

Wormholes and black universes communicated with extra dimensions

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Abstract. In 6D general relativity with a phantom scalar field as a source of gravity, we present solutions that implement a transition from an effective 4D geometry times small extra dimensions to an effectively 6D space-time where the physical laws are different from ours. We consider manifolds with the structure $\mathbb{M}_0 \times \mathbb{M}_1 \times \mathbb{M}_2$, where \mathbb{M}_0 is 2D Lorentzian space-time while each of $\mathbb{M}_{1,2}$ can be a 2-sphere or a 2-torus. Some solutions describe wormholes with spherical symmetry in our space-time and toroidal extra dimensions. Others are of black universe type: at one end there is a 6D asymptotically anti-de Sitter black hole while beyond the horizon the geometry tends to a 4D de Sitter cosmology times a small 2D spherical extra space.

1. Introduction

Multidimensional theories suggest a great variety of geometries, topologies and compactification schemes (for reviews see, e.g., [1–4] and references therein). One of the opportunities of interest is that the geometry can be effectively four-dimensional in some space-time region but have a greater dimension in other regions. We here try to obtain examples of such space-times in the framework of 6D general relativity with a minimally coupled scalar field as a source of gravity.

More specifically, we consider a 6D space-time of the form $\mathbb{M} = \mathbb{M}_0 \times \mathbb{M}_1 \times \mathbb{M}_2$, where \mathbb{M}_0 is 2D Lorentzian space-time, while each of $\mathbb{M}_{1,2}$ is a two-sphere or a two-torus. As $x \rightarrow -\infty$, we assume an asymptotically flat or asymptotically de Sitter 4D geometry times small extra dimensions, while at the other end, $x \rightarrow +\infty$, we expect a geometry with large extra dimensions.

We give three explicit examples of such solutions. In two of them the 4D subspace has a wormhole geometry,¹ while the third example corresponds to the notion of a black universe, that is, a black hole with an expanding universe instead of a singularity beyond its horizon [11–13].

In all these cases, the solutions can only be obtained with a phantom scalar field should, having a wrong sign of kinetic energy. Such fields naturally emerge in some unification theories, there are theoretical arguments both *pro et contra* their possible existence, see, e.g., discussions in [5, 6, 14, 15]. In this paper, as in many others, we admit it as a working hypothesis.

¹ See, e.g., [1, 5, 6] for reviews and also more recent papers, e.g., [7–10].



2. Equations in 4+2 dimensions

We consider 6D GR with a minimally coupled scalar field ϕ having a potential $V(\phi)$ as the only source of gravity. The total action is

$$S = \frac{m_6^2}{2} \int \sqrt{|g_6|} \left[R_6 + 2\varepsilon_\phi g^{AB} \partial_A \phi \partial_B \phi - 2V(\phi) \right], \quad (1)$$

where m_6 is the 6D Planck mass, R_6 and g_6 are the 6D Ricci scalar and metric determinant, respectively, ε_ϕ is +1 for a normal, canonical scalar field and -1 for a phantom one, and $A, B, \dots = \overline{0}, 5$. The corresponding equations of motion are the scalar field equation $2\varepsilon_\phi \square_6 \phi + dV/d\phi = 0$ and the Einstein equations which can be written as

$$R_B^A = -\tilde{T}_B^A \equiv -T_B^A - \frac{1}{4} \delta_B^A T_C^C \equiv -2\varepsilon_\phi \partial^A \phi \partial_B \phi + \frac{1}{2} V(\phi) \delta_B^A, \quad (2)$$

where R_B^A is the 6D Ricci tensor and T_B^A is the stress-energy tensor (SET) of the scalar field.

