in Critical Dimensions

For a massless particle the momentum is light-like, so that

$$p\,\bar{p} = \mathrm{Det}\,p \;=\; 0 \tag{20}$$

and the hermitian matrix p factorizes in the form

$$p \;=\; \psi\,\psi^\dagger\;,\quad \psi \;=\; \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}\;,\quad \psi^\dagger \;=\; (\bar{\psi}_1 \quad \bar{\psi}_2)\;. \tag{21}$$

Here the components ψ_1 and ψ_2 of the spinor ψ are elements of the division algebras. The bar denotes division algebra conjugation. Under a Lorentz transformation

$$\psi' \;=\; L\,\psi\;. \tag{22}$$

In the octonionic case $(d\,=\,10)$ it is understood that the automorphism group G_2 of octonions is adjoined to the left multiplication by L. The real dimensions of these spinors is $2D = 2(d-2)$ where D is the dimension of the division algebra. Let us also display the dimensions of the vector combined with a spinor, a combination that appears for representing a point in superspace.

Minkowski (vector dimension d)	3	4	6	10
Spinor dimension	2	4	8	16
Superspace dimension	5	8	14	26

The spinor dimensions correspond to those of a real spinor in $d\,=\,3$, a Weyl or Majorana spinor in $d\,=\,4$, a Weyl spinor in $d\,=\,6$ and finally a spinor that is both Weyl and Majorana in $d\,=\,10$. The superspace dimensions are the same as those for traceless 3×3 Jordan algebras of the first line of the magic square. This suggest a correspondence between a superspace point z (with vector component x and spinor component θ) and a matrix F of Jordan form

$$F = \begin{pmatrix} x & \theta\\ \theta^\dagger & a\end{pmatrix} \tag{23}$$

where the constant a is determined by a Lorentz invariant condition in terms of x and θ, such as

$$\text{Det}\ F = 1\ . \tag{24}$$

At this stage the correspondence is suggestive but not exact, as the components of θ for a superspace point are Grassmann numbers while they are real or complex numbers for an element of a Jordan algebra. To make the correspondence more precise we shall have to generalize the Jordan algebra to a graded Jordan algebra[13]. On the other hand, the 26-dimensional bosonic string vector can be represented by an element of the usual exceptional Jordan algebra of the form

$$J = \begin{pmatrix} y & u \\ u^\dagger & b \end{pmatrix}, \quad \text{Det}\ J\ =\ 1\ , \tag{25}$$

where y is a 2×2 hermitian matrix, b is a scalar and u a 2×1 matrix over octonions. When the octonions are real we have seen that J is a (27) module for $E_{6(-26)}$. J^{-1} and hence $J\times J$ is a $(\bar{27})$ module. It is known that the determinant of J is a multiplicative cubic norm for the exceptional Jordan algebra. We have

$$I\ \text{Det}\ J = (J\times J).\ J, \qquad \text{Det}\ J = \frac{1}{3}\ \text{Tr}\ [(J\times J).\ J]\ . \tag{26}$$

With the vector-spinor decomposition of J given in Eq. (25), using Eq. (8), one finds

$$J\times J\ =\ \begin{pmatrix} b\bar{y} - \overline{u u}^\dagger & -\bar{y}u \\ -u^\dagger\bar{y} & y\bar{y} \end{pmatrix} \tag{27}$$

and

$$\text{Det}\ J = b\ \bar{y}y - \text{Sc}\ (u^\dagger\bar{y}u)\ . \tag{28}$$

Note that the last formula is unambiguous because of the identity

$$\text{Sc}\ [u^\dagger\ (\bar{y}u)]\ =\ \text{Sc}\ [(u^\dagger\bar{y})\ u]\ =\ \frac{1}{2}\ (u^\dagger\bar{y})\ u\ +\ \frac{1}{2}\ u^\dagger\ (\bar{y}u)\ . \tag{29}$$

The bar notation over a 2×2 hermitian matrix has been defined as in eq.(17) by

$$y = \begin{pmatrix} y_+ & y_\perp \\ \bar{y}_\perp & y_- \end{pmatrix}\ , \qquad \bar{y} = \begin{pmatrix} y_- & -y_\perp \\ -\bar{y}_\perp & y_+ \end{pmatrix}\ , \tag{30}$$

and $u^\dagger$ has the same meaning as in eq.(21).

Let us emphasize that when J is a hermitian matrix over real octonions it corresponds to the (27) representation of $E_{6(-26)}$ and Det J is invariant under this noncompact group. Hence, it is also invariant with respect to its $O(9,1)$ subgroup under which y transforms like a vector, u like a Weyl-Majorana spinor and b like a scalar. There is an independent real $(\bar{27})$ representation of $E_{6(-26)}$ that has the same vector-spinor-scalar decomposition in 10-dimensional Minkowski space. Since the $(\bar{27})$ involves a right handed spinor instead of the left handed spinor u, these two representations will be assigned different chiralities.

We will leave out some further discussion on the Poincaré group and superPoincaré group in (D+1,1), the covariant superstring action, bosonic string formalism, and compactification and discretization for another publication due to lack of space. For further works on octonions we refer our reader to references at the end of this paper.

3. Remarks

We hope to have shown that there is a one to one correspondence between a graded matrix representation of the super Poincaré group and division algebras for Minkowski dimensions $(D{+}1,1)$ in which classical superstring theories and classical super Yang-Mills theories exist, D denoting the dimension of the division algebra. On the other hand classical bosonic strings can exist in any dimension. However, if closed bosonic strings are associated with superstrings to form classical heterotic strings, then the dimension of the bosonic string must be $3D + 2$. The internal symmetry group of the classical heterotic string becomes a group of rank $2D$. Quantization only works for $D = 8$ (octonionic case) but paraquantization might be possible in the other cases[2]. The internal symmetry group was obtained by discretizing $2D$ elements of a certain subgroup of $O(3D + 1, 1)$. If a bosonized ghost dimension is added as in the case of covariant treatment, then the Lorentz group is extended to $O(3D + 1, 2)$. For $D = 8$, the complex Lorentz group admits as subgroup a real form of E_6 that has the same Lorentz subgroup $O(9, 1)$ as $O(25, 2)$. The free bosonic Lagrangian than becomes invariant under both $O(25, 2)$ and $E_{6(-26)}$. This approach establishes a link between classical heterotic strings and the groups of the magic square associated with Jordan algebras, and also allows a uniform treatment of the bosonic and superstring sectors of the heterotic string.

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