IOP Conf. Series: Journal of Physics: Conf. Series **802** (2017) 012005 doi:10.1088/1742-6596/802/1/012005

The evolution of gauge couplings and the Weinberg angle in 5 dimensions for an SU(3) gauge group

Mohammed Omer Khojali[a,1], **A. S. Cornell**[a,2] **and Aldo Deandrea**[b,3]

[a] National Institute for Theoretical Physics; School of Physics and Mandelstam Institute for Theoretical Physics, University of the Witwatersrand, Wits 2050, South Africa
[b] Universite de Lyon, F-69622 Lyon, France; Universite Lyon 1, CNRS/IN2P3, UMR5822 IPNL, F-69622 Villeurbanne Cedex, France

E-mail: [1]khogali11@gmail.com, [2]alan.cornell@wits.ac.za, [3]deandrea@ipnl.in2p3.fr

Abstract. We test in a simplified 5-dimensional model with SU(3) gauge symmetry, the evolution equations of the gauge couplings of a model containing bulk fields, gauge fields and one pair of fermions. In this model we assume that the fermion doublet and two singlet fields are located at fixed points of the extra-dimension compactified on an S^1/Z_2 orbifold. The gauge coupling evolution is derived at one-loop in 5-dimensions, for the gauge group $G = SU(3)$, and used to test the impact on lower energy observables, in particular the Weinberg angle. The gauge bosons and the Higgs field arise from the gauge bosons in 5 dimensions, as in a gauge-Higgs model. The model is used as a testing ground as it is not a complete and realistic model for electroweak interactions.

1. Introduction

A gauge theory defined in more than four dimensions has many attractive features, where interactions at low energies may be truely unified and some of the distinct fields in four dimensions can be integrated as a single multiplet in higher dimensions, like in gauge-Higgs models, where the Higgs fields can be a component of 5-dimensional gauge fields. Note also that the topology and structure of the extra-dimensional space provides new ways of breaking symmetries [1]. The simplest theories of this type have problems in reproducing low energy observables, such as the Weinberg angle, the SM fermion content and where Yukawa couplings are different from the gauge couplings [2].

In this paper we shall discuss the gauge coupling evolution for a model which contains a pair of left-handed and right-handed fermion matter fields located at different fixed points. Our model shall contain a heavy bulk fermion, which has quantum numbers allowing couplings to both the boundary fermions. This matter field is introduced as a representation of the unified group $G = SU(3)$, where this gauge group is broken to a subgroup H. This unified $SU(3)$ group is the gauge symmetry of the electroweak interaction and a Higgs doublet field, and it is not the gauge group of the strong interaction [3].

Note that H (the subgroup of G) is not broken by the vacuum expectation value (vev) of the scalar fields (under which the vev of the scalar fields is invariant). So we can correspondingly divide the generators of G into two sets: the unbroken ones, that annihilate the vacuum, and the broken ones, the orthogonal set [4]. According to the Goldstone theorem each broken generator in the coset G/H is associated to an independent massless scalar (Goldstone bosons), carrying the same quantum numbers as the generators [5]:

$$G/H = \frac{SU(3)}{SU(2)\times U(1)} \Rightarrow \dim(G/H) = 8-(3+1) = 4. \tag{1}$$

In the case of the bulk fields, the standard Yukawa coupling can originate only from higher-dimensional gauge couplings, but in the case of the boundary localised matter fields the standard Yukawa coupling cannot be directly introduced [6]. The gauge bosons arise from the 4-dimensional components of the 5-dimensional gauge fields, whilst the Higgs field arises from the internal components of the gauge group $G = SU(3)$ compactified on an S^1/Z_2 orbifold; and it can be described as a circle of radius R where the opposite points can be identified by the action of the Z_2 orbifold [7]. This orbifold is given as $Z_2 : y \to -y$, so our physical space is the interval $y \in [0, \pi R]$ and it has two fixed points at $y = 0$ and $y = \pi R$. The generic 5-dimensional bulk fields, for example $A_\mu(x^\mu, y)$, have the following transformation property:

