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Towards NNLO accuracy for ε'/ε

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Abstract. The quantity ε'/ε measures direct CP violation in Kaon decays. Recent SM predictions show a 2.9σ tension with data, with the theoretical uncertainty dominating. As rapid progress on the lattice is bringing nonperturbative long-distance effects under control, a more precise knowledge of short-distance contributions is needed. We describe the first NNLO results for ε'/ε and discuss future prospects, as well as issues of scheme dependence and the separation of perturbative and nonperturbative effects. Finally we also comment on the solution of the renormalisation-group evolution in one of the talks at this conference and present the correct solution.

1. Introduction

CP violation (CPV) is one of the most fascinating phenomena of high energy physics and is a natural place to search for physics beyond the Standard Model (SM). For instance, the SM is unable to account for the observed matter-antimatter asymmetry in the universe, which, in thermal baryogenesis scenarios, requires CPV. New sources of CPV, however, generally modify the SM predictions for CPV in flavour-violating decays. In the case of CPV in $K_L \to \pi\pi$, the SM predicts a particularly strong suppression due to mass and CKM hierarchies, which imply powerful Glashow-Iliopoulos-Maiani (GIM) cancellations. This mechanism need not apply to models of new physics. Therefore kaons are particularly promising for shedding light on physics at a more fundamental level. High-precision CP-violating kaon observables offer the exciting possibility of establishing the presence of new physics (NP).

While CPV in K_L decays was discovered 50 years ago, and direct CPV—quantified by the parameter ε'—has been firmly established for more than a decade, the theoretical treatment of the latter suffered from nonperturbative uncertainties which until very recently could not be computed in a controlled approximation. While short-distance contributions from scales $\sim m_c$ and above have been known to next-to-leading-order in perturbation theory since the mid-1990s [1–6], the long-distance corrections, usually represented as $K \to \pi\pi$ matrix elements of local four-quark operators, have been for the first time determined with controlled systematics only in 2015 [7,8]. This achievement, when combined with the short-distance calculations, opens the possibility of a precision theory prediction of ε'. A state-of-the-art analysis at NLO within the SM [9] gives

$$\left(\frac{\varepsilon'}{\varepsilon}\right)_{\text{SM}} = (1.9 \pm 4.5) \times 10^{-4}. \tag{1}$$

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(The complex phase is very close to zero.) This is to be contrasted with the current experimental world average based on measurements at NA48 [10] and KTeV [11, 12] of

$$\mathrm{Re}\left(\frac{\varepsilon'}{\varepsilon}\right)_{\mathrm{exp}} = (16.6 \pm 2.3) \times 10^{-4}. \tag{2}$$

The 2.9σ difference between experiment and theory could have one of several origins, the most exciting among them being a possible contribution from new particles beyond the SM. Clearly, a refined SM prediction is necessary. While at the moment the error budget [9] is dominated by the uncertainty on a single non-perturbative matrix element, this error is expected to shrink in the near future with further progress in lattice QCD. The next most important source of uncertainty at present is due to missing higher-order perturbative contributions, making NNLO accuracy imperative. In this proceedings, we preview (partial) results of an ongoing NNLO calculation [13], which roughly cuts the (total) perturbative error in half. We further describe a renormalisation-group-invariant formalism for separating perturbative and non-perturbative effects and briefly discuss future perspectives. Finally, we comment on and correct the expressions for the mixed QCD-QED renormalisation-group evolution [14] that were presented in another talk at this conference.

2. $\Delta S = 1$ Effective Hamiltonian

The key tool to disentangle the physics of the different scales ($\Lambda_{\mathrm{QCD}}, m_c, m_b, M_W$ and possibly M_{NP}) is the weak effective $\Delta S = 1$ Hamiltonian [1–6]. It takes the general form

$$\mathcal{H}_{\mathrm{eff}} = \sum C_i(\mu) Q_i(\mu), \tag{3}$$

where the process-independent Wilson coefficients $C_i(\mu)$ contain the physics of scales (virtualities) above μ (μ^2), while the degrees of freedom below μ are encoded in process-dependent hadronic matrix elements of local operators Q_i, in the present case $\langle\pi\pi|Q_i(\mu)|K\rangle$. These hadronic matrix elements are non-perturbative, and the lack of systematic predictions for them has precluded controlled predictions of ε' for a long time. This bottleneck has begun to be removed by recent developments in lattice QCD, in particular by the RBC and UKQCD collaborations [7,8].

