IOP Conf. Series: Journal of Physics: Conf. Series **804** (2017) 012004 doi:10.1088/1742-6596/804/1/012004

Lax representation and soliton solutions for the (2+1) -dimensional two-component complex modified Korteweg-de Vries equations

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Abstract. Under investigation in this paper is the (2+1) -dimensional two-component complex modified Korteweg-de Vries (tcmKdV) equations. Based on the Ablowitz-Kaup-Newell-Segur system, the a new Lax representation is derived of the tcmKdV equations. By Darboux transformation method soliton solutions for the tcmKdV equations are obtained, respectively.

1. Introduction

The most typical and well-studied integrable evolution equations which describe nonlinear wave phenomena for a range of dispersive physical systems are the nonlinear Schrodinger (NLS) equation , Korteweg-de Vries (KdV) equation , modified Korteweg-de Vries (mKdV) equation. The complex modified Korteweg-de Vries (cmKdV) equation is one of generalized form of the KdV equation. It admits soliton solutions. In soliton theory, (cmKdV) equation possesses all the basic characters of integrable models. From a physical point of view, cmKdV equation has been derived for, e.g. the dynamical evolution of nonlinear lattices, fluid dynamics, plasma physics, nonlinear transmission lines and so on [1]. With regard to exact solutions, many classical solving methods, such as Hirotas bilinear method [2], the IST [3, 4], Darboux transformation [5-15] has been used to solve integrable evolution equations.

In this paper, we consider the new (2+1) -dimensional two-component complex modified

Korteweg-de Vries (tcmKdV) equations as follows:

$$iq_{1t} + iq_{1yxx} + i(w_1q_1)_x + i(f_1q_2)_x - \upsilon_1q_1 - u_1q_2 = 0, \tag{1}$$
$$iq_{2t} + iq_{2yxx} + i(w_2q_2)_x + i(f_2q_1)_x - \upsilon_2q_2 - u_2q_1 = 0, \tag{2}$$
$$\upsilon_{1x} + i(2q_1^*q_{1xy} - 2q_{1xy}^*q_1 + q_2^*q_{2xy} - q_{2xy}^*q_2) + f_2q_2^*q_1 - f_1q_1^*q_2 = 0, \tag{3}$$
$$\upsilon_{2x} + i(2q_2^*q_{2xy} - 2q_{2xy}^*q_2 + q_1^*q_{1xy} - q_{1xy}^*q_1) + f_1q_1^*q_2 - f_2q_2^*q_1 = 0, \tag{4}$$
$$u_{1x} + i[(q_2^*q_{1yx} - q_{2yx}^*q_1) - f_1(|q_1|^2 + |q_2|^2) + (w_2 - w_1)q_2^*q_1] = 0, \tag{5}$$
$$u_{2x} + i[(q_1^*q_{2yx} - q_{1yx}^*q_2) - f_2(|q_1|^2 + |q_2|^2) + (w_1 - w_2)q_1^*q_2] = 0, \tag{6}$$
$$w_{1x} - (2|q_1|^2 + |q_2|^2)_y = 0, \tag{7}$$
$$w_{2x} - (|q_1|^2 + 2|q_2|^2)_y = 0, \tag{8}$$
$$f_{1x} - (q_2^*q_1)_y = 0, \tag{9}$$
$$f_{2x} - (q_1^*q_2)_y = 0, \tag{10}$$

where q_1, q_2 are complex functions, υ_1, υ_2, u_1, u_2, w_1, w_2, f_1, f_2 are real functions. In one-component case cmKdV equations has been studied in [16].

The paper is prepared as follows. In Section 2, the Lax pair is presented. In Section 3, we apply Darboux transformation method for the two-component complex modified Korteweg-de Vries (tcmKdV) equations. Section 4, soliton solutions are obtained by Darboux transformation, and in Section 5 conclusions are given.