$$A_\mu(x^\mu, -y) \to PA_\mu(x^\mu, y). \tag{2}$$

It also has a $U(1)$ transformation, under the boundary condition of the fifth coordinate $y \to y + \pi R$ [8]:

$$A_\mu(x^\mu, y+\pi R) \to PA_\mu(x^\mu, y). \tag{3}$$

The $A_5(x^\mu, y)$ gauge field has the following transformation property:

$$A_5(x^\mu, y) \to PA_5(x^\mu, y)P^{-1} - iP^{-1}\partial_y P, \tag{4}$$

where our Z_2 orbifold boundary condition is given in the following way:

$$P = e^{i\vartheta^a(x^\mu,y)T^a}. \tag{5}$$

Here we are considering that the gauge transformation ϑ^a is linear in y.

Taking our gauge field to be along the third isospin direction, which means that the Z_2 orbifold boundary condition can then be written in the following way:

$$P = e^{i\pi\lambda_3} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{6}$$

where λ_a are the standard $SU(3)$ Gell-Mann matrices, normalised as $Tr(\lambda_a\lambda_b) = 2\delta_{ab}$. The group Z_2 acts on the tours as π rotations, the orbifold projection P breaking the gauge group G in 4-dimensions to the subgroup $H = SU(2)\times U(1)$ of the projection P. The massless 4-dimensional fields are the gauge bosons A^a_μ in the adjoint of H and the charged scalar doublet arises from the internal components A^a_5 of the gauge field [9].

The brane fields of the model we shall focus on consist of a left-handed fermion doublet $Q_L = (u_L, d_L)$, and two right-handed fermion singlets u_R and d_R. We are going to assume that the doublet and the two singlet fields are located respectively at positions y_1 and y_2, which equals to either 0 or πR. In this model we are neglecting the bulk to boundary couplings, because we are trying to control the Yukawa coupling terms, which are present in the model. We are also trying to simplify the model.

The Lagrangian for the bulk fields, gauge fields and the pair of fermions is given by:

$$\begin{aligned}
\mathcal{L}_{matter} &= \sum_a \Bigg[i\bar{\psi}_a(x,y) \not{D}_5 \psi_a(x,y) + i\bar{\tilde{\psi}}_a(x,y) \not{D}_5 \tilde{\psi}_a(x,y) + \bar{\psi}_a(x,y) M_a \tilde{\psi}_a(x,y) \\
&\quad + \bar{\tilde{\psi}}_a(x,y) M_a \psi_a(x,y) \Bigg] + \delta(y-y_1) \Bigg[i\tilde{Q}_L(x,y) \not{D}_\mu Q_L(x,y) \Bigg] \\
&\quad + \delta(y-y_2) \Bigg[i\bar{d}_R(x,y) \not{D}_\mu d_R(x,y) + i\bar{u}_R(x,y) \not{D}_\mu u_R(x,y) \Bigg], \qquad (7)
\end{aligned}$$

where $\not{D}_\mu$ and $\not{D}_5$ are the 4-dimensional and 5-dimensional covariant derivatives respectively, and are related by the following equality

$$\not{D}_5 = \not{D}_\mu + i\gamma_5 D_5. \qquad (8)$$

$$\not{D}_M = \gamma^M \partial_M - i\gamma^M g_M A^a_M T^a, \qquad (9)$$

where $M = (\ \mu$ or 5), the Hermitian matrix $\gamma_5 = i\gamma_\mu$, T^a are the generators of the Lie algebra of the gauge group G, A^a_μ are the 4-dimensional gauge bosons and the scalar fields A^a_5 are identified with the components of the Higgs field [10].

