

# Robustness and cognition in stabilization problem of dynamical systems based on asymptotic methods

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**Abstract.** The problem of synthesis of stabilizing systems based on principles of cognitive (logical-dynamic) control for mobile objects used under uncertain conditions is considered. This direction in control theory is based on the principles of guaranteeing robust synthesis focused on worst-case scenarios of the controlled process. The guaranteeing approach is able to provide functioning of the system with the required quality and reliability only at sufficiently low disturbances and in the absence of large deviations from some regular features of the controlled process. The main tool for the analysis of large deviations and prediction of critical states here is the action functional. After the forecast is built, the choice of anti-crisis control is the supervisory control problem that optimizes the control system in a normal mode and prevents escape of the controlled process in critical states. An essential aspect of the approach presented here is the presence of a two-level (logical-dynamic) control: the input data are used not only for generating of synthesized feedback (local robust synthesis) in advance (off-line), but also to make decisions about the current (on-line) quality of stabilization in the global sense. An example of using the presented approach for the problem of development of the ship tilting prediction system is considered.

## 1. Introduction

One of the main problems in control theory is the problem of motion stabilization under disturbed conditions. Despite long history of the issue, it is not less important than in the middle of the last century, when such problems were set on the agenda for the first time, primarily, in the interests of aircraft and marine vessels control. Meanwhile, an analysis of the statistics of flight accidents, both in military and civil aviation, indicates that the share of aircraft accidents due to falling into stall and spin modes is increasing over the years. Accidents statistics in the maritime transport is similar. The fact is that the existing experience in control problems is mainly restricted by local stabilization of some most common modes of motion, allowing linear equations in the process description that allows obtaining a solution in the form of a feedback (robust synthesis). In the nonlinear case, everything is much more complicated – usually there is no solution in the form of synthesis. Here, however, it should be noted that we are talking about solving a mathematical problem, for example, the optimal control problem. There is an alternative in the form of systems based on the principles of mathematical informatics [1, 2], that is, systems that are based on knowledge, or cognitive systems. In this approach, the stabilizing feedback is generated from the experimental data in the form of some expert system [3]. The difficulty, however, is that the concepts of the lattice and the scale of notions, [1, 2] generated during training of such systems, are very complex in motion control problems, they are difficult even implementing modern means of fuzzy logic and artificial neural networks. But the main thing is that



the system supposedly based on "knowledge" is ultimately based on the absolutely certain "lack of knowledge". During its training, according to experts (operators, pilots, steering, etc.), there is no way of taking into account the laws of motion of the object (dynamic equations), and we have to abandon them.

In this paper, an asymptotic approach to the control design, combining a two-level synthesis mentioned in a certain sense, is proposed. Note that there is a two-level scheme in [4], that uses entirely different solutions; on the other hand, the use of asymptotic techniques of the synthesis has been scheduled in [5].

But the main source of the algorithm, proposed here, is a Wentzel-Freidlin method of analysis of large deviations [6], which is particularly intensively developed in English literature, that yielded some interesting results, forming whole directions, such as «risk-sensitive control» and «aiming control» [7-11]. It should be noted that in this case the used asymptotic scheme with a small perturbations parameter of  $\varepsilon$  does not mean that perturbations are small in amplitude (it is not true for the "white noise"), it just means that large deviations begin to appear in the system, initially located in a small neighborhood of the equilibrium state, only after longer time intervals of order  $\varepsilon^{-1}$ . In problems of local stabilization (robust synthesis), other methods of asymptotic analysis are used – methods of singular perturbations, based primarily on the system decomposition and the construction of a stabilizing control in the form of a composition of simple controls, which does not depend on perturbation parameter  $\varepsilon$  [12-15] that gives the system robust properties with respect to  $\varepsilon$ . In this case, the stabilization problem can be formulated as a problem of parametric synthesis aimed at achieving stability of a non-original perturbed system, but a family of systems, parameterized by values  $\varepsilon$  in some interval  $(0, \varepsilon^*)$ , and the criteria will be to maximize the upper bound of  $\varepsilon^*$  [16].