2. Lax representation

In this section, we would like to present a new Lax pair of the (2+1)-dimensional tcmKdV equations. According to the Ablowitz-Kaup-Newell- Segur (AKNS) formalism, Lax pair for equations (1)-(10) is written as

$$\Psi_x = A\Psi, \tag{11}$$
$$\Psi_t = 4\lambda^2\Psi_y + B\Psi, \tag{12}$$

with the following matrices:

$$\begin{aligned} A &= i\lambda\Sigma + A_0, \\ B &= \lambda B_1 + B_0, \end{aligned}$$

where

$$\Sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad A_0 = \begin{pmatrix} 0 & q_1 & q_2 \\ -q_1^* & 0 & 0 \\ -q_2^* & 0 & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} a_{11} & 2iq_{1y} & 2iq_{2y} \\ 2iq_{1y}^* & a_{22} & -2if_2 \\ 2iq_{2y}^* & -2if_1 & a_{33} \end{pmatrix},$$

$$B_0 = \begin{pmatrix} b_{11} & -q_{1yx} - w_1q_1 - f_1q_2 & -q_{2yx} - w_2q_2 - f_2q_1 \\ q_{1yx}^* + w_1q_1^* - f_2q_2^* & b_{22} & iu_2 \\ q_{2yx}^* + w_2q_2^* - f_1q_1^* & iu_1 & b_{33} \end{pmatrix}.$$

Moreover, $\Psi = (\Psi_1(\lambda, x, y, t), \Psi_2(\lambda, x, y, t), \Psi_3(\lambda, x, y, t))^T$ (T mean a matrix transpose) denotes the eigenfunction of Lax pair equations (11)-(12) associated with λ. The compatible condition of equations (1)-(10) is

$$A_t - B_x + AB - BA - 4\lambda^2 A_y = 0. \tag{13}$$

In fact, by direct calculation of the compatible condition (13) we can obtain next equations

$$iq_{1t} + iq_{1yxx} - \frac{1}{2}(a_{2211}q_1)_x + i(f_1q_2)_x + ib_{2211}q_1 - u_1q_2 = 0 \tag{14}$$

$$iq_{2t} + iq_{2yxx} - \frac{1}{2}(a_{3311}q_2)_x + i(f_2q_1)_x + ib_{3311}q_2 - u_1q_1 = 0 \tag{15}$$

$$a_{11x} = 2i(|q_1|^2 + |q_2|^2)_y \tag{16}$$

$$a_{22x} = -2i(|q_1|^2)_y \tag{17}$$

$$a_{33x} = -2i(|q_2|^2)_y \tag{18}$$

$$b_{11x} = (q^*_{1yx}q_1 - q^*_1q_{1yx} + q^*_{2yx}q_2 - q^*_2q_{2yx}) \tag{19}$$

$$b_{22x} = (q^*_1q_{1yx} - q^*_{1yx}q_1) + i(f_1q^*_1q_2 - f_2q^*_2q_1) \tag{20}$$

$$b_{33x} = (q^*_2q_{2yx} - q^*_{2yx}q_2) - i(f_1q^*_1q_2 - f_2q^*_2q_1) \tag{21}$$

where

$$b_{2211} = b_{22} - b_{11} = iv_1, \tag{22}$$

$$b_{3311} = b_{33} - b_{11} = iv_2, \tag{23}$$

$$a_{2211} = a_{22} - a_{11} = -2iw_1, \tag{24}$$

$$a_{3311} = a_{33} - a_{11} = -2iw_2. \tag{25}$$

After doing some simplifications of equations (14)-(25) we can yield the (2+1)-dimensional tcmKdV equations (1)-(10).

3. Darboux transformation

Based on the Darboux transformation for AKNS system, we consider the following transformation of equations (11)-(12)

$$\Psi' = T\Psi = (\lambda I - M)\Psi, \tag{26}$$

where

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix}. \tag{27}$$

New function Ψ' is supposed to satisfy

$$\Psi'_x = A'\Psi', \tag{28}$$

$$\Psi'_t = 4\lambda^2\Psi'_y + B'\Psi'. \tag{29}$$

where A' and B' depend on $q_1^{[1]}, q_2^{[1]}, u_1^{[1]}, u_2^{[1]}, v_1^{[1]}, v_2^{[1]}, w_1^{[1]}, w_2^{[1]}, f_1^{[1]}, f_2^{[1]}$ and λ. In order to hold equations (28)-(29) the T must satisfy

$$T_x = A'T - TA, \tag{30}$$

$$T_t = B'T - TB + 4\lambda^2 T_y. \tag{31}$$

One can choose $T = \lambda I - M$. Then the relation between $q_1, q_2, u_1, u_2, v_1, v_2, w_1, w_2, f_1, f_2$ and $q_1^{[1]}, q_2^{[1]}, u_1^{[1]}, u_2^{[1]}, v_1^{[1]}, v_2^{[1]}, w_1^{[1]}, w_2^{[1]}, f_1^{[1]}, f_2^{[1]}$ can be reduced from equations (30)-(31), which is the Darboux transformation of the (2+1)-dimensional tcmKdV equations. Similar to the case of AKNS, comparing the coefficient of $\lambda^i (i = 0, 1, 2)$ of the two sides of equations (30), we have

$$\begin{aligned} \lambda^0 &: M_x = A_0' M - M A_0, && (32)\\ \lambda^1 &: A_0' = A_0 + i[\Sigma, M], && (33)\\ \lambda^2 &: iI\Sigma = i\Sigma I. && (34) \end{aligned}$$