In the fundamental representation of the gauge group G, the mode expansion for the left-handed ψ_L and the right-handed ψ_R bulk fermion is

$$\psi_{aL}(y) = \sum_{-\infty}^{\infty} \eta_n \frac{1}{\sqrt{2\pi R}} \sin\left(\frac{ny}{R}\right) \psi^n_{aL}(x), \qquad (10)$$

$$\psi_{aR}(y) = \sum_{-\infty}^{\infty} \eta_n \frac{1}{\sqrt{2\pi R}} \cos\left(\frac{ny}{R}\right) \psi^n_{aR}(x). \qquad (11)$$

By adding equations (10) and (11) one can get the corresponding Fourier decomposition of a generic bulk fermion

$$\psi_a(y) = \sum_{n=-\infty}^{\infty} \eta_n \frac{1}{\sqrt{2\pi R}} \left[\sin\left(\frac{ny}{R}\right) \psi^n_{aL}(x) + \cos\left(\frac{ny}{R}\right) \psi^n_{aR}(x) \right], \qquad (12)$$

where the factor η_n is defined to be 1 for n $= 0$ and $1/\sqrt{2}$ for $n \neq 0$, which means we can rewrite the bulk fermion in equation (12) as

$$\psi_a(y) = \frac{1}{\sqrt{2\pi R}} \psi^0_{aR}(x) + \frac{1}{2\sqrt{\pi R}} \sum_{n=1}^{\infty} \left[\sin\left(\frac{ny}{R}\right) \psi^n_{aL}(x) + \cos\left(\frac{ny}{R}\right) \psi^n_{aR}(x) \right]. \qquad (13)$$

The 4-dimensional Lagrangian for the bulk fermion ψ_a is written as

$$\mathcal{L}^{\psi_a}_{4D} = \int_0^{\pi R} \left[\bar{\psi}_a(y) i \not{D}_5 \psi_a(y) \right] dy, \qquad (14)$$

where integrating out the y coordinate one can get

$$\begin{aligned}
\mathcal{L}_{4D}^{\psi_a} &= \frac{1}{2}\Bigg[\bar{\psi}_{aR}^{0}(x)(\gamma^\mu\partial_\mu - \gamma_\mu\partial_5)\psi_{aR}^{0}(x) + i\gamma_\mu g_5 T^a \bar{\psi}_{aR}^{0}(x)A_5^a\psi_{aR}^{0}(x) \\
&\quad -i\gamma^\mu g_\mu T^a\bar{\psi}_{aR}^{0}(x)A_\mu^a\psi_{aR}^{0}(x) + \frac{1}{4}\Bigg(\sum_{n=1}^{\infty}\bar{\psi}_{aL}^{n}(x)(\gamma^\mu\partial_\mu - \gamma_\mu\partial_5)\psi_{aL}^{n}(x) \\
&\quad +i\gamma_\mu g_5 T^a\sum_{n=1}^{\infty}\bar{\psi}_{aL}^{n}(x)A_5^a\psi_{aL}^{n}(x) - i\gamma^\mu g_\mu T^a\sum_{n=1}^{\infty}\bar{\psi}_{aL}^{n}(x)A_\mu^a\psi_{aL}^{n}(x) \\
&\quad +\sum_{n=1}^{\infty}\bar{\psi}_{aR}^{n}(x)(\gamma^\mu\partial_\mu - \gamma_\mu\partial_5)\psi_{aR}^{n}(x) + i\gamma_\mu g_5 T^a\sum_{n=1}^{\infty}\bar{\psi}_{aR}^{n}(x)A_5^a\psi_{aR}^{n}(x) \\
&\quad -i\gamma^\mu g_\mu T^a\sum_{n=1}^{\infty}\bar{\psi}_{aR}^{n}(x)A_\mu^a\psi_{aR}^{n}(x)\Bigg)\Bigg]. \qquad (15)
\end{aligned}$$

We can obtain the 4-dimensional Lagrangian for the bulk fermion, $\tilde{\psi}_a$, in similar way as in the case of the bulk fermion, ψ_a, by replacing ψ_a by $\tilde{\psi}_a$ in equation (15).

