Flight Control System Synthesis for High-Speed Unmanned Vehicle, Considering its Elastic Properties

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Abstract. The paper presents the method of building, under certain assumptions, of an accurate dynamic model of the elastic link in the form of the transfer of meromorphic functions and their expansions in series of tones fluctuations. This allows one to compare the exact and approximate characteristics of the elastic link and, therefore, it is reasonable to impose appropriate simplification.

1. Introduction

Elastic properties of high-speed unmanned vehicles (UV) may occur in their dynamic motion. Circuit stabilization transients, occurring under the influence of aerodynamic forces, are conducted with elastic deformation of the UV’s hull, which affect the signals of measurement facilities. For instance, bending angles of the hull bring an additional component in gyroscope’s signals and accelerations of elastic vibrations occur in the signals of accelerometers. Thus, the inherent feedback appears in stabilization loop, in short, elastic linkages, which should be taken into account while analyzing the outline property. Mathematical description of these linkages as well as further analysis with regard to their stabilization loop is a very complicated problem [1-4]. Usually, one is satisfied with approximate phenomenon models in the form of one or more oscillatory links which correspond to bases of elastic vibrations [5-7]. The problem of accuracy of the approximate models is usually solved by experimental procedures.

We consider an issue of designing, under certain assumptions, of an adequate dynamic model of an elastic linkage in the form of meromorphic transfer functions and their expansion in terms of the fluctuation mode. This allows comparing accurate and approximate responses of the elastic lineage and, consequently, adding certain conditions on a reasonable basis.

2. Background

The model of elastic vibrations with equal distribution of mass along the length of UV and constant (medium) along the length of elasto-mechanical characteristics of the hull is adopted as basic assumption, which allows us to obtain this result. Due to this assumption, the model of elastic vibrations of UV in the coordinate system which is connected with the strainless hull can be obtained [8,9] in the form of a partial differential equation for deflection $Z(x,t)$

$$\frac{\partial^4 Z(x,t)}{\partial x^4} + k^4 \frac{\partial^2 Z(x,t)}{\partial t^2} = R(x,t)\,, \tag{1}$$

Boundary conditions $Z''(0,t)=0,\quad Z''(\ell,t)=0,\ Z'''(0,t)=0,\quad Z'''(\ell,t)=0,$ and also initial conditions $Z(x,0)=u(x),\quad \dot{Z}(x,0)=v(x)$, which thereafter are accepted as zero (i.e. free oscillations of the elastic hull are not considered as they do not affect build-up inherent feedback).

The following representations were taken in equation (1): $k^4 = \rho/EJ$ - elasto-mechanical constant; ρ - linear mass density of the vehicle, where $\rho = m/\ell$ (m - mass, ℓ - length of the vehicle); E - Young's modulus; J - second area moment; $R(x,t)$ - distribution function of the aerodynamic force, which is taken as equal to

$$R(x,t) = R^{\beta}(x)\beta(t) + R^{\delta}(x)\delta(t)\,,$$

where $\beta(t)$, $\delta(t)$ - angles of slide and deviation of the rudder, respectively.

$$R^{\beta}(x) = \mathrm{P}^{\beta}(x)\big/EJ\,,\ R^{\delta}(x) = P^{\delta}(x)\big/EJ\,,$$

$$P^{\beta}(x) = D^{\beta}(x) - \frac{\rho}{m}Z^{\beta} - \frac{(x-x_c)}{J_y}M_y^{\beta}\,,\ P^{\delta}(x) = D(x) - \frac{\rho}{m}Z^{\delta} - \frac{(x-x_c)}{J_y}M_y^{\delta}\,.$$

Here, $D^{\beta}(x)$, $D^{\delta}(x)$ - densities of distributions of aerodynamic forces which are defined by the angles of slide and deviation of the rudder respectively, in which connection

$$\int_0^{\ell} D^{\beta}(x)dx = Z^{\beta} = \frac{1}{2}C_Z^{\beta}\rho_b SV^2\,,$$

$$\int_0^{\ell} D^{\delta}(x)dx = Z^{\delta} = \frac{1}{2}C_Z^{\delta}\rho_b SV^2\,,$$

where C_Z^{β}, C_Z^{δ} - aerodynamic force coefficients, ρ_b - mass density of the airflow,

$$\int_0^{\ell} (x-x_c)D^{\beta}(x) = M_y^{\beta} = \frac{1}{2}m_y^{\beta}\rho_b SV^2\,,$$

$$\int_0^{\ell} (x-x_c)D^{\delta}(x) = M_y^{\delta} = \frac{1}{2}m_y^{\delta}\rho_b SV^2\,,$$

where m_y^{β}, m_y^{δ} - coefficients of the aerodynamic moment. For functions $P^{\beta}(x), P^{\delta}(x)$, there are conditions

$$\int_0^{\ell} P^{\beta}(x)dx = 0\,, \int_0^{\ell} P^{\delta}(x)dx = 0\,, \int_0^{\ell} xP^{\beta}(x)dx = 0\,, \int_0^{\ell} xP^{\delta}(x)dx = 0\,.$$