From the above equations we obtain

$$\begin{aligned} q_1^{[1]} &= q_1 - 2im_{12}, && (35)\\ q_2^{[1]} &= q_2 - 2im_{13}, && (36)\\ q_1^{*[1]} &= q_1^* - 2im_{21}, && (37)\\ q_2^{*[1]} &= q_2^* - 2im_{31}. && (38) \end{aligned}$$

Hence we get $m_{21} = -m_{12}^*, m_{31} = -m_{13}^*$.

Then comparing the coefficients of λ^i of the two sides of the equation (31) gives us

$$\begin{aligned} \lambda^0 &: M_t = B_0' M - M B_0, && (39)\\ \lambda^1 &: B_0' = B_0 - M B_1 + B_1' M, && (40)\\ \lambda^2 &: B_1' = B_1 + 4M_y. && (41) \end{aligned}$$

It leads to

$$\begin{aligned} B_0' &= B_0 - M B_1 + (B_1 + 4M_y) M, && (42)\\ B_1' &= B_1 + 4M_y. && (43) \end{aligned}$$

Using the equations (42)-(43) we can found

$$\begin{aligned} v_1^{[1]} &= v_1 + 2(2q_{1y}^* m_{12} + 2q_{1y} m_{12}^* + q_{2y} m_{13}^* + q_{2y}^* m_{13}) - 4i(m_{22} m_{22y} + m_{32} m_{23y} - \\ &\quad - m_{11} m_{11y} + m_{13}^* m_{13y}) - 2(f_2 m_{32} - f_1 m_{23})), && (44)\\ v_2^{[1]} &= v_1 + 2(2q_{2y}^* m_{13} + 2q_{2y} m_{13}^* + q_{1y} m_{12}^* + q_{1y}^* m_{12}) - 4i(m_{33} m_{33y} + m_{23} m_{32y} - \\ &\quad - m_{11} m_{11y} + m_{12}^* m_{12y}) - 2(f_1 m_{23} - f_2 m_{32})), && (45)\\ u_1^{[1]} &= u_1 + 2(q_{2y}^* m_{12} + q_{1y} m_{13}^*) - 4i(m_{22} m_{32y} + m_{32} m_{33} - m_{12} m_{13y}^*) - \\ &\quad - 2f_1(m_{22} - m_{33}) + 2(w_1 - w_2) m_{32}, && (46)\\ u_2^{[1]} &= u_2 + 2(q_{1y}^* m_{13} + q_{2y} m_{12}^*) - 4i(m_{33} m_{23y} + m_{23} m_{22} - m_{13} m_{12y}^*) - \\ &\quad + 2f_2(m_{22} - m_{33}) - 2(w_1 - w_2) m_{23}, && (47) \end{aligned}$$

$$\begin{aligned} w_1^{[1]} &= w_1 + 2i(m_{22} - m_{11})_y, && (48)\\ w_1^{[1]} &= w_2 + 2i(m_{33} - m_{11})_y, && (49)\\ f_1^{[1]} &= f_1 + 2im_{32y}, && (50)\\ f_2^{[1]} &= f_2 + 2im_{23y}. && (51) \end{aligned}$$

The key step is to find the concrete form of M expressed by the column solution of equations (11)-(12). Let

IOP Conf. Series: Journal of Physics: Conf. Series **804** (2017) 012004 doi:10.1088/1742-6596/804/1/012004

$$M = H\Lambda H^{-1}, \tag{52}$$

where

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix},$$

$$H = \begin{pmatrix} \Psi_1(\lambda_1,x,y,t) & \Psi_1(\lambda_2,x,y,t) & \Psi_1(\lambda_3,x,y,t) \\ \Psi_2(\lambda_1,x,y,t) & \Psi_2(\lambda_2,x,y,t) & \Psi_2(\lambda_3,x,y,t) \\ \Psi_3(\lambda_1,x,y,t) & \Psi_3(\lambda_2,x,y,t) & \Psi_3(\lambda_3,x,y,t) \end{pmatrix} := \begin{pmatrix} \Psi_{1,1} & \Psi_{1,2} & \Psi_{1,3} \\ \Psi_{2,1} & \Psi_{2,2} & \Psi_{2,3} \\ \Psi_{3,1} & \Psi_{3,2} & \Psi_{3,3} \end{pmatrix}$$