Consider a 6D manifold being a direct product of three 2D spaces, $\mathbb{M} = \mathbb{M}_0 \times \mathbb{M}_1 \times \mathbb{M}_2$, where \mathbb{M}_0 is 2D space-time with the coordinates $x^0 = t$ and $x^1 = x$, while \mathbb{M}_1 and \mathbb{M}_2 are compact 2D spaces of constant nonnegative curvature, i.e., each of them can be a sphere or a torus. The metric is taken in the form:

$$ds^2 = A(x)dt^2 - dx^2/A(x) - R(x)d\Omega_1^2 - P(x)d\Omega_2^2, \quad (3)$$

where $A(x)$, $R = r^2(x)$, $P = p^2(x)$ are functions of the “radial” coordinate x , chosen under the condition $g_{tt}g_{xx} = -1$ (the so-called quasiglobal gauge [1]), while $d\Omega_1^2$ and $d\Omega_2^2$ are x -independent metrics on 2D manifolds \mathbb{M}_1 and \mathbb{M}_2 of unit size. We also assume $\phi = \phi(x)$.

We do not fix which of $\mathbb{M}_{1,2}$ belongs to our 4D space-time and which is “extra”: everything depends on their size. Thus, if \mathbb{M}_1 is large and spherical while \mathbb{M}_2 is small and toroidal, we have a static, spherically symmetric configuration in 4D and a toroidal extra space, and so on.

Due to the symmetry of the problem and the properties of the scalar field SET, there are four independent equations, which may be written as follows (the prime denotes d/dx):

$$R_t^t = -\tilde{T}_t^t \Rightarrow -(PR)^{-1}(A'PR)' = V(\phi), \quad (4)$$

$$R_t^t - R_x^x = -\tilde{T}_t^t + \tilde{T}_x^x \Rightarrow \frac{r''}{r} + \frac{p''}{p} = -\varepsilon_\phi \phi'^2, \quad (5)$$

$$R_t^t - R_a^a = 0 \Rightarrow [P(AR' - A'R)]' = 2\varepsilon_1 P, \quad (6)$$

$$R_t^t - R_m^m = 0 \Rightarrow [R(AP' - A'P)]' = 2\varepsilon_2 R, \quad (7)$$

where $a = 2, 3$ (belong to \mathbb{M}_1 , $m = 4, 5$ (belong to \mathbb{M}_2 , there is no summing over an underlined index, $\varepsilon_1 = 1$ if \mathbb{M}_1 is a sphere and $\varepsilon_1 = 0$ if it is a torus, and similarly for ε_2 and \mathbb{M}_2 .

Equations (6) and (7) contain only the metric functions $A(x), P(x), R(x)$. Therefore, considering them separately, these are two equations for three unknown functions, so there is arbitrariness in one function. If the metric functions are known, the other two Einstein equations can be used to find the scalar ϕ and the potential V . From equation (5) it follows that solutions with $r > 0$ and $p > 0$ in the whole range $x \in \mathbb{R}$ can only exist with $\varepsilon_\phi = -1$, i.e., a phantom field, since such solutions require $r'' > 0$ and $p'' > 0$.

3. Possible asymptotic behaviour of the metric

The metric under consideration describes the following types of geometries:

- (i) SS (double spherical) space-times: the case $\varepsilon_1 = \varepsilon_2 = 1$.

Table 1. Asymptotic behaviours compatible with equations (6) and (7). Notations: a plus or minus mean that a particular behaviour is possible or impossible, respectively, \pm that it is possible under special conditions for the functions involved. The symbol “fin” means a positive constant, while an asterisk indicates that this asymptotic is only possible with $A < 0$.

Line No.	Asymptotic behaviour			6D geometries			Comments
	$A(x)$	$R(x)$	$P(x)$	SS	ST	TT	
0	fin	fin	fin	–	–	\pm	$\mathbb{M}^2 \times \mathbb{T}^2 \times \mathbb{T}^2$
1	fin	fin	x^2	–	–	–	none
2	fin	x^2	fin	–	+	–	$\mathbb{M}^4 \times \mathbb{T}^2$
3	fin	x^2	x^2	+	–	–	\mathbb{M}^6
4	x^2	fin	fin	+	*	–	$\text{dS}_2 \times \mathbb{S}^2 \times \mathbb{S}^2$
5	x^2	fin	x^2	\pm *	\pm *	–	$\text{dS}_4 \times \mathbb{S}^2$
6	x^2	x^2	fin	\pm *	–	–	$\text{dS}_4 \times \mathbb{S}^2$
7	x^2	x^2	x^2	\pm	\pm	\pm	(A)dS ₆

(ii) ST (spherical-toroidal) space-times: the case $\varepsilon_1 = 1$, $\varepsilon_2 = 0$ or vice versa. If M_1 is large and M_2 small, we have static spherical symmetry in our space-time and a small toroidal extra space. The opposite situation is also possible as well as a total observable 6D geometry.