3. Transfer functions and frequency characteristics of the high-speed Unmanned Vehicle as an elastic linkage

The use of the Laplace transformation adapted to the model [10], which is shown above, allows us to receive the following representation for the deflections in figures with respect to t:

$$Z(x,s) = U^{\beta}(x,s)\beta(s) + U^{\delta}(x,s)\delta(s)\,, \tag{2}$$

where

$$U^{\beta}(x,s)=\frac{1}{\alpha^3}\int_0^x R^{\beta}(x-\xi)k_4(\alpha\xi)d\xi+\int_0^{\ell}\frac{g(\xi,s)}{D(s)}R^{\beta}(\ell-\xi)d\xi\,, \tag{3}$$

$$U^{\delta}(x,s)=\frac{1}{\alpha^3}\int_0^x R^{\delta}(x-\xi)k_4(\alpha\xi)d\xi+\int_0^{\ell}\frac{g(\xi,s)}{D(s)}R^{\delta}(\ell-\xi)d\xi \tag{4}$$

are represented by the transfer functions of the UV as the elastic linkage. In these relationships:

$$D(s)=16\alpha^4[k^2(\alpha\ell)-k_2(\alpha\ell)k_4(\alpha\ell)]\,,\, g(\xi,s)=k_1(\alpha x)a(\xi,s)+k_2(\alpha x)b(\xi,s)\,,$$

$$a(\xi,s)=4\alpha[k_3(\alpha\ell)k_2(\alpha\xi)-k_4(\alpha\ell)k_1(\alpha\xi)]\,,\, b(\xi,s)=4\alpha[k_3(\alpha\ell)k_1(\alpha\xi)-k_2(\alpha\ell)k_2(\alpha\xi)]\,,$$

$$\alpha^4=k^4s^2/4\,,\, k^4=\rho/EJ\,,$$

$$k_1(x)=ch(x)\cos(x)\,,\, k_2(x)=\frac{1}{2}(ch(x)\sin(x)+sh(x)\cos(x))\,,\, k_3(x)=\frac{1}{2}sh(x)\sin(x)\,,$$

$$k_4(x)=\frac{1}{4}(ch(x)\sin(x)-sh(x)\cos(x))\,.$$

The relationship (2) allows us to introduce the transfer functions concerning measured variables. For example, if there is acceleration gauge in-line stabilization along with free oscillations, the auxiliary signal of the acceleration gauge is emerged:

$$\sigma_a(s)=s^2U^{\beta}(x,s)\beta(s)+s^2U^{\delta}(x,s)\delta(s)\,.$$

In a similar way, in the presence of the displacement gyroscope, the auxiliary signal of this sensory organ appears, as it will trace the bending angle of UV's hull. With small swaying, this angle can be denoted as

$$\mu(x,s)=\frac{\partial}{\partial x}Z(x,s)\bigg|_{x=x_G}\,,$$

where x_G - abscissa of the attaching point of the gyroscope.

Differentiating with respect to x expression for deflection, we can find

$$\mu(x,s)=V^{\beta}(x,s)\beta(s)+V^{\delta}(x,s)\delta(s)\,, \tag{5}$$

where $V^{\beta}(x,s)=\frac{\partial}{\partial x}U^{\beta}(x,s)\,,\, V^{\delta}(x,s)=\frac{\partial}{\partial x}U^{\delta}(x,s)$.

Differentiating equations (3) and (4), we can find

$$V^{\beta}(x,s)=-\frac{1}{\alpha^2}\int_0^x R^{\beta}(x-\xi)k_3(\alpha\xi)d\xi+\int_0^{\ell}\frac{g_x(\xi,s)}{D(s)}R^{\beta}(\ell-\xi)d\xi\,, \tag{6}$$

$$V^{\delta}(x,s)=-\frac{1}{\alpha^2}\int_0^x R^{\delta}(x-\xi)k_3(\alpha\xi)d\xi+\int_0^{\ell}\frac{g_x(\xi,s)}{D(s)}R^{\delta}(\ell-\xi)d\xi\,, \tag{7}$$

where

$$g_x(\xi,s)=\frac{\partial}{\partial x}g(\xi,s)=-4\alpha k_4(\alpha x)a(\xi,s)+\alpha k_1(\alpha x)b(\xi,s)\,. \tag{8}$$