Several choices of the scale μ are possible. So far, the state of the art [9] consisted of evaluating the Wilson coefficients in a three-flavour ($n_f = 3$) theory, obtained by integrating out the weak scale, as well as the bottom and charm quarks. Hadronic matrix elements with fully quantified systematic errors are available in the $n_f = 3$ theory from RBC and UKQCD [7,8] and agree well with earlier calculations in realistic models [15–17]. In the future, nonperturbative calculations with dynamical charm quark will likely become available, allowing to work based on a weak Hamiltonian in a theory with $n_f = 4$ quark flavours. It is convenient to split the Wilson coefficients as $C_i = \frac{4G_F}{\sqrt{2}} V_{ud} V_{us}^* (z_i + \tau y_i)$, where z_i and y_i are real and $\tau = -(V_{td} V_{ts}^*)/(V_{ud} V_{us}^*)$ is the phase-convention-independent CP-violating CKM combination governing CPV in K decays. For $n_f = 4$ (dynamical charm quark), we then have[1]

$$\mathcal{H}_{\mathrm{eff}} = \frac{4G_F}{\sqrt{2}} V_{ud} V_{us}^* \left(\sum_{i=1}^{2} z_i(\mu)[Q_i^u - Q_i^c] + \tau \left[\sum_{i=1}^{2} z_i(\mu) Q_i^c + \sum_{i=3..10,7\gamma,8g} y_i(\mu) Q_i \right] \right), \tag{4}$$

where we have made use of the fact that $z_{1,2}^c = y_{1,2}^c = -z_{1,2}$, while GIM cancellations ensure $z_{3...10} = 0$.

---

[1] Using the projection operators P_L and P_R instead of the traditional $V \pm A \times V \pm A$ introduces a factor of 4 in the overall normalisation of the Hamiltonian.

If the charm quark is integrated out ($n_f = 3$), we have instead

$$\mathcal{H}_{\text{eff}} = \frac{4G_F}{\sqrt{2}} V_{ud} V_{us}^* \left(\sum_{i=1}^{2} z_i(\mu) Q_i^u + \sum_{i=3\ldots10,7\gamma,8g} z_i(\mu) Q_i + \sum_{i=3\ldots10,7\gamma,8g} \tau y_i(\mu) Q_i \right). \qquad (5)$$

The operators $Q_{1,2}^c$ involving the charm quark have disappeared. The hard GIM breaking in the $n_f = 3$ theory now entails nonzero values also for the "real parts" $z_{3\ldots10,7\gamma,8g}$ of $C_{3\ldots10,7\gamma,8g}$.

The Wilson coefficients $C_i(\mu)$, or equivalently $z_i(\mu)$ and $y_i(\mu)$, are determined using renormalisation-group-improved perturbation theory. Schematically,

$$C_i(\mu_3) = \left[U(\mu_3, \mu_c) M^c(\mu_c) U(\mu_c, \mu_b) M^b(\mu_b) U(\mu_b, \mu_W) \right]_{ij} C_j(\mu_W), \qquad (6)$$

which is easily decomposed into expressions for z_i and y_i. The evolution operators $U(\mu_1, \mu_2)$ resum large logarithms $\ln \mu_2/\mu_1$, while the matching matrices $M(\mu)$ connect the Wilson coefficients across a flavour threshold, in particular:

$$C_i^{(n_f=3)}(\mu_c) = M_{ij}^c(\mu_c) C_j^{(n_f=4)}(\mu_c). \qquad (7)$$

The operators can be classified as current-current ($Q_{1,2}^u$, $Q_{1,2}^c$), QCD penguin ($i = 3\ldots6, 8g$), and electroweak penguin ($i = 7\ldots10, 7\gamma$). At present, all ingredients in (6) are known to NLO accuracy (with the exception of the dipole operators Q_{8g} and $Q_{7\gamma}$), and some of them to NNLO accuracy. The work described here completes the QCD-penguin part to NNLO, which removes about half of the current perturbative uncertainty on ε'.