## 2. Statement of the Problem

Let the  $r$ -vector of control  $U = U(t)$  be needed to be selected for some object, motion of which is described by weakly perturbed differential equations for the  $n$ -vector of state  $x = x(t)$ :

$$\dot{x} = \alpha(x, U) + \varepsilon \sigma(x)w, \quad x(0) = x_0 \in E, \quad (1)$$

where  $\varepsilon > 0$  – small parameter,  $w$  –  $k$  – vector of disturbances such as "white noise",  $\alpha, \sigma$  – smooth matrix functions.

Control  $U$  in (1) is formed as feedback

$$U = Kx \quad (2)$$

to provide equilibrium state  $\chi$  (attractor) of the unperturbed system, which is obtained from (1) with  $\varepsilon = 0$ :

$$\dot{x} = \alpha(x, U), \quad x(0) = x_0 \quad (3)$$

with attraction region  $O_\chi$ . Further, it is considered that  $\chi = 0$ , and the quality of stabilizing control (2) is determined by the linearized system:

$$\dot{x} = Ax + BU, \quad (4)$$

where matrixes  $A, B$  are the matrixes of partial derivatives  $\alpha(x, U)$  with respect to arguments at zero. As a result of the closure of original system (1) with such a stabilizing control, we obtain autonomous system

$$\dot{\tilde{x}} = a(\tilde{x}) + \varepsilon \sigma(\tilde{x})w. \quad (5)$$

Along with system (5) let write an ordinary differential equation – the system of deviation [3, 4]:

$$\dot{\varphi} = a(\varphi) + \sigma(\varphi)v, \quad \varphi(0) = x_0 \in E, \quad (6)$$

and functional

$$S_{t_0, t_f}(\varphi, v) = \int_{t_0}^{t_f} v^T v dt. \quad (7)$$

Let us also introduce region  $D$ , such that  $E \subset D \subset O_\chi$  and the condition of the path belonging to set  $F_D$  (implements event  $\partial_D$ , the probability of which is estimated) from the family of functions, continuous on an interval:  $F_D = \{\varphi \in C_{t_0, t_f}(R^n) : \varphi_t \in D \cup \partial_D \forall t \in [t_0, t_f]\}$ . For set  $F_D$  and systems (6) the following equality holds [6, 7]:

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^2 \ln P\{\tilde{x} \in D\} = -\min_{\varphi \in F_D} S_{t_0, t_f}(\varphi, v), \quad (8)$$

where functional  $S_{t_0, t_f}$  with restriction (5) is defined based on the solutions of control system (6), for which one can write the boundary condition of getting into the critical state:

$$\varphi(t_f) \in \Delta \subset \partial_D. \quad (9)$$

Thus, to evaluate the probability of event  $\partial_D$ , associated with process (6), we get the optimal control problem of Lagrange-Pontryagin (LP) (6),(7),(9) in contrast to the corresponding variational problem formulated in [6, 7].

### 3. The solving of the LP problem

The result of the solving of the LP problem is a triple  $(\bar{v}, \bar{\varphi}, \bar{t}_f)$ , that is, extremal  $\bar{\varphi} = \bar{\varphi}(t, \Delta)$ ,  $t \in [t_0, t_f]$  determines minimum value (7) and action functional (AF):  $\bar{I} = \bar{I}_{t_0, \bar{t}_f}(\bar{\varphi}, \bar{v}) = \varepsilon^{-2} \bar{S}_{t_0, \bar{t}_f}(\bar{\varphi}, \bar{v})$ . Quasipotential of system (6) is determined by AF [6, 7] and represents the function of point  $x$  and the equilibrium state:

$$V(\chi, x) = \inf\{S_{t_0, t_f}(\varphi) : \varphi \in C_{t_0, t_f}(R^n), \varphi_{t_0} = \chi, \varphi_{t_f} = x\}.$$

Corresponding extremal  $\bar{\varphi}$ , satisfying (6) and leading out of equilibrium  $\chi$ , is called quasipotential extremal.

Assume the following terminology: let  $\bar{\varphi}_1, \bar{\varphi}_2$  be two solutions of the LP problem corresponding to two conditions (9):  $\bar{\varphi}_i = \bar{\varphi}(t, \Delta_i)$ ,  $i = 1, 2$ , where  $\Delta_1 \subset \Delta_2 \subset \partial_D$ . In view of (8), corresponding solution (6) is called the prediction of the critical state of system (6). If  $\Delta$  is a subset of border  $\partial_D$ , then - solution  $\bar{\varphi}_2$ , we assume a weaker prediction compared to  $\bar{\varphi}_1$ . Finally, an extremely poor prediction corresponds to the case of one-point  $\Delta = \Delta_* = x_*$ .