(iii) TT (double toroidal) space-times: if $\varepsilon_1 = \varepsilon_2 = 0$, we have the same as before but both M_1 and M_2 are toroidal.

We seek configurations where $x \in \mathbb{R}$ and there are different geometries in the two asymptotic regions $x \rightarrow \pm\infty$. In particular, there can be a 4D flat asymptotic region at large negative x times small extra dimensions and something different at the other end. In this section we do not consider the properties of the scalar field but only analyze which kinds of asymptotic behaviour are compatible with equations(6) and (7) for each of the types 1–3 of 6D geometry.

As an example, let us consider an asymptotically flat 4D space-time with constant extra dimensions in SS geometry. Without loss of generality, this means that

$$A(x) \rightarrow \text{fin}, \quad R(x) \sim x^2, \quad P(x) \rightarrow \text{fin} \quad \text{as } x \rightarrow \infty. \quad (8)$$

Let us substitute these conditions to equations(6) and (7). According to (8), $R' \sim x$, $A' \sim x^{-2}$ or even smaller (due to the expansion $A = A_- + A_{-1}/x + \dots$), and the l.h.s. of (6) tends, in general, to a nonzero constant, which agrees with the requirement to P that stands on the r.h.s.. However, in (7) the expression in square brackets tends to a constant, hence its derivative vanishes, while the r.h.s., equal to $2R$, should behave as x^2 . We conclude that *the asymptotic conditions (8) are incompatible with the field equations*.

On equal grounds we could consider $x \rightarrow -\infty$ and/or exchange $R(x)$ and $P(x)$. Other opportunities are considered in the same manner, and the results are summarized in table 1.

The table shows that the choice of possible 6D geometries is rather restricted. For example, SS space-times cannot have a flat Minkowski asymptotic times a finite sphere, but a de Sitter behaviour times a finite sphere can be realized. In ST geometry, in addition to effectively 6D asymptotics, there can exist an asymptotically flat spherically symmetric 4D space-time with constant extra dimensions. Evidently, at the two ends, $x \rightarrow \pm\infty$, one can have similar or different admissible asymptotics. In what follows we will give three examples of such solutions to the field equations where on one end there is a 4D geometry with small extra dimensions and on the other there is an effectively 6D space-time.

4. Examples

Example 1: ST geometry, wormholes with a massless scalar

Of interest are space-times with asymptotically flat spherically symmetric geometry in one asymptotic region and something different in the other. The first example is an ST wormhole geometry with a strongly different size of extra dimensions at the two ends, which exists among well-known solutions for a massless scalar, $V = 0$ [16,17]. One of such solutions has the form [18] (note that the coordinate z is different from x used in the rest of the paper)

$$ds^2 = dt^2 - e^{-4nu} [dz^2 + (z^2 + k^2)d\Omega_1^2] - e^{2nu}d\Omega_2^2, \\ \phi = Cu \equiv (C/k) \cot^{-1}(-z/\bar{k}), \quad (9)$$

where $n > 0$, $C, k > 0$ are integration constants related by $2C^2 = k^2 + 3n^2$. It describes a spherically symmetric, twice asymptotically flat wormhole in the 4D subspace $\mathbb{M}_0 \times \mathbb{M}_1$ with a toroidal extra space \mathbb{M}_2 having a unit size, p_- at $u = 0$ (that is, $z = -\infty$) and the size $p_+ = e^{n\pi/k}p_-$ at the other end, $u = \pi/k$, corresponding to $z = +\infty$.²

Suppose that the size of extra dimensions p_- on the left end, $z = -\infty$, is small enough to be invisible by modern instruments, say, $p_- = 10^{-17}$ cm. The size p_+ on the other end is much larger if n/k is large enough. For example, to obtain $p_+ \sim 1$ m, one should take $n/k \approx 14$.