Now let us move to the case of the 4-dimensional left-handed fermion doublet, where the Fourier decomposition for that field is written as

$$Q_L(y) = \frac{1}{\sqrt{2\pi R}}Q_L^0(x) + \frac{1}{2\sqrt{\pi R}}\sum_{n=1}^{\infty}\left[\cos\left(\frac{ny}{R}\right)Q_L^n(x) + \sin\left(\frac{ny}{R}\right)Q_R^n(x)\right]. \qquad (16)$$

The 4-dimensional Lagrangian for the left-handed fermion doublet is given by

$$\mathcal{L}_{4D}^{Q_L} = \int_0^{\pi R} dy\delta(y-y_1)\left[\bar{Q}_L i\not{D}_\mu Q_L\right], \qquad (17)$$

where as we mentioned before, the $\delta(y-y_1)$ is needed as the left-handed fermion doublet is located at position y_1, which is equal to either 0 or πR. By integrating out the y coordinate one can get

$$\mathcal{L}_{4D}^{Q_L} = \frac{1}{4\pi R}\left[\bar{Q}_L^0(x)[i\gamma^\mu\partial_\mu + \gamma^\mu g_\mu A_\mu^a T^a]Q_L^0(x) + \frac{1}{2}\sum_{n=1}^{\infty}\bar{Q}_L^n(x)[i\gamma^\mu\partial_\mu + \gamma^\mu g_\mu A_\mu^a T^a]Q_L^n(x)\right]. \qquad (18)$$

Finally, we can see the case of the two singlet fields which are located at position y_2, the Fourier decomposition for those fields are written as

$$d_R(y) = \frac{1}{\sqrt{2\pi R}}d_R^0(x) + \frac{1}{2\sqrt{\pi R}}\sum_{n=1}^{\infty}\left[\cos\left(\frac{ny}{R}\right)d_R^n(x) + \sin\left(\frac{ny}{R}\right)d_L^n(x)\right], \qquad (19)$$

$$u_R(y) = \frac{1}{\sqrt{2\pi R}}u_R^0(x) + \frac{1}{2\sqrt{\pi R}}\sum_{n=1}^{\infty}\left[\cos\left(\frac{ny}{R}\right)u_R^n(x) + \sin\left(\frac{ny}{R}\right)u_L^n(x)\right]. \qquad (20)$$

The 4-dimensional Lagrangian for the two singlet fields d_R and u_R is written as

$$\mathcal{L}_{4D}^{singlet} = \int_0^{\pi R} dy\delta(y-y_2)\left[\bar{d}_R i\not{D}_\mu d_R + \bar{u}_R i\not{D}_\mu u_R\right], \qquad (21)$$

where by integrating out the y coordinate one can get

$$\begin{aligned}\mathcal{L}_{4D}^{singlet} &= \frac{1}{4\pi R}\Bigg[\bar{d}_R^0(x)[i\gamma^\mu\partial_\mu + \gamma^\mu g_\mu A_\mu^a T^a]d_R^0(x) + \bar{u}_R^0(x)[i\gamma^\mu\partial_\mu + \gamma^\mu g_\mu A_\mu^a T^a]u_R^0(x) \\ &+ \frac{1}{2}\sum_{n=1}^{\infty}\bar{d}_R^n(x)[i\gamma^\mu\partial_\mu + \gamma^\mu g_\mu A_\mu^a T^a]d_R^n(x) + \frac{1}{2}\sum_{n=1}^{\infty}\bar{u}_R^n(x)[i\gamma^\mu\partial_\mu + \gamma^\mu g_\mu A_\mu^a T^a]u_R^n(x)\Bigg].\end{aligned} \quad (22)$$

2. The gauge coupling evolution equations

Our goal is to discuss the gauge coupling evolution for the model presented in the previous section. In order to do so, we need to introduced the β-functions. This crucial object is needed to determine the evolution of the coupling constants. In general, in a theory with n-couplings g_i, we have to solve a set of coupled differential equations of the form

$$\beta_i = \mu\frac{dg_i}{d\mu} = \frac{dg_i}{dt}, \quad (23)$$

where $t = (\ln[\mu/M_Z])$. In general the β-functions depend on all the couplings and masses of the theory. We can get rid of the masses by focusing only on the universal UV relevant coefficients. For example, one can focus on the gauge coupling evolution equations, where we can write the general term for the gauge interaction of the fermions and the gauge bosons as $g\bar{\psi}\gamma^\mu\psi A_\mu$. In terms of renormalisable quantities (by rescaling)

$$\bar{\psi} = Z_{\bar{\psi}}^{1/2}\bar{\psi}^R, \quad (24)$$

$$\psi = Z_{\psi}^{1/2}\psi^R, \quad (25)$$

$$A_\mu = Z_{A_\mu}^{1/2}A_\mu^R, \quad (26)$$

where $Z_{\psi}^{1/2}$, $Z_{\bar{\psi}}^{1/2}$ and $Z_{A_\mu}^{1/2}$ are the renormalisation constants.