3. NNLO matching for the QCD penguins

To obtain the QCD-penguin Wilson coefficients to NNLO accuracy, we require the submatrices of M^c and M^b for $Q_{1\ldots6,8g}$ and $Q_{1,2}^c$. We leave aside the dipole operator for the time being, which is formally suppressed by m_s/Λ_{QCD}. The new ingredients are then related to the removal of the operators Q_1^c and Q_2^c at a scale $\mu_c \sim m_c$, although we have recalculated the remaining, known [18] elements of the matching matrices M^c, M^b. The anomalous dimension matrices are known from [19].

To determine these threshold corrections, we match Green's functions with operator insertions in the n_f- and $n_f + 1$-flavour theories at the matching scale $\mu_i = \mathcal{O}(m_i)$ ($m_i = m_c, m_b$).

Novel effects at NNLO include current-current operator insertions getting contributions from virtual charm/bottom-quarks and the strong coupling constant being discontinuous at the threshold scale. In our calculation we use dimensional regularization and avoid the appearance of traces over γ_5 by employing the so-called "modern basis" [20, 21]; however this requires extra steps to connect with the "traditional" basis employed in the existing literature on ε', and in the lattice-QCD calculations, which is nontrivial at NNLO. We expand in the external momenta (as appropriate for a matching onto dimension-six operators) and set the masses of the n_f light quarks to zero. After renormalisation, the $n_f + 1$-flavour result still contains infrared divergences in the form of poles in $\epsilon = (4 - d)/2$, which have to be reproduced in the n_f-flavour theory. As the Green's functions in the n_f-flavour theory contain only massless tadpole loop diagrams (after expansion in the external momenta), they are given entirely in terms of the ultraviolet counterterms in the n_f-flavour theory, which are related to known anomalous dimensions. The cancellation of divergences constitutes an important check of our calculation. In the end, the matching results in finite threshold corrections for the Wilson coefficients

$$C_i^{(n_f)}(\mu) = \left[M_{n_f, n_f+1}(\mu) \right]_{ij} C_j^{(n_f+1)}(\mu). \qquad (8)$$

4. Scale and scheme independence

Physical observables cannot depend on the dimensional renormalisation scale μ. This means that the μ-dependence has to cancel between the Wilson coefficients $C_i(\mu)$ and matrix elements $\langle Q_i(\mu)\rangle$ order by order in perturbation theory. However, in the present case the matrix elements are found from non-perturbative calculations and their explicit μ-dependence is not known. Instead, we can verify explicitly that our Wilson coefficients $C_i(\mu)$ satisfy the NNLO renormalisation-group equations up to $\mathcal{O}(\alpha_s^3)$ remainder terms, which they do. However, this still does not allow to give an estimate of the impact of missing higher-order perturbative corrections, part of which are contained in the published lattice-QCD matrix elements and inseparable from the non-perturbative uncertainties from dynamics at scales $\sim \Lambda_{\mathrm{QCD}}$.

To allow for a full separation of scales, we note that the evolution operator $U(\mu_1, \mu_2)$ in fact factorizes,

$$U(\mu_1, \mu_2) = J(\mu_1)U^{(0)}(\mu_1)\left(U^{(0)}(\mu_2)\right)^{-1}J^{-1}(\mu_2), \tag{9}$$

where

$$U^{(0)}(\mu) = \exp\left(-\frac{\gamma^{0T}}{2\beta_0}\ln\alpha_s(\mu)\right), \tag{10}$$

$$J(\mu) = \hat{1} + \frac{\alpha_s(\mu)}{4\pi}J^{(1)} + \left(\frac{\alpha_s(\mu)}{4\pi}\right)^2 J^{(2)} + \ldots. \tag{11}$$

$U^{(0)}(\mu)$ is just the leading-order evolution operator, with the usual ratio $\alpha_s(\mu_2)/\alpha_s(\mu_1)$ replaced by $1/\alpha_s(\mu)$.