Let us in (6) denote  $a(x) = A_{CL}x$ ,  $A_{CL} = A + BK$ ,  $\sigma(x) = \sigma$ , so

$$\dot{\varphi} = A_{CL}\varphi + \sigma v. \quad (10)$$

Also, let  $x_f \in \Delta$  mean that some of the system output  $y = Cx_f$  reaches the critical value. Problems (7), (9) and (10) in this case are well known. It is the basis of a constructive proof of system (10) controllability under the appropriate conditions [17,18], and the construction of its attainability set [18]. Without going into details, we present the result [17], where  $\psi$  denotes the  $n$ - vector of conjugate variables for the system of deviation (10) and  $\psi_f = \psi(t_f)$ : on condition that  $A_{CL}$  is a Hurwitz-stable matrix and pair  $(A_{CL}, \sigma)$  is controllable, the quasi-potential of system (10) is equal to

$$V(0, x_f) = \psi_f^T D \psi_f = x_f^T D^{-1} x_f, \quad (11)$$

where positive definite matrix  $D$  satisfies the Lyapunov equation:

$$\sigma \sigma^T = -A_{CL} D - D A_{CL}^T,$$

and quasipotential extremal (QE) for problems (7), (9), (10) is defined by:

$$\tilde{\varphi}(t) = D \exp\{A_{CL}^T(t_f - t)\} C^T y (C D C^T)^{-1}, \quad (12)$$

and normalized AF (7) for system (6) in linear case (10) is equal to

$$S_{t_0, t_f}(\tilde{\varphi}) = V(0, x_f) - V(0, x_0).$$

Thus, the motion control in terms of perturbations can be represented in two levels:

1) local – the stabilization problem is solved on the basis of deterministic system (4); provided with Hurwitz-stable matrix  $A_{CL} = A + BK$ ;

2) global – control of trajectory misses of the perturbed system in the neighborhood of QE (12).

#### 4. Evaluation of ship seaworthiness

Let us consider the problem of estimating the probability of occurrence of certain critical values of the roll angle. Rolling of the vessel in waves and wind can be represented by system (6) of the second order, unless you consider a filters equation, forming a wind-wave perturbations given the spectra of "white noise". We will not take into account the forming filters, and take into account the effect associated with the increment shoulders of static stability in the wave [17], as a result, a pitching model is bilinear. Equation (6) in this case can be represented as

$$\dot{x} = \gamma(x, u) = Ax + Bu + \{xM\}u, \quad (13)$$

where  $x = (x_1, x_2, \dots, x_n)^T \in R^n$ ,  $u \in R^m$ ,  $\{xM\} = \sum_{i=1}^n x_i M_i$ ,  $x = (x_1, x_2)^T = (\theta, \omega_x)^T$  – roll angle and angular velocity, rad and rad/s,

$$A = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & -2h \end{pmatrix}, B = \begin{pmatrix} 0 & 0 & \varepsilon \\ 0 & \gamma & 0 \end{pmatrix}, M_1 = \begin{pmatrix} 0 & 0 & 0 \\ \mu & 0 & 0 \end{pmatrix},$$

in this case, provided that:  $\omega_0^2 = 0.36$ ,  $h = 0.0315$ ,  $\gamma = \mu = 0.1$ ,  $\varepsilon = 0.001$ . The problem is to predict the critical value of roll angle  $y = Cx_f = \theta_f$ ,  $\theta_f = 0.5$  rad.

Results (11) (12) can not be used directly, but they can be used for numerical calculations. So, for the construction of limiting forecast, we have the estimate of the boundary conditions for the conjugate system:

$$\psi_f = C^T \psi_{f1} = \begin{pmatrix} \psi_{f1} \\ 0 \end{pmatrix}, \psi_{f1} = 0.5(CDC^T)^{-1}, \quad (14)$$

And when estimating covariance matrix  $D$  based on the linear approximation and the Lyapunov equation, we get:

$$D = \begin{pmatrix} D_1 & D_2 \\ D_2 & D_3 \end{pmatrix}, D_3 \cong \frac{\gamma^2}{4h}, D_1 \cong D_3 / \omega_0^2.$$