Derived transfer functions of the UV as the elastic linkage, presented by equations (3), (4) and (6), (7) are meromorphic function which pole sets comprise imaginary zeros of the characteristic function of elastic vibration $D(s)$. Point s=0 falls outside the limits of a set. Characteristic function $D(s)$ can be represented in terms of

$$D(s) = 2k^8\ell^4 s^4 \delta(s)\,, \delta(s) = \sum_{n=0}^{\infty} \frac{1}{(4n+4)!} c^{4n} s^{2n}\ , c = \sqrt{2}k\ell\,,$$

where entire function $\delta(s)$ has only imaginary zeros. Transfer functions $U^{\beta}(x,s), U^{\delta}(x,s)$ with regard to this, can be reported in terms of

$$U^{\beta}(x,s) = \frac{1}{\alpha^3}\int_0^x R^{\beta}(x-\xi)k_4(\alpha\xi)d\xi + \frac{1}{2k^4\ell^4}\cdot\frac{\int_0^{\ell}\psi(\xi,s)R^{\beta}(\ell-\xi)d\xi}{\delta(s)}\,,$$

$$U^{\delta}(x,s) = \frac{1}{\alpha^3}\int_0^x R^{\delta}(x-\xi)k_4(\alpha\xi)d\xi + \frac{1}{2k^4\ell^4}\cdot\frac{\int_0^{\ell}\psi(\xi,s)R^{\delta}(\ell-\xi)d\xi}{\delta(s)}\,,$$

where $\psi(s)$ - function of even degrees s. From the previous equations, it follows that the pole set of the transfer functions make up a zero function, i.e. imaginary zeros of function $D(s)$. The same representations are valid for functions $V^{\beta}(x,s), V^{\delta}(x,s)$.

In the expression for transfer functions $s = j\omega$, we will have response characteristics of the elastic device. For example, for response characteristics $U^{\beta}(x,j\omega) = U^{\beta}(x,\omega)$ from equation (3), one can receive

$$U^{\beta}(x,\omega) = \frac{1}{2k^3\omega\sqrt{\omega}}\int_0^x R^{\beta}(x-\xi)(sh(k\sqrt{\omega}\xi) - \sin(k\sqrt{\omega}\xi))d\xi - \frac{1}{4k^3\omega\sqrt{\omega}}\int_0^{\ell}\frac{r(\xi,\omega)}{\varphi(k\ell\sqrt{\omega})}R^{\beta}(\ell-\xi)d\xi, \quad (9)$$

where $r(\xi,\omega) = (\sin(\lambda x) + sh(\lambda x))Y(\lambda,\xi) - (\cos(\lambda x) + ch(\lambda x))X(\lambda,\xi)\,, \lambda = k\sqrt{\omega}$.

Functions $Y(\lambda,\xi),\ X(\lambda,\xi)$ are defined by formulas

$$Y(\lambda,\xi) = (sh(\lambda\xi) + \sin(\lambda\xi))(sh(\lambda\ell) + \sin(\lambda\ell)) + (ch(\lambda\xi) + \cos(\lambda\xi))(\cos(\lambda\ell) - ch(\lambda\ell)),$$

$$X(\lambda,\xi) = (ch(\lambda\xi) + \cos(\lambda\xi))(\sin(\lambda\ell) - sh(\lambda\ell)) + (sh(\lambda\xi) + \sin(\lambda\xi))(ch(\lambda\ell) - \cos(\lambda\ell)).$$

Moreover,

$$\varphi(z) = ch(z)\cos(z) - 1\,.$$

We will have exactly the same equation with the substitution of $R^{\beta}(x)$ for $R^{\delta}(x)$ in functions $U^{\delta}(x,j\omega) = U^{\delta}(x,\omega)$.

Similarly, for response characteristic $V^{\beta}(x,j\omega) = V^{\beta}(x,\omega)$, there is formula

$$V^{\beta}(x,\omega) = -\frac{1}{2k^2\omega}\int_0^{\infty} R^{\beta}(x-\xi)(ch(k\sqrt{\omega}\xi) - \cos(k\sqrt{\omega}\xi))d\xi - \frac{2}{k^2\omega}\int_0^{\ell}\frac{F(\xi,\omega)}{\varphi(k\ell\sqrt{\omega})}R^{\beta}(\ell-\xi)d\xi, \quad (10)$$

where

$$F(\xi,\omega) = (ch(\lambda x) + \cos(\lambda x))Y(\xi,\lambda) - (sh(\lambda x) - \sin(\lambda x))X(\xi,\lambda)\,.$$

Response characteristic $V^{\delta}(x,j\omega) = V^{\delta}(x,\omega)$ follows from equation (10) with function replacement of $R^{\beta}(x)$ with function $R^{\delta}(x)$. Equations (3), (4) and (6), (7) for the transfer function and (9), (10) for the frequency characteristics can be used as datum points for building up multiple views of the transfer function and response characteristics under any given model of the distribution function of forces $R^{\beta}(x)$, $R^{\delta}(x)$.

