

Cosmology in nonlinear multidimensional gravity and the Casimir effect

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Abstract. We study the possible cosmological models in Kaluza-Klein-type multidimensional gravity with a curvature-nonlinear Lagrangian and a spherical extra space, taking into account the Casimir energy. First, we find a minimum of the effective potential of extra dimensions, leading to a physically reasonable value of the effective cosmological constant in our 4D space-time. In this model, the huge Casimir energy density is compensated by a fine-tuned contribution of the curvature-nonlinear terms in the original action. Second, we present a viable model with slowly evolving extra dimensions and power-law inflation in our space-time. In both models, the results formulated in Einstein and Jordan frames are compared.

1. Introduction. Basic equations

The idea of extra dimensions is now firmly established in theoretical physics in many contexts, such as Kaluza-Klein theories, supergravity, strings and M-theory, brane-world theories. This idea provides a powerful methodological framework for many crucial problems: geometric unification of physical interactions, the hierarchy problem, possible variations of fundamental constants as well as searches for realistic cosmological scenarios including inflationary, string or brane backgrounds, see, e.g., [1, 2] and references therein.

One of the important problems in multidimensional models is inclusion of quantum vacuum effects due to the compact topology of extra dimensions (the Casimir effect [3–5]).

In this paper, in the framework of a Kaluza-Klein-type theory including quadratic curvature invariants, we try to construct viable cosmological models taking into account the Casimir energy of massless fields. For simplicity we restrict ourselves to vacuum models with the geometry $\mathbb{M}^4 \times \mathbb{S}^n$, where \mathbb{S}^n is an n -dimensional sphere of sufficiently small radius. We thus consider a $(D = 4 + n)$ -dimensional manifold with the metric

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu - e^{2\beta(x^\mu)} d\Omega_n^2 \quad (1)$$

where x^μ are the observable four space-time coordinates, and $d\Omega_n^2$ is the metric on a unit sphere \mathbb{S}^n . In this space-time, we consider a curvature-nonlinear theory of gravity with the action

$$S = \frac{1}{2} m_D^{D-2} \int \sqrt{g_D} d^D x (F(R) + c_1 R^{AB} R_{AB} + c_2 R^{ABCD} R_{ABCD} + L_m), \quad (2)$$



where capital Latin indices cover all D coordinates, $g_D = |\det(g_{MN})|$, $F(R)$ is a smooth function of D -dimensional scalar curvature R , c_1, c_2 are constants, L_m is a matter Lagrangian, and $m_D = 1/r_0$ is the D -dimensional Planck mass, thus r_0 is a fundamental length in this theory.

We use the system of units with $c = \hbar = 1$. As L_m , we consider the Casimir energy density in the geometry (1). Our goal is to find viable models describing the present-day Universe. To this end, following the methodology of [1, 6, 7], we simplify the problem as follows:

(a) Integrate over \mathbb{S}^n , reducing all quantities to 4D variables and $\beta(x^\mu)$; thus we have

$$R = R_4 + \phi + f_1, \quad f_1 = 2n\Box\beta + n(n+1)(\partial\beta)^2, \quad (3)$$

where $R_4 = R_4[g_{\mu\nu}]$ is the 4D scalar curvature, $\Box = \nabla^\mu \nabla_\mu$ is the 4D d'Alembert operator, $(\partial\beta)^2 = g^{\mu\nu} \partial_\mu \beta \partial_\nu \beta$, and the effective scalar field is equal to the Ricci scalar of \mathbb{S}^n :

$$\phi(x^\mu) = m_D^2 n(n-1) e^{-2\beta(x^\mu)}. \quad (4)$$

(b) Suppose slow variations of all quantities as compared with the D -dimensional Planck scale, i.e., associate each derivative ∂_μ with a small parameter ε and neglect all quantities of orders higher than $O(\varepsilon^2)$ (see [1, 6]). This approximation is justified in almost all thinkable situations.