The wormhole throat is a minimum of $g_{22} = e^{-4nu}(z^2 + k^2)$, its radius is equal to

$$r_{\min} = \sqrt{k^2 + 4n^2} \exp\left(\frac{2n}{k} \cot^{-1} \frac{2n}{k}\right). \quad (10)$$

To make this radius large enough for passing of a macroscopic body, large values of k are required: e.g., to obtain $r_{\min} = 10$ m, one has to suppose $k \sim 10^{18}$.

Example 2: ST geometry, asymptotically AdS wormholes

With nonzero potentials $V(\phi)$, in most cases solutions can be found only numerically. An exception appears if after integrating equation (7) we turn the appearing constant to zero. It then follows $P = cA$, $c = \text{const}$, and equation (6) takes the form $A^3(R/A)'' = 2A$. It is a single equation for two functions $A(x)$ and $R(x)$, which can be solved by quadratures if one specifies $A(x)$ since it can be rewritten as

$$\left(\frac{R}{A}\right)' = \frac{2}{A^3} \int A(x) dx. \quad (11)$$

To obtain an example with an asymptotically flat 4D space-time on the left end and an AdS asymptotic on the right (a transition from line 2 to line 7 in table 1), we should suppose $A \rightarrow 1$ as $x \rightarrow -\infty$ and $A \sim x^2$ as $x \rightarrow +\infty$. It is hard to find $A(x)$ with such properties that would lead to good analytic expressions of other quantities. Therefore, an example has been obtained with a piecewise smooth function $A(x)$ [18]:

$$A(x) = 1, \quad x \leq 0; \quad A(x) = 1 + 3x^2/a^2, \quad x \geq 0 \quad (a = \text{const} > 0). \quad (12)$$

The resulting solution has a C^1 smooth metric but jumps in ϕ' and V (figure 1). We have $P = cA$ in the whole space and the following behavior of $R(x)$:

$$R(x) = x^2 + b^2 \quad (x \leq 0), \quad R(x) = \left(1 + \frac{3x^2}{a^2}\right) \left[b^2 + \frac{x^2(1 + 2x^2/a^2)}{(1 + 3x^2/a^2)^2}\right] \quad (x \geq 0), \quad (13)$$

with $b = \text{const} > 0$ (thus $x = 0$ is a throat of radius b). At negative x we have $V(x) \equiv 0$ and $\phi(x) = \arctan(x/b)$ while at positive x the expressions are rather cumbersome.

² Both ends correspond to line 2 in table 1. In the trivial case $n = 0$ we obtain the well-known 4D Ellis wormhole [19,20] times a toroidal extra space of constant size.

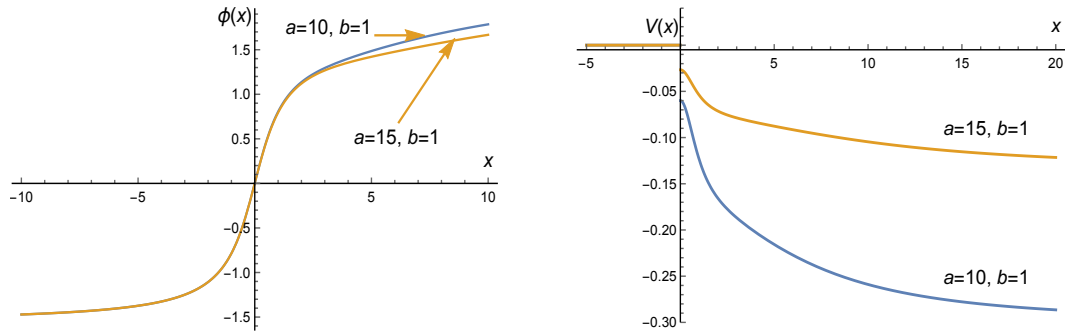


Figure 1. The scalar field $\phi(x)$ (left) and the potential $V(x)$ (right) in example 2

The resulting configuration is asymptotically flat times constant (arbitrarily small) extra dimensions on the left end and a 6D AdS asymptotic on the right end, with V tending to a negative constant playing the role of a cosmological constant.