We can then define the following object appearing in the intermediate stages in our calculation:

$$\hat{M}(\mu_i) = \left(U_{n_f}^{(0)}(\mu_i)\right)^{-1}\left(J_{n_f}(\mu_i)\right)^{-1}M_{n_f,n_f+1}(\mu_i)J_{n_f+1}(\mu_i)U_{n_f+1}^{(0)}(\mu_i). \tag{12}$$

Note that the matrix $\hat{M}(\mu_i)$ contains *all* the dependence on the scale μ_i in the expression (6) for the Wilson coefficient, and depends on no other scales. As a consequence, it must by itself be independent of the renormalisation scale. The way our calculation is set up, we obtain the matching matrix with dependence on both $\alpha_s^{(n_f)}(\mu)$ and on $\alpha_s^{(n_f+1)}(\mu)$. Expressing $\alpha_s^{(n_f+1)}$ in terms of $\alpha_s^{(n_f)}$ and taking into account both the explicit μ-dependence and the μ-dependence of the quark mass appearing in the matching, we observe that the scale dependence cancels, up to terms of order $\mathcal{O}(\alpha_s^3)$. The residual μ-dependence, which is of order α_s^3, gives a complete estimate of higher-order effects at the threshold μ_i. We find that these residual scale dependences are at the percent level and that the value of ε'/ε is only mildly shifted when going from NLO to NNLO. This points to an excellent behaviour of the perturbation series not only at the scale μ_b, but also at the scale μ_c. This means at least for the dominant contribution the perturbation theory seems to work.

Our results are best expressed in terms of renormalisation-group and scheme-independent Wilson coefficients and matrix elements,

$$\hat{C}_i^{(n_f)} = \left(U_{n_f}^{(0)}(\mu)\right)_{ik}^{-1}\left(J_{n_f}(\mu)\right)_{kj}^{-1}C_j^{(n_f)}(\mu), \tag{13}$$

$$\langle\pi\pi|\hat{Q}_i^{(n_f)}|K\rangle = \langle\pi\pi|Q_j^{(n_f)}(\mu)|K\rangle\left(J_{n_f}(\mu)\right)_{jk}\left(U_{n_f}^{(0)}(\mu)\right)_{ki}. \tag{14}$$

Note that we have not suppressed any μ-dependence: each hatted object is separately scale- and scheme-independent[2]. For this reason, relating the "modern" basis to the "traditional" basis is

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[2] The explicit NNLO transformations of J, C and $\langle Q\rangle$ under a change of scheme and the resulting scheme cancellation can be found in Ref. [19].

trivial at the level of the hatted objects. Perturbative uncertainties are fully contained in the hatted coefficients, while non-perturbative uncertainties reside in the hatted matrix elements. The two matching steps together with RG evolution are then expressed as

$$\hat{C}_i^{(3)} = \hat{M}_{ik}^c \hat{M}_{kj}^b \hat{C}_j^{(5)}. \tag{15}$$

For dynamical charm, one simply removes the last matching step.

In fact, the hatted hadronic matrix elements could in principle be obtained nonperturbatively on the lattice, obviating or reducing the need to resort to technically demanding lattice-continuum matching. If this is not done, the resultant extra "perturbative" uncertainty may be considered as part of the lattice error budget.

5. ε'/ε within the Standard Model

Lattice computations of the relevant matrix elements for the analysis of ε'/ε are currently performed only in the isospin limit ($\alpha = 0$, degenerate masses), and the Standard-Model prediction is based on the following expression

$$\frac{\varepsilon'}{\varepsilon} = -i\frac{\omega_+}{\sqrt{2}|\varepsilon_K|}e^{i(\delta_2 - \delta_0 - \phi_{\varepsilon_K})}\left[\frac{\mathrm{Im}(\mathcal{A}_0)}{\mathrm{Re}(\mathcal{A}_0)}(1 - \Omega_{\mathrm{eff}}) - \frac{1}{a}\frac{\mathrm{Im}(\mathcal{A}_2)}{\mathrm{Re}(\mathcal{A}_2)}\right], \tag{16}$$

which is accurate up to $\mathcal{O}(1/\omega_+)$ corrections. Here $\mathcal{A}_I \equiv \langle(\pi\pi)_I|\mathcal{H}_{\mathrm{eff}}|K\rangle$ ($I = 0, 2$) are the amplitudes for the two isospin states with their strong phases $\delta_{0,2}$ removed. The latter, as well as the phase ϕ_{ε_K} and magnitude $|\varepsilon_K|$ of ε_K and the isospin ratio ω_+ are all determined from experimental data. The coefficients a and Ω_{eff} are a (partial) parameterization of corrections to the isospin limit, with values computed in chiral perturbation theory in [22–24].