By virtue of the fact that these equations are quite difficult, the search of their acceptable simplification is of interest. Such simplifications can be obtained with the help of the instrument which decomposes meromorphic functions to partial fractions. It is commonly known that for accurate meromorphic function $f(z)$ with poles a_k, $k=1,2,\ldots$ and principal parts

$$G(z,a_k)=\sum_{m=1}^{m_k}\frac{A_{mk}}{(z-a_k)^m}$$

of its expansions to Laurent series near these poles, the following expression is valid:

$$f(z)=\lim_{\nu\to\infty}\sum_{k=1}^{\nu}G(z,a_k)$$

with the equal pursuance of the limit in any final domain, which does not have points a_k. Nevertheless, for the accurate meromorphic function, the existence of such number of sequence r_ν is assumed, for which

$$M(r_\nu)\to 0,\ \nu\to\infty,\ M(r)=\max_{|z|=r}\left|f(z)\right|.$$

If poles a_k are simple, then $m_k=1$ and

$$G(z,a_k)=\frac{A_{1k}}{z-a_k},\ A_{1k}=\lim_{z\to a_k}(z-a_k)G(z,a_k).$$

Assuming that for chosen models of the distribution function of forces $R^{\beta}(x)$, $R^{\delta}(x)$, the transfer functions are the meromorphic functions of specified type, one can find their decimal expansion. Grouping poles $s=\pm j\omega_n$ in pairs, for transfer function $U^{\beta}(x,s)$ we will have, for example, the following representation:

$$U^{\beta}(x,s)=\sum_{n=1}^{\infty}\left(\frac{A_{1n}}{s-s_n}+\frac{\bar{A}_{1n}}{s-\bar{s}_n}\right),\ s_n=j\omega_n,\ \bar{s}_n=-j\omega_n,$$

where

$$A_{1n}=\lim_{s\to s_n}U^{\beta}(x,s)=\int_0^{\ell}\frac{g(\xi,s_n)}{D^{(1)}(s_n)}R^{\beta}(\ell-\xi)d\xi.$$

Using the value for $D^{(1)}(j\omega)$ in the form of

$$D^{(1)}(j\omega)=-2j\frac{D(j\omega)}{\omega}-j\frac{k^5\omega^2\ell}{\sqrt{\omega}}k_4(k\ell\sqrt{\omega}),$$

we will discover that

$$A_{1n}=j\frac{\sigma_n}{\omega_n},$$

$$\frac{\sigma_n}{\omega_n}=\int_0^{\ell}\frac{g(\xi,\omega_n)R^{\beta}(\ell-\xi)}{k^5\ell\omega_n\sqrt{\omega_n}k_4\mu_n}d\xi.$$

Hence, since $\bar{s}_n=j\omega_n$, then

$$\bar{A}_{1n}=-j\frac{\sigma_n}{\omega_n},$$

and

$$U^{\beta}(x,s) = -2\sum_{n=1}^{\infty} \frac{\sigma_n}{s^2+\omega_n^2}, \tag{11}$$

where

$$\sigma_n = \frac{1}{k^5 \ell \sqrt{\omega_n} k_4 \mu_n} \int_0^{\ell} g(\xi,\omega_n) R^{\beta}(\ell-\xi) d\xi .$$

We can shape the received expansion into

$$U^{\beta}(x,s) = \sum_{n=1}^{\infty} \frac{\delta_n^{\beta}(x)}{s^2+\omega_n^2},$$

$$\delta_n^{\beta}(x) = \frac{4X_n(x)\int_0^{\ell} X_n(x) R^{\beta}(x) dx}{k^4 \ell X_n^2(\ell)},$$

$$X_n(x) = (ch(\mu_n \frac{x}{\ell}) + \cos(\mu_n \frac{x}{\ell}))(\sin(\mu_n) - sh(\mu_n)) + (sh(\mu_n \frac{x}{\ell}) + \sin(\mu_n \frac{x}{\ell}))(ch(\mu_n) - \cos(\mu_n))$$

and

$$\omega_n = \mu_n^2 / k^2 \ell^2 ,\ 0 < \mu_1 < \mu_2 < \cdots < \mu_n < \mu_{n+1} < \cdots,$$

where μ_n, $n = 1, 2, \ldots$ - zeroes of function $\varphi(z)$.

Similar considerations are fair for the housing of the bending angle.

4. Conclusion

The transfer functions of occasional dynamic links were built, affected by elastic deformations in stabilization loop through gyroscopic devices and accelerometers using Laplace transformation into a differential equation in partial derivatives for UV's deflections of the hull.

Determined analytical representations of transfer functions for meromorphic functions are valid under substantial assumptions towards elastic and mechanical properties of UV's hull.

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