(c) The 4D formulation of the theory has the form of a scalar-tensor theory in a Jordan conformal frame. We perform a transition to the Einstein frame, more suitable for studying the dynamics of the scalar field ϕ since in this frame it is minimally coupled to the 4D curvature.

In the expression (3), only ϕ has the order $O(1)$ while f_1 and R_4 are $O(\varepsilon^2)$. Therefore,

$$F(R) = F(\phi + R_4 + f_1) \simeq F(\phi) + F' \cdot (R_4 + f_1) + o(\varepsilon^2), \quad (5)$$

where $F' = dF/d\phi$. Thus the 4D (Jordan-frame) action obtained from (2) takes the form

$$S = \frac{1}{2} \mathcal{V} m_D^2 \int \sqrt{g_4} d^4x \left[e^{n\beta} F' R_4 + K_J (\partial\beta)^2 - 2V_J(\phi) \right], \quad (6)$$

where $\mathcal{V}(n) = 2\pi^{(n+1)/2} / \Gamma(\frac{1}{2}(n+1))$ is the volume of a unit sphere \mathbb{S}^n , and

$$\begin{aligned} K_J &= e^{n\beta} [n(n-1)(4\phi F'' - F') + 4(c_1 + c_2)\phi], \\ V_J(\phi) &= \frac{1}{2} e^{n\beta} [-F(\phi) - c_J e^{-4\beta} + 2C_n r_0^{-2} F' e^{-(n+4)\beta}]. \end{aligned} \quad (7)$$

The dimensionless constants C_n are factors characterizing the Casimir energy density [8],¹ written in such a way that it contributes to the T_0^0 component of the total stress-energy tensor of matter in the theory (6). The constant c_J is defined as

$$c_J = r_0^{-4} n(n-1) [(n-1)c_1 + 2c_2] \quad (8)$$

A transition to the Einstein frame is carried out using the conformal mapping

$$g_{\mu\nu} \mapsto \tilde{g}_{\mu\nu} = |f(\phi)| g_{\mu\nu}, \quad f(\phi) = e^{n\beta} F_\phi. \quad (9)$$

¹ This expression for the Casimir energy density is only valid for odd n , while for even n the results are not so confident because of an additional logarithmic divergence [4]. We therefore consider only odd n .

The resulting action in the Einstein frame has the form

$$S = \frac{1}{2} \mathcal{V}(n) m_D^2 \int \sqrt{\tilde{g}} d^4x \left[\tilde{R}_4 + K_E (\partial\beta)^2 - 2V_E(\phi) \right], \quad (10)$$

$$K_E = 6\phi^2 \left(\frac{F''}{F'} \right)^2 - 2n\phi \frac{F''}{F'} + \frac{1}{2} n(n+2) + \frac{4(c_1 + c_2)\phi}{F'}, \quad (11)$$

$$V_E(\phi) \equiv \frac{W(x)}{r_0^2} = \frac{e^{-n\beta}}{2F'^2} \left[-F(\phi) - c_J e^{-4\beta} + 2C_n r_0^{-2} \mathcal{V}^{-1} e^{-(n+4)\beta} \right], \quad (12)$$

where the tilde marks quantities obtained from or with $\tilde{g}_{\mu\nu}$. In what follows we will try to build viable cosmologies in the theory (10) that follows from (2) under the above assumptions. To be consistent with observations in the present-day Universe, we should require:

1. The space-time is described classically, therefore the size $r = r_0 e^\beta$ of the extra dimensions should exceed the fundamental length scale $r_0 = 1/m_D$, i.e., $e^\beta \gg 1$, or $e^{-\beta} \equiv x \ll 1$.
2. The extra dimensions should not be directly observable, hence $r = r_0 e^\beta \lesssim 10^{-17}$ cm, which corresponds to the TeV energy scale.
3. The model parameters should conform to observations: so, our 3D space should expand with acceleration, and the famous Cosmological Constant Problem should be somehow addressed.

We will discuss two kinds of models: one based on a minimum of V_E and another one with slowly evolving extra dimensions.