By using equations (24), (25) and (26) one can write the gauge interaction of the fermions and the gauge bosons in terms of the renormalisable quantities [11]

$$gZ_{\bar{\psi}}^{1/2}Z_{\psi}^{1/2}Z_{A_\mu}^{1/2}\bar{\psi}^R\gamma^\mu\psi^R A_\mu^R = Z_g^{1/2}g^R\bar{\psi}^R\gamma^\mu\psi^R A_\mu^R. \quad (27)$$

From the above equation one can see that

$$gZ_{\bar{\psi}}^{1/2}Z_{\psi}^{1/2}Z_{A_\mu}^{1/2} = Z_g^{1/2}g^R. \quad (28)$$

As we discussed earlier, the couplings g_i are determined by noticing that physics cannot depend on our arbitrary choice of scale μ. We have, therefore,

$$\frac{d\ln g^R}{dt} = \frac{1}{2}\frac{d\ln Z_{\bar{\psi}_R}}{dt} + \frac{1}{2}\frac{d\ln Z_{\psi_R}}{dt} + \frac{1}{2}\frac{d\ln Z_{A_\mu}}{dt} - \frac{1}{2}\frac{d\ln Z_g}{dt}. \quad (29)$$

We then need to calculate the renormalisation constants. When doing so, we usually ignore the mass terms in the propagators, since they have nothing to do with the divergent part of the one-loop diagrams. We are going to focus on the UV regime where we can neglect the m/μ dependence of β [11].

The general formula of the β-functions for the gauge couplings is given by [12]

$$16\pi^2 \frac{dg_i}{dt} = b_i^{SM} g_i^3 + (b_i + S(t)\tilde{b_i}) g_i^3, \tag{30}$$

where $t = \ln(S(t)/M_Z R)$, $S(t) = \mu R$ for $M_Z < \mu < \ln(1/M_Z R)$. The numerical coefficients appearing in equation (30) are given by:

$$b_i^{SM} = \left[\frac{41}{10}, -\frac{19}{6}, -7\right], \tag{31}$$

are the SM β-functions coefficients.

$$b_i = \left[\frac{10}{3}, -\frac{51}{16}, -\frac{20}{3}, \frac{3}{8}\right], \tag{32}$$

are the 0-mode β-functions coefficients.

$$\tilde{b_i} = \left[\frac{45}{16\pi}, -\frac{15}{16\pi}, -\frac{5}{\pi}, 0\right]. \tag{33}$$

are the n-mode β-functions coefficients.