The evaluation of the ratios $\mathrm{Im}(\mathcal{A}_I)/\mathrm{Re}(\mathcal{A}_I)$ is the central ingredient to the theory prediction of ε'/ε and the formalism used for their determination is explained in reference [9]. The quantities

$$\mathrm{Re}(\mathcal{A}_0) \approx \frac{4G_F}{\sqrt{2}}V_{ud}V_{us}^*(z_+\langle Q_+\rangle_0 + z_-\langle Q_-\rangle_0), \quad \mathrm{Re}(\mathcal{A}_2) \approx \frac{4G_F}{\sqrt{2}}V_{ud}V_{us}^*z_+\langle Q_+\rangle_2 \tag{17}$$

are completely dominated by the two operators $Q_{1,2}^u$. ($Q_\pm = \frac{1}{2}(Q_2^u \pm Q_1^u)$.) Operator identities among the $V - A \times V - A$ operators imply that the ratios

$$\left(\frac{\mathrm{Im}(\mathcal{A}_2)}{\mathrm{Re}(\mathcal{A}_2)}\right)_{V-A} = \mathrm{Im}(\tau)\frac{3(y_9 + y_{10})}{2z_+} \tag{18}$$

$$\left(\frac{\mathrm{Im}(\mathcal{A}_0)}{\mathrm{Re}(\mathcal{A}_0)}\right)_{V-A} = \mathrm{Im}(\tau)\frac{2y_4}{(1 + q)z_-} + \mathcal{O}(p_3), \tag{19}$$

where in the numerators only the terms involving the $V - A \times V - A$ operators are kept, are almost free from hadronic uncertainties. The factor q in Eq.(19) is given in terms of the current-current Wilson coefficients and operators, $q \equiv (z_+(\mu)\langle Q_+(\mu)\rangle_0)/(z_-(\mu)\langle Q_-(\mu)\rangle_0)$. The small ratio z_+/z_- implies that only a modest accuracy is needed on the hadronic matrix elements entering the isospin-0 ratio through q. The term $\mathcal{O}(p_3)$ also involves a small Wilson coefficient, which moreover multiplies a colour-suppressed hadronic matrix element.

Conversely, the contributions coming from the $(V - A) \times (V + A)$ operators are very sensitive to long-distance effects (hadronic matrix elements). To minimize the nonperturbative uncertainties, one notes that $\mathrm{Re}(\mathcal{A}_{0,2})$ govern CP-conserving $K \to \pi\pi$ decays and are well determined by data. One then has

$$\left(\frac{\mathrm{Im}(\mathcal{A}_0)}{\mathrm{Re}(\mathcal{A}_0)}\right)_{V+A} = -\frac{4G_F}{\sqrt{2}}V_{ud}V_{us}^*\,\mathrm{Im}(\tau)\,y_6\frac{\langle Q_6\rangle_0}{\mathrm{Re}(\mathcal{A}_0)} + \mathcal{O}(p_5) \tag{20}$$

$$\left(\frac{\mathrm{Im}(\mathcal{A}_2)}{\mathrm{Re}(\mathcal{A}_2)}\right)_{V+A} = -\frac{4G_F}{\sqrt{2}}V_{ud}V_{us}^*\,\mathrm{Im}(\tau)\,y_8^{\mathrm{eff}}\frac{\langle Q_8\rangle_2}{\mathrm{Re}(\mathcal{A}_2)}. \tag{21}$$

Again, the omitted terms are suppressed by small Wilson coefficients and/or colour-suppressed operator matrix elements. As a result, the prediction for ε' involves predominantly two hadronic matrix elements (often parameterised in terms of parameters $B_6^{(1/2)}$ and $B_8^{(3/2)}$), as well as perturbative Wilson coefficients $z_{1,2}$ and $y_{6,8}$. Our new calculation essentiall removes the perturbative uncertainty on y_6. The uncertainties on $z_{1,2}$ are already tiny, leaving y_8 and an improved treatment of isospin-breaking corrections as the main objectives for the future.

Note that using $\mathrm{Re}(\mathcal{A}_{0,2})$ from data also in the $V-A \times V-A$ operators introduces dependence on (mainly) the matrix element of the operator Q_4, and its Wilson coefficient, and should be avoided. This is the main reason why the prediction of [9] is more accurate than that of RBC and UKQCD, and leads to a more pronounced tension with the data, in spite of employing the same nonperturbative matrix elements and including an error estimate for isospin-breaking corrections.