2. A possible stationary state

Following [7], let us suppose, for simplicity, $F(R) = -2\Lambda_D + R$, hence $F(\phi) = -2\Lambda_D + \phi$, so that the nonlinearity of the theory is only contained in terms with c_1 and c_2 in the action (2), $F' = 1$, $F'' = 0$, and $f(\phi) = e^{n\beta}$. Let us also choose for specific calculations $n = 3$, then

$$W(x) = \lambda x^3 - 3x^5 - k_2 x^7 + k_3 x^{10}, \quad (13)$$

where $x \equiv e^{-\beta}$; the quantity $W(x)$ is dimensionless as well as the constants

$$\lambda = r_0^2 \Lambda_D, \quad k_1 = n(n-1)/2, \quad k_2 = r_0^2 c_J/2, \quad k_3 = C_n/\mathcal{V}. \quad (14)$$

We seek a minimum of $W(x)$ at some $x = x_0 > 0$, corresponding to a stable stationary state of β provided that $K_E > 0$.

Let us estimate k_3 in the Casimir term. According to [4], $C_3 = 7.5687046 \times 10^{-5}$ for the vacuum density of a single massless scalar field. Furthermore, $\mathcal{V}(3) = 2\pi^2$, and one should recall that there is a number of degrees of freedom in 7D space-time, able to increase the Casimir energy density by a factor of $\lesssim 100$. Thus we can take $k_3 \sim 10^{-4}$ for our qualitative estimates. Since the expected values of x_0 are small (say, ~ 0.1), the term with k_3 weakly affects our search for a minimum. Next, without this term but with $k_2 > 0$, there cannot be a minimum of $W(x)$ at $x > 0$, so we take $k_2 < 0$. Examples of such minima for $k_2 = -200$ are shown in figure 1, where the right panel shows the dependence of $W(x_0)$ on the parameter λ . One can also verify that $K_E > 0$ at $x \lesssim 0.1$. Thus one can easily obtain an arbitrarily small positive value of $W(x_{\min})$ by properly choosing the value of λ .

In cosmology this leads to viable models with $\Lambda > 0$, in particular, to the de Sitter isotropic model in the absence of matter other than the Casimir vacuum contribution.

For further estimates, we should decide which conformal frame corresponds to observations, and this in turn depends on how fermions should be described in a (so far unknown) underlying

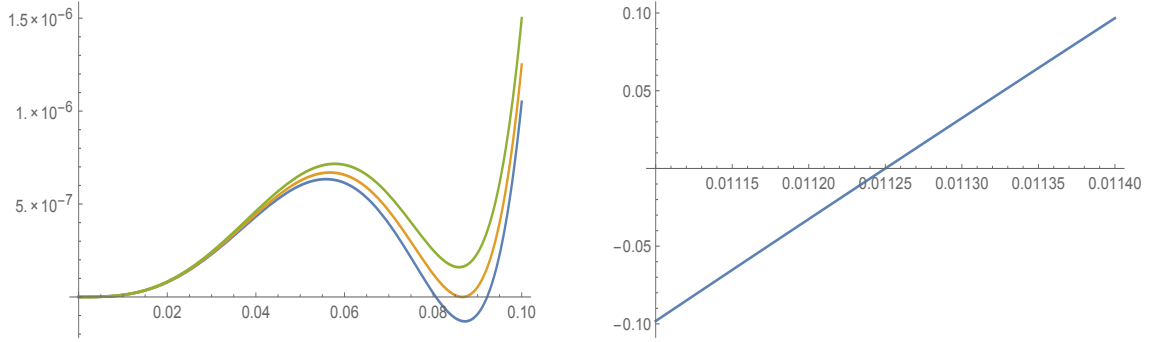


Figure 1. (a) Plots of $W(x)$ for $n = 3$, $F_2 = 0$, $k_2 = -200$, $k_3 = 10^{-4}$, and $\lambda = 0.01105, 0.01125, 0.0115$ (bottom-up); (b) Plot of $W(x_{\min})$ as a function of λ for the same values of n, F_2, k_2, k_3 .

theory of all interactions. We consider two opportunities, the Einstein frame with the action (10), and the Jordan frame, directly obtained from the D-dimensional theory.