and $det(H) \neq 0$. We take

$$\lambda_3 = \lambda_2 = \lambda_1^*, \quad H = \begin{pmatrix} \Psi_1(\lambda_1,x,y,t) & \Psi_2^*(\lambda_1,x,y,t) & \Psi_3^*(\lambda_1,x,y,t) \\ \Psi_2(\lambda_1,x,y,t) & -\Psi_1^*(\lambda_1,x,y,t) & 0 \\ \Psi_3(\lambda_1,x,y,t) & 0 & -\Psi_1^*(\lambda_1,x,y,t) \end{pmatrix} \tag{53}$$

and then

$$H^{-1} = \frac{1}{(|\Psi_1|^2+|\Psi_2|^2+|\Psi_3|^2)\Psi_1^*} \begin{pmatrix} \Psi_1^{2*} & \Psi_1^*\Psi_2^* & \Psi_1^*\Psi_3^* \\ \Psi_2\Psi_1^* & -|\Psi_1|^2-|\Psi_3|^2 & \Psi_2\Psi_3^* \\ \Psi_3\Psi_1^* & \Psi_3\Psi_2^* & -|\Psi_1|^2-|\Psi_2|^2 \end{pmatrix}. \tag{54}$$

So taking equation (53) and (54) back into equation (52) we get matrix M in the following form

$$M = \frac{1}{\Delta} \begin{pmatrix} \lambda_1|\Psi_1|^2+\lambda_1^*(|\Psi_2|^2+|\Psi_3|^2) & \Psi_1\Psi_2^*(\lambda_1-\lambda_1^*) & \Psi_1\Psi_3^*(\lambda_1-\lambda_1^*) \\ \Psi_2\Psi_1^*(\lambda_1-\lambda_1^*) & \lambda_1|\Psi_2|^2+\lambda_1^*(|\Psi_1|^2+|\Psi_3|^2) & \Psi_2\Psi_3^*(\lambda_1-\lambda_1^*) \\ \Psi_3\Psi_1^*(\lambda_1-\lambda_1^*) & \Psi_3\Psi_2^*(\lambda_1-\lambda_1^*) & \lambda_1|\Psi_3|^2+\lambda_1^*(|\Psi_1|^2+|\Psi_2|^2) \end{pmatrix}, \tag{55}$$

where

$$\Delta = |\Psi_1|^2+|\Psi_2|^2+|\Psi_3|^2.$$

4. Soliton solution

Having the explicit form of the Darboux transformation, we are ready to construct exact solutions of the (2+1)-dimensional tcmKdV equations. As an example, let us present the one-soliton solution. To get the one-soliton solutions we assume trivial seed solutions of equations (1)-(10) as

$$q_1 = 0; q_2 = 0; \upsilon_1 = 0; \upsilon_2 = 0; u_1 = 0; u_2 = 0; w_1 = 0, w_2 = 0; f_1 = 0; f_2 = 0. \tag{56}$$

From equations (11)-(12), the corresponding eigenfunctions of this seed become

$$\Psi_1 = e^{-i\lambda x+ib_1y+4\lambda^2b_1t+i\delta_1}, \tag{57}$$

$$\Psi_2 = e^{i\lambda x+ib_2y+i4\lambda^2b_3t+i\delta_2}, \tag{58}$$

$$\Psi_3 = e^{i\lambda x+ib_3y+i4\lambda^2b_3t+i\delta_3}, \tag{59}$$

where $\delta_1, \delta_2, \delta_3$ are real constants. If $\lambda = \alpha + i\beta$, $b_j = \nu_j + in_j$ $(j = 1, 2, 3)$ then we can rewrite equations (57)-(59) as

$$\begin{aligned} \Psi_1 &= e^{\theta_1 + i\chi_1}, && (60) \\ \Psi_2 &= e^{\theta_2 + i\chi_2}, && (61) \\ \Psi_3 &= e^{\theta_3 + i\chi_3}, && (62) \end{aligned}$$