3. Result and discussion

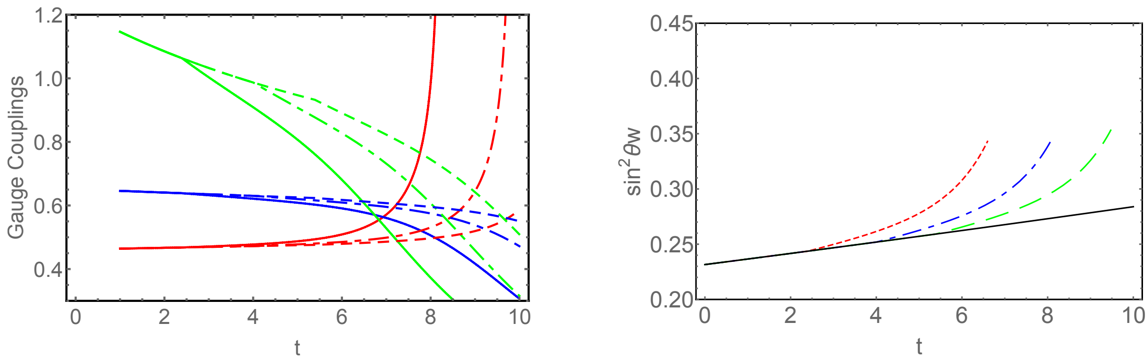

Figure 1. Left panel: Evolution of the gauge couplings g_1 (red), g_2 (blue) and g_3 (green), for three values of the $R^{-1} = 1$ TeV (solid line), 5 TeV (dot-dashed line), 20 TeV (dashed line) as a function of t. Right panel: Evolution of the Weinberg angle $\sin^2\theta_W$ with the bulk fermions, with the doublet located at position y_1 and two singlets located at position y_2, for $R^{-1} = 1$ TeV (red), $R^{-1} = 5$ TeV (blue) and $R^{-1} = 20$ TeV (green) as a function of t.

We have chosen the cut-off for our effective theory, these cut-offs are where $g_2 = g_3$, as shown in Table 1. In Figure 1, left panel, we present the evolution of the gauge couplings for the one-loop β-functions, by assuming that the bulk fermion is the top quark. We see that the three gauge couplings unify at some value of t. In the right panel we present the evolution of the Weinberg angle for the one-loop β-functions, for different values of compactification scales, for the model discussed in the previous section. When the fifth dimension contributions switch on there are large changes in the Weinberg angle up until we reach the cut-off. We can conclude that with this model the estimate of the Weinberg angle is closer to the group theoretically predicted Weinberg angle.

Scenario	t	R^{-1}	$\sin^2\theta_W$
5D $SU(3)$ gauge group	6.59	1TeV	0.335
5D $SU(3)$ gauge group	8.14	5TeV	0.345
5D $SU(3)$ gauge group	9.47	20TeV	0.348

Table 1. The cut-off and Weinberg angle for the model presented in the previous sections for the three different compactification radii $R^{-1} = 1$, 5 and 20 TeV, where $t = \ln(\mu/M_Z)$.

Acknowledgments
This work is supported by the National Research Foundation (South Africa).

References

[1] N. Haba, M. Harada, Y. Hosotani and Y. Kawamura, Nucl. Phys. B **657**, 169 (2003) [Nucl. Phys. B **669**, 381 (2003)] [hep-ph/0212035].
[2] C. Csaki, C. Grojean and H. Murayama, Phys. Rev. D **67**, 085012 (2003) [hep-ph/0210133].
[3] N. Kitazawa and Y. Sakai, Mod. Phys. Lett. A **31**, no. 07, 1650041 (2016) [arXiv:1509.03409 [hep-ph]].
[4] A. Romanino, The Standard model of particle physics, Proceedings, International Baikal Summer School, Bolshie Koty, Russia, July 23-30, 2009.
[5] Y. Nambu, Phys. Rev. 117 (1960) 648. J. Goldstone, Nuovo Cim. 19 (1961) 154. J. Goldstone, A. Salam and S. Weinberg, Phys. Rev. 127 (1962) 965.
[6] G. Burdman and Y. Nomura, Nucl. Phys. B **656**, 3 (2003) [hep-ph/0210257].
[7] I. Antoniadis, K. Benakli and M. Quiros, Phys. Lett. B **331**, 313 (1994) [hep-ph/9403290].
[8] L. J. Hall, H. Murayama and Y. Nomura, Nucl. Phys. B **645**, 85 (2002) [hep-th/0107245].
[9] C. A. Scrucca, M. Serone and L. Silvestrini, Nucl. Phys. B **669**, 128 (2003) [hep-ph/0304220].
[10] A. Masiero, C. A. Scrucca, M. Serone and L. Silvestrini, Phys. Rev. Lett. **87**, 251601 (2001) [hep-ph/0107201].
[11] M. Serone. Notes on Quantum Field Theory, Lecture notes from SISSA 2015.
[12] G. Bhattacharyya, A. Datta, S. K. Majee and A. Raychaudhuri, Nucl. Phys. B **760**, 117 (2007) [hep-ph/0608208].