6. Summary and outlook

The experimental data for ε'/ε is in mild tension with the SM. This conclusion is possible because of recent progress in lattice QCD, which has reduced the (still dominant) uncertainties on long-distance non-perturbative contributions. With further progress expected in the near future, currently subleading uncertainties, most importantly higher-order perturbative short-distance contributions, will start to dominate. We have described a calculation at NNLO accuracy of the short-distance contributions to the QCD penguin amplitudes, sufficient to achieve NNLO accuracy on the isospin-zero $K \to \pi\pi$ decay amplitude ratio. Our results leave the central value of the theoretical prediction nearly unchanged, while greatly reducing the perturbative uncertainty on this piece, leaving unknown NNLO corrections to the $I=2$ amplitude ratio to dominate the perturbative uncertainty on ε', at about half the previous NLO error.

Several further steps will be needed before the theoretical uncertainty matches the experimental one, and perhaps a discrepancy with the SM can be confirmed. One of them is the computation of the $I=2$ amplitude ratio to NNLO accuracy. Another is a better treatment of isospin-breaking corrections, and electromagnetic corrections in particular. The latter affect both the Wilson coefficients and the hadronic matrix elements, and require both perturbative and lattice calculations, as well as a consistent interfacing of the two.

Comment on the solution of the renormalisation group evolution

At this conference one of the authors of Ref. [14] presented a solution to the renormalisation group equation

$$\mu \frac{d}{d\mu} C = \gamma^T C \tag{22}$$

for the Wilson coefficients C. Here

$$\gamma = \sum_{m+n \geq 1} \gamma^{(m,n)} \left(\frac{\alpha_s}{4\pi}\right)^m \left(\frac{\alpha_e}{4\pi}\right)^n \tag{23}$$

represent the anomalous dimensions relevant for ε'/ε. While the presented solution incorporates some corrections of $\mathcal{O}(\alpha_e^2/\alpha_s^2)$ not all effects of this order have been take into account in this work. In the following we will give the correct solution for the evolution matrix (9) in the case of combined QCD and QED corrections, where

$$
J(\mu) = \left[1 + \frac{\alpha_e}{4\pi} \left(J_0^{(1,1)} + J_1^{(1,1)} \ln \alpha_s + J_2^{(1,1)} (\ln \alpha_s)^2 \right) \right] \left[1 + \frac{\alpha_s}{4\pi} \left(J_0^{(1,0)} + J_1^{(1,0)} \ln \alpha_s \right) \right] \times
$$
$$
\left[1 + \frac{\alpha_e}{\alpha_s} \left(J_0^{(0,1)} + J_1^{(0,1)} \ln \alpha_s \right) + \frac{\alpha_e^2}{\alpha_s^2} \left(J_0^{(0,2)} + J_1^{(0,2)} \ln \alpha_s \right) \right]
\tag{24}
$$

follows the form chosen in Ref. [14]. To determine a differential equation for $J(\mu)$ we use the scale invariance $\mu(d/d\mu)\hat{C} = 0$ of the Wilson coefficients defined in (13) and trade the scale dependence in the total derivative for an α_s dependence as

$$
\mu \frac{d}{d\mu} = -2\alpha_s^2 \frac{\beta_s}{4\pi} \left(\frac{\partial}{\partial \alpha_s} - \frac{\alpha_e^2 \beta_e}{\alpha_s^2 \beta_s} \frac{\partial}{\partial \alpha_e} \right) .
\tag{25}
$$

Here the implicit μ dependence in α_e is now understood as $\alpha_e(\mu) = \alpha_e(\alpha_s(\mu))$ and β_s and β_e are defined via the QCD $\mu(d/d\mu)\alpha_s = -2\alpha_s^2\beta_s/(4\pi)$ and QED $\mu(d/d\mu)\alpha_e = +2\alpha_e^2\beta_e/(4\pi)$ beta functions. The resulting equation for J then reads:

$$
\frac{\partial J}{\partial \alpha_s} = \frac{\alpha_e^2 \beta_e}{\alpha_s^2 \beta_s} \frac{\partial J}{\partial \alpha_e} - \frac{4\pi \gamma^T J}{2\alpha_s^2 \beta_s} + \frac{J\gamma^{(1,0)^T}}{2\alpha_s \beta_s^{(0,0)}}
\tag{26}
$$

and which implies a set of simple algebraic equations for the matrices $J_l^{(i,j)}$ which read

$$
0 = \frac{\left[J_1^{(1,0)}, \gamma^{(1,0)^T} \right]}{2\beta_s^{(0,0)}} - J_1^{(1,0)}
$$

$$
0 = \frac{\left[J_0^{(1,0)}, \gamma^{(1,0)^T} \right]}{2\beta_s^{(0,0)}} + \frac{\beta_s^{(1,0)}\gamma^{(1,0)^T}}{2\beta_s^{(0,0)2}} - \frac{\gamma^{(2,0)^T}}{2\beta_s^{(0,0)}} - J_0^{(1,0)} - J_1^{(1,0)}
$$

$$
0 = \frac{\left[J_1^{(0,1)}, \gamma^{(1,0)^T} \right]}{2\beta_s^{(0,0)}} + J_1^{(0,1)}
$$

$$
0 = \frac{\left[J_0^{(0,1)}, \gamma^{(1,0)^T} \right]}{2\beta_s^{(0,0)}} - \frac{\gamma^{(0,1)^T}}{2\beta_s^{(0,0)}} + J_0^{(0,1)} - J_1^{(0,1)}
$$

$$
0 = \frac{\left[J_2^{(1,1)}, \gamma^{(1,0)^T} \right]}{2\beta_s^{(0,0)}}
\tag{27}
$$

$$
0 = \frac{\left[J_1^{(1,0)}, \gamma^{(0,1)^T} \right]}{2\beta_s^{(0,0)}} + \frac{\left[J_1^{(1,1)}, \gamma^{(1,0)^T} \right]}{2\beta_s^{(0,0)}} - 2J_2^{(1,1)}
$$

$$
0 = \frac{\left[J_0^{(1,0)}, \gamma^{(0,1)^T} \right]}{2\beta_s^{(0,0)}} + \frac{\left[J_0^{(1,1)}, \gamma^{(1,0)^T} \right]}{2\beta_s^{(0,0)}} + \frac{\beta_s^{(1,0)}\gamma^{(0,1)^T}}{2\beta_s^{(0,0)2}} + \frac{\beta_s^{(0,1)}\gamma^{(1,0)^T}}{2\beta_s^{(1,0)2}} - \frac{\gamma^{(1,1)^T}}{2\beta_s^{(0,0)}} - J_1^{(1,1)}
$$

$$
0 = \frac{\left[J_1^{(0,2)}, \gamma^{(1,0)^T} \right]}{2\beta_s^{(0,0)}} - \frac{\gamma^{(0,1)^T} J_1^{(0,1)}}{2\beta_s^{(0,0)}} + \frac{J_1^{(0,1)}\beta_e^{(0,0)}}{\beta_s^{(0,0)}} + 2J_1^{(0,2)}
$$

$$
0 = \frac{\left[J_0^{(0,2)}, \gamma^{(1,0)^T} \right]}{2\beta_s^{(0,0)}} - \frac{\gamma^{(0,1)^T} J_0^{(0,1)}}{2\beta_s^{(0,0)}} + \frac{J_0^{(0,1)}\beta_e^{(0,0)}}{\beta_s^{(0,0)}} + 2J_0^{(0,2)} - J_1^{(0,2)} .
$$

Here the expansion coefficients $\beta_s^{(n,m)}$ and $\beta_e^{(n,m)}$ of the beta functions are defined via

$$\beta_s = \sum_{n,m\geq 0} \beta_s^{(m,n)} \left(\frac{\alpha_s}{4\pi}\right)^m \left(\frac{\alpha_e}{4\pi}\right)^n \quad \text{and} \quad \beta_e = \sum_{n,m\geq 0} \beta_e^{(m,n)} \left(\frac{\alpha_e}{4\pi}\right)^m \left(\frac{\alpha_s}{4\pi}\right)^n . \tag{28}$$

In the original version of Ref. [14] terms proportional to $\beta_e^{(0,0)}$ and $\beta_s^{(0,1)}$ were absent. Our results agree with their revised version.

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