where

$$\begin{aligned} \theta_1 &= \beta x - n_1 y + 4(\beta^2 n_1 - \alpha^2 n_1)t - 8\alpha\beta\nu_1 t, \\ \chi_1 &= -\alpha x + \nu_1 y + 4(\alpha^2 \nu_1 - \beta^2 \nu_1)t - 8\alpha\beta n_1 t, \\ \theta_2 &= \beta x - n_2 y + 4(\beta^2 n_2 - \alpha^2 n_2)t - 8\alpha\beta\nu_2 t, \\ \chi_2 &= -\alpha x + \nu_2 y + 4(\alpha^2 \nu_2 - \beta^2 \nu_2)t - 8\alpha\beta n_2 t, \\ \theta_3 &= \beta x - n_3 y + 4(\beta^2 n_3 - \alpha^2 n_3)t - 8\alpha\beta\nu_3 t, \\ \chi_3 &= -\alpha x + \nu_3 y + 4(\alpha^2 \nu_3 - \beta^2 \nu_3)t - 8\alpha\beta n_3 t. \end{aligned}$$

By taking equations (60)-(62) into equations (35),(36),(44)-(51) and doing some simplifications, we obtain the one-soliton solutions of the (2+1)-dimensional tcmKdV equations in the following form

$$\begin{aligned} q_1^{[1]} &= -2im_{12}, \\ q_2^{[1]} &= -2im_{13}, \\ v_1^{[1]} &= -4i(m_{22}m_{22y} + m_{32}m_{23y} - m_{11}m_{11y} + m_{13}^* m_{13y}), \\ v_2^{[1]} &= -4i(m_{33}m_{33y} + m_{23}m_{32y} - m_{11}m_{11y} + m_{12}^* m_{12y}), \\ u_1^{[1]} &= -4i(m_{22}m_{32y} + m_{32}m_{33} - m_{12}m_{13y}^*), \\ u_2^{[1]} &= -4i(m_{33}m_{23y} + m_{23}m_{22} - m_{13}m_{12y}^*), \\ w_1^{[1]} &= 2i(m_{22} - m_{11})_y, \\ w_1^{[1]} &= 2i(m_{33} - m_{11})_y, \\ f_1^{[1]} &= 2im_{32y}, \\ f_2^{[1]} &= 2im_{23y}, \end{aligned}$$

where

$$\begin{aligned} m_{11} &= \alpha + \frac{i\beta(e^{2\theta_1} - e^{2\theta_2} - e^{2\theta_3})}{e^{2\theta_1} + e^{2\theta_2} + e^{2\theta_3}}, \\ m_{22} &= \alpha + \frac{i\beta(e^{2\theta_2} - e^{2\theta_1} - e^{2\theta_3})}{e^{2\theta_1} + e^{2\theta_2} + e^{2\theta_3}}, \\ m_{33} &= \alpha + \frac{i\beta(e^{2\theta_3} - e^{2\theta_1} - e^{2\theta_2})}{e^{2\theta_1} + e^{2\theta_2} + e^{2\theta_3}}, \\ m_{12} &= \frac{2i\beta e^{\theta_1 + \theta_2 + i(\chi_1 - \chi_2)}}{e^{2\theta_1} + e^{2\theta_2} + e^{2\theta_3}}, \\ m_{13} &= \frac{2i\beta e^{\theta_1 + \theta_3 + i(\chi_1 - \chi_3)}}{e^{2\theta_1} + e^{2\theta_2} + e^{2\theta_3}}, \\ m_{23} &= \frac{2i\beta e^{\theta_2 + \theta_3 + i(\chi_2 - \chi_3)}}{e^{2\theta_1} + e^{2\theta_2} + e^{2\theta_3}}, \\ m_{32} &= \frac{2i\beta e^{\theta_2 + \theta_3 + i(\chi_3 - \chi_2)}}{e^{2\theta_1} + e^{2\theta_2} + e^{2\theta_3}}. \end{aligned}$$

5. Conclusion

In this letter, we present a new Lax pair of the (2+1)-dimensional two-component complex modified Korteweg-de Vries equations. One fold Darboux transformation is constructed. The one-soliton solutions are obtained be means of the Darboux transformation. In a recursive manner, one can obtain the multi-soliton solution from the two seeds in previous. We will study other different kind solutions of the (2+1)-dimensional tcmKdV equations in future.

Acknowledgment

The work of G.B., G.Sh., K.Y., and M.R. is supported by the Ministry of Education and Science of Republic of Kazakhstan (the base part of the state task No.0888/GF4)

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