In the **Einstein frame**, by (10), the 4D Planck mass is $m_4 = \sqrt{\mathcal{V}(n)}m_D$, and $r_0 = \sqrt{\mathcal{V}(n)}/m_4$. Hence for $x_0 \sim 0.1$ the size of extra dimensions is $r(x_0) = r_0/x_0 \sim 10\sqrt{\mathcal{V}(n)}m_4^{-1}$, close to the Planck length $1/m_4 \approx 8 \times 10^{-33}$ cm as long as $\mathcal{V}(n)$ is not far from unity.

The effective cosmological constant is $\Lambda_{\text{eff}} = W(x_0)/r_0^2 = W(x_0)m_4^2/\mathcal{V}(n)$, and to conform to observations which require $\Lambda_{\text{eff}} \sim 10^{-120}$, fine tuning is necessary: λ should be close to the value at which $W(x_0) = 0$ (see figure 1, right) with an accuracy close to 10^{-118} .

The Casimir contribution to $W(x)$ is very large as compared to 10^{-120} : e.g., with the parameter values used in the figure, this contribution is $k_3x_0^{10} \approx 1.4 \times 10^{-15}$. This comparatively large value is compensated by fine-tuned values of other parameters of the theory.

In the **Jordan frame**, the 4D Planck mass is related to m_4 by

$$m_D^2 = 1/r_0^2 = m_4^2\mathcal{V}(n)^{-1}x_0^n. \quad (15)$$

Since $x_0 \ll 1$, r_0 is in general a few orders of magnitude larger than the Planck length, which at large enough n may be in tension with the invisibility of extra dimensions.

The effective cosmological constant in the Jordan frame is obtained if we present the integrand in (6) as $\sqrt{g_4}e^{n\beta}F_\phi[R_4 - 2\Lambda_{\text{eff}} + \text{kinetic term}]$, which leads to $\Lambda_{\text{eff}} = F_\phi x_0^{-n}W(x_0)/r_0^2$. However, expressing r_0 in terms of m_4 , we arrive again at the expression $\Lambda_{\text{eff}} = W(x_0)m_4^2/\mathcal{V}(n)$. Thus we need the same fine tuning as in the Einstein frame and have the same estimate of the Casimir contribution to $W(x)$, despite another value of the fundamental length $r_0 = 1/m_D$.

3. A model with evolving extra dimensions

A viable model with slowly evolving extra dimensions can be obtained, by analogy with [9, 10], from the same Einstein-frame action (10)–(12), where now we should focus on the minimum of $W(x)$ at $x = 0$ (see figure 1, left panel for illustration) and a slow decrease of $x(t) \equiv e^{-\beta(t)} \equiv 1/b(t)$ to zero, so that $b(t) \equiv 1/x \rightarrow \infty$, still remaining small enough to conform to observations. Consider 4D isotropic, spatially flat cosmologies with n -dimensional spherical extra space:

$$ds^2 = dt^2 - a^2(t)d\vec{x}^2 - b^2(t)d\Omega_n^2. \quad (16)$$

We now keep n arbitrary. Two independent field equations for the unknowns $a(t)$ and $\beta(t)$ are

$$\frac{3\dot{a}^2}{a^2} = K_E\dot{\beta}^2 + V_E(\beta), \quad (17)$$

$$2K_E\left(\ddot{\beta} + \frac{3\dot{a}}{a}\dot{\beta}\right) + \frac{dK_E}{d\beta}\dot{\beta}^2 + \frac{dV_E}{d\beta} = 0. \quad (18)$$