The jumps in $V(x)$ and $\phi'(x)$ at $x = 0$ could be easily removed by choosing $A(x)$ smoother than C^1 at $x = 0$, which is possible by making a suitable arbitrarily small addition to (12).

Example 3: SS geometry, a black universe with AdS₆ at the far end

In SS geometry, corresponding to $\varepsilon_1 = \varepsilon_2 = 1$, it is hard to solve equations (6) and (7) analytically, but examples of interest can be obtained numerically. One such solution, which has been found under the assumption $R = 1 + x^2$, realizes a transition from line 6 to line 7 of table 1. The metric functions are plotted in figure 2, the corresponding $\phi(x)$ and $V(x)$ are not shown but are easily obtained using (4) and (5). On the left end ($x \rightarrow -\infty$) there is a 4D de Sitter metric (describing an exponentially expanding or contracting spherically symmetric space) times small extra dimensions; on the right end there is an AdS static space-time. The configuration as a whole is a black universe [11,12] where the cosmological expansions starts from a horizon, the static region belonging to the remote past; in this case this expanding universe has a multidimensional origin.

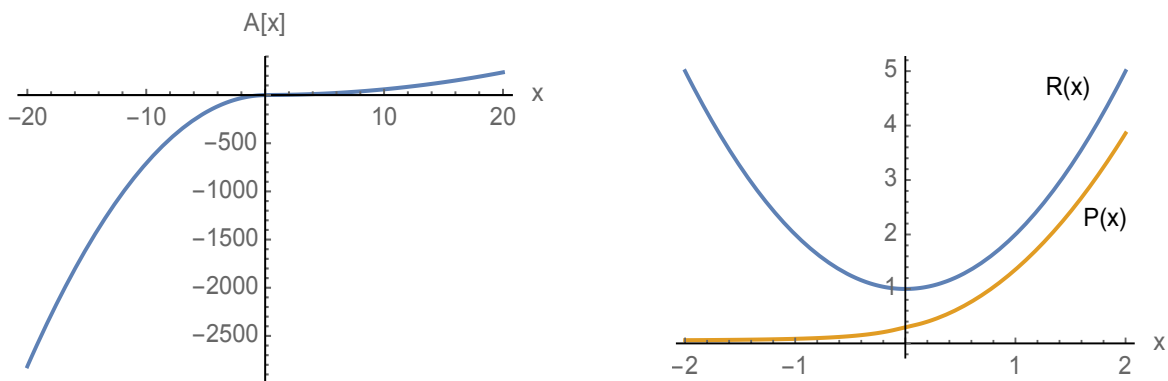


Figure 2. The metric functions in example 3. Left: $A(x)$, with the asymptotics $A(-\infty) \approx -7.02x^2$ and $A(+\infty) \approx 0.59x^2$. Right: $R = 1 + x^2$, $P \rightarrow 0.0474$ as $x \rightarrow -\infty$ and $P(+\infty) \approx 0.8x^2$.

5. Concluding remarks

The results can be summarized as follows. In 6D GR with a minimally coupled scalar field as a source of gravity, we have constructed examples of space-times which are effectively 4D on the

left end and effectively 6D on the right end. Some of them have the nature of wormholes, others (black universes) actually represent expanding cosmological models of multidimensional origin. The existence of such configurations or their analogs with a different number of extra dimensions in our universe cannot be a priori excluded, and their possible astrophysical consequences could be a subject of further studies.

It should be noted that our analysis certainly did not cover all opportunities: other, more complicated cases are also possible. For instance, of particular, there can be models where different subspaces exchange their roles as those belonging to a large observable space, like that described in [21]. One more subject of a future study can be a relationship between the present scalar-vacuum system and multidimensional gravity with curvature-nonlinear actions [1, 21] in different conformal frames in application to space-times of the types considered here and in [21].

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