Now, from (14) we have:

$$\psi_{f1} = 0.5D_1^{-1} = 2h \left( \frac{\omega_0}{\gamma} \right)^2 \cong 2.27. \quad (15)$$

For simulation, the results of which are shown in figure 1, we finally selected  $\psi_{f1} = 2.15$ ; in nonlinear problems, the choice of the boundary conditions for the conjugate variables is the most time-consuming operation, so evaluations (22), (23) are very useful. In figure 1,  $u_1, u_2$  – correspondingly the disturbance of the sea waves and wind fluctuation. We have already noted that extremals LP are a significantly smoother function than the original trajectory, and the impact of stochastic systems – oscillograms in figure 1 demonstrates this fact. The lower curve is the AF, giving, in view of (8), the estimation of the logarithm of critical roll probability.

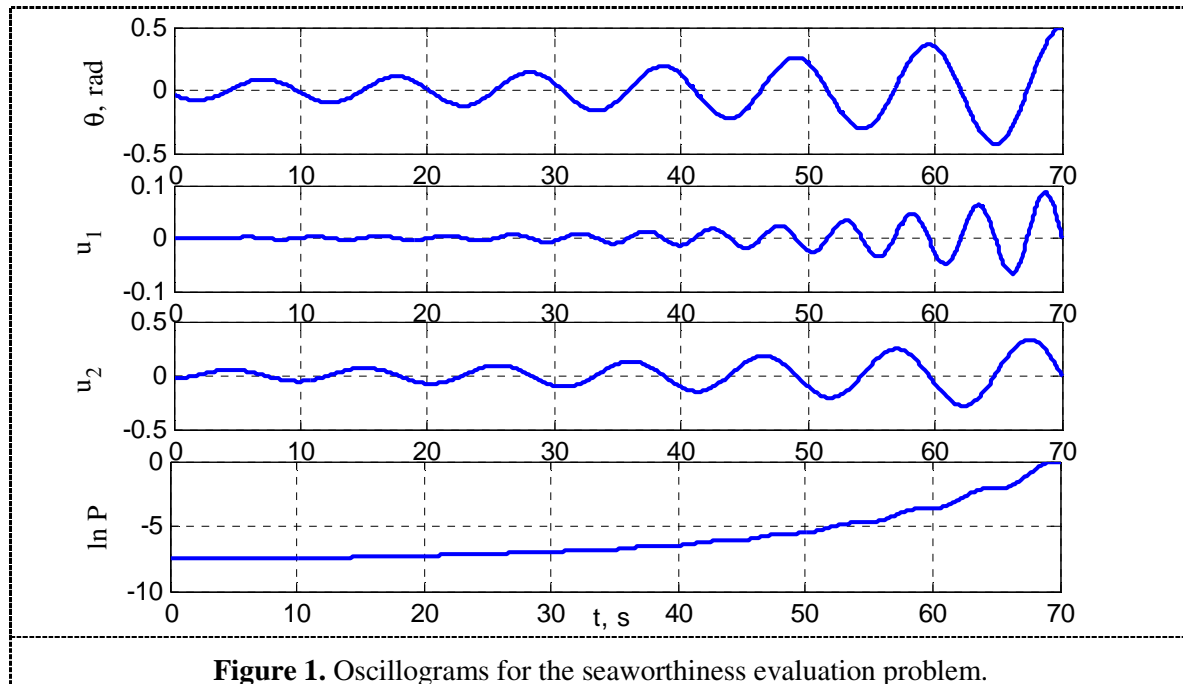
#### 5. Conclusion

The asymptotic approach allows minimizing sophisticated stochastic analysis - finding QE, which is also reduced for solving a deterministic problem, in this case – the LP problem.

An essential aspect of the method presented here is the presence of cognitive control relay pulse components: the input data are used not only for generating of synthesized feedback (local robust

synthesis) in advance (off-line), but also to make decisions about the current (on-line) quality of stabilization in the global sense, that is, in terms of the system state membership in a given domain of attraction of the equilibrium state (lattice and concepts of the scale).

Thus, the experience of object control embodied in these concepts of a quite compact knowledge base is perfectly combined with the usual methods of synthesis adopted in control theory.



**Figure 1.** Oscillograms for the seaworthiness evaluation problem.

## 6. Acknowledgments

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