It turns out that the assumption $F = R - 2\Lambda_D$ now does not lead to a good solution. Instead, following [9, 10], we can choose $F = -2\Lambda_D + R^2$, which leads to the following expressions for K_E and $W = V_E$:

$$\begin{aligned} K_E &= K_0 = \frac{1}{4}[n^2 - 2n + 12 + 4(c_1 + c_2)], \\ V_E &= \frac{1}{8n^2(n-1)^2} \left[2\lambda e^{(4-n)\beta} - [c_J + n^2(n-1)^2]e^{-n\beta} + \frac{2C_n}{\mathcal{V}(n)}e^{-2n\beta} \right], \end{aligned} \quad (19)$$

where we have put for convenience $r_0 = 1$, so that $\lambda = \Lambda_D$. One sees that the required minimum of V_E at $\phi = 0$ takes place for $n > 4$, and it is clear that the Casimir term with $C_n \sim 10^{-4}$ again can only weakly affect the solution. Neglecting also the second term in V_E (with $e^{-n\beta}$), it is easy to find a slow-rolling solution, satisfying the conditions $|\ddot{\beta}| \ll 3(\dot{a}/a)\dot{\beta}$ and $K_E\dot{\beta}^2 \ll V_E(\beta)$ [9, 10]:

$$a(t) = a_1(t + t_1)^p, \quad b(t) = b_0 \left(\frac{t + t_1}{t_0 + t_1} \right)^{1/\bar{d}}, \quad (20)$$

where

$$p = \frac{K_0}{\bar{d}^2}, \quad \bar{d} = \frac{n-4}{2}, \quad b_0 = \left(\frac{\sqrt{\lambda/12}}{H_0 n(n+1)} \right)^{1/\bar{d}}; \quad (21)$$

a_1 and t_1 are integration constants, t_0 is the present age of the Universe, and H_0 is the Hubble constant: $H_0 = \dot{a}/a$ at $t = t_0$. The solution satisfies the slow-rolling conditions if $p \gg 1$ (which in turn requires $c_1 + c_2 \gg 1$), and in this case it describes power-law inflation and a very slow increase of the size $b(t)$ of extra dimensions.

The 4D Planck mass is $m_4 = \sqrt{\mathcal{V}(n)}m_D$, and $r_0 = \sqrt{\mathcal{V}(n)}/m_4$. If n is not too large (say, $n < 25$), we can suppose $\sqrt{\mathcal{V}(n)} \approx 1$, so we work with approximately the conventional 4D Planck units. From (20) we can estimate the input parameter λ (the initial dimensionless cosmological constant) in terms of the present-day size $b_0 = e^{\beta(t_0)}$ [7]:

$$\lambda \approx \frac{3}{16} n^2 (n-1)^2 b_0^{n-4} \times 10^{-120}. \quad (22)$$

To keep the extra dimensions invisible, we should put $1 \ll b_0 < 10^{16}$. It is clear that at some admissible values of n and b_0 the quantity λ will be of the order of unity, for instance, it happens at $n = 13$ and $b_0 = 10^{13}$. Thus the Cosmological Constant Problem can be solved in this framework without fine tuning.

These estimates have been made in the Einstein frame. As to Jordan's, it turns out [10] that the same solution leads to a model incompatible with observations: the cosmological expansion is too strongly accelerated (for the effective equation-of-state parameter w we have $-w \gg 1$ while observations imply $w \approx -1$) and leads to a big rip.

4. Concluding remarks

In curvature-nonlinear multidimensional gravity with spherical extra dimensions, we have presented two simple examples of cosmological models approximately describing the present stage of the Universe evolution: one with a suitable minimum of the effective potential, corresponding to a stable stationary size b of the extra dimensions, the other in which this size is very slowly growing. The second model differs from the first one in the following features: (i) a larger dimension n is required, (ii) no fine tuning of the initial parameters is necessary, (iii) the Casimir contribution to the total energy density is insignificant, and (iv) only the Einstein frame is viable.

Among possible interesting extensions of this work there is a study of variations of fundamental constants depending on the size of extra dimensions, including the Newtonian gravitational constant G and the electromagnetic coupling constant α , a possible application of such models to the early (inflationary) Universe with maybe quantum tunneling between different minima of the potential, and, certainly, a consideration of other geometries of extra dimensions, e.g., in the form of products of spherical, toroidal and/or hyperbolic factor spaces.

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