

Jin–Xin relaxation solution to the spatially-varying Burgers equation in one dimension

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Abstract. We consider the spatially-varying Burgers equation in one dimension. We take the Lax–Friedrichs finite volume method and Jin–Xin relaxation method in solving the equation. According to our research, the Jin–Xin relaxation method produces a more accurate solution, as they produce smaller error than the Lax–Friedrichs finite volume method. However, the Lax–Friedrichs finite volume method is faster in computation than the Jin–Xin method.

1. Introduction

A partial differential equation is an equation which contains its partial derivatives involving two or more independent variables. There are a large number of examples of partial differential equations in mathematical modelling, such as shallow water wave equations [1-3], advection equations [4-5], gas dynamics [6], elasticity equation [7] and Burgers equation [8].

In this paper, we focus on numerical solutions $u(x, t)$ to the spatially-varying Burgers equation in one dimension with the flux function depends on the space variable x and time variable t , where the flux function is of the form $F(x, u) = k(x)f(u)$ for some function f . Details of this problem is found in the work of Prnić [9].

A number of numerical methods are available in the literature, as computer simulations are desired for large scale computations [10-12]. In this paper numerical methods are used to solve the spatially-varying Burgers equation. We focus on the Lax–Friedrichs finite volume method [4-5] and Jin–Xin relaxation method [13-14]. We choose Lax–Friedrichs finite volume method because of its simplicity. The Jin–Xin relaxation method offers linearity in the model, so they are easy to compute. The main task is to compare our results between the Lax–Friedrichs finite volume method and Jin–Xin relaxation method. We research for which method resulting in better performance than the other.

The paper structure is as follows. Section 2 writes the mathematical model that we want to solve. Section 3 presents numerical methods that we use to solve the model. Section 4 provides numerical results. Conclusion is drawn in Section 5.

2. Problem formulation

We consider the spatially-varying Burgers equation in one dimension

$$\frac{\partial u}{\partial t} + \frac{\partial(k(x)f(u))}{\partial x} = 0, \quad (1)$$

where



$$f(u) = \frac{u^2}{2}, \quad (2)$$

and

$$k(x) = \frac{1}{1+x^2}. \quad (3)$$

The initial condition for $x > 0$ is

$$u(x, 0) = \frac{1}{\sqrt{1+x^2}}. \quad (4)$$

Here x is the space variable, t is the time variable, u is the conserved quantity and $F(x, u) = k(x)f(u)$ is the flux function. The boundary condition is supposed to be known.

3. Numerical methods

Numerical methods that we use in this paper are the Lax–Friedrichs finite volume method and Jin–Xin relaxation method.

3.1. Lax–Friedrichs finite volume method

Burgers equation (1) can be written in a compact form as

$$u_t + [k f(u)]_x = 0. \quad (5)$$

Equation (5) is solved using the finite volume method with an explicit numerical scheme

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x} (F_{i+\frac{1}{2}}^n - F_{i-\frac{1}{2}}^n) \quad (6)$$

where $U_i^n \approx u(x_i, t^n)$ is an approximation of the conserved quantity and $F_{i+1/2}^n \approx f(u(x_{i+1/2}, t^n))$ is the flux function (see LeVeque [4-5]). Here Δt is the time step, Δx is the cell-width, i is defined as the index of space and n is the index of time. Formulas for calculation of fluxes in equation (6) using the Lax–Friedrichs finite volume method are given by

$$F_{i+\frac{1}{2}}^n = \frac{1}{2} [k_{i+1} f(U_{i+1}^n) + k_i f(U_i^n)] - \frac{\Delta x}{2\Delta t} (U_{i+1}^n - U_i^n), \quad (7)$$

and

$$F_{i-\frac{1}{2}}^n = \frac{1}{2} [k_i f(U_i^n) + k_{i-1} f(U_{i-1}^n)] - \frac{\Delta x}{2\Delta t} (U_i^n - U_{i-1}^n). \quad (8)$$

3.2. Jin–Xin relaxation method

The Jin–Xin relaxation system for equation (1) is

$$\frac{\partial u}{\partial t} + \frac{\partial v}{\partial x} = 0, \quad (9)$$

and

$$\frac{\partial v}{\partial t} + a \frac{\partial u}{\partial x} = -\frac{1}{\varepsilon} (v - k f(u)), \quad (10)$$

where v is an artificial variable defined by $v = k f(u)$, with a is a positive constant of the relaxation system which can be given by $a = k (f'(u))^2$ and ε is a small positive constant.

Equations (9) and (10) with the discretised space can be written as

$$\frac{\partial}{\partial t} u_i + \frac{1}{\Delta x} (v_{i+\frac{1}{2}} - v_{i-\frac{1}{2}}) = 0, \quad (11)$$

and

$$\frac{\partial}{\partial t} v_i + \frac{1}{\Delta x} a (u_{i+\frac{1}{2}} - u_{i-\frac{1}{2}}) = -\frac{1}{\varepsilon} (v_i - k f(u_i)). \quad (12)$$

Furthermore, the first order fully explicit scheme of the Jin–Xin relaxation method (refer to Jin and Xin [13]) to solve the Burgers equation is

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{2\Delta x} \left(v_{i+1}^n - v_i^n - \sqrt{a} (u_{i+1}^n - 2u_i^n + u_{i-1}^n) \right), \quad (13)$$

and

$$v_i^{n+1} = v_i^n - \frac{\Delta t}{2\Delta x} \left(a (u_{i+1}^n - u_i^n) - \sqrt{a} (v_{i+1}^n - 2v_i^n + v_{i-1}^n) \right) - \frac{1}{\varepsilon} (v_i^n - k f(u_i^n)). \quad (14)$$

In this paper, we have used the Euler's time stepping method [15] for the integration with respect to the time variable.

4. Numerical results

In this section we present our research results and give some discussion. All quantities are assumed to have SI units.

4.1. Results of the Lax–Friedrichs finite volume method

For computation in the Lax–Friedrichs finite volume method, we take $x \in [0, 2]$ with cell width $\Delta x = 0.02$. We calculate the numerical solution from the initial time $t = 0$ until the final time $t = 1.5$ with time step $\Delta t = 0.5\Delta x$. For the initial condition we assume $u(x, 0) = 1/\sqrt{1+x^2}$. The boundary condition at the left-end is $u(0, t) = 1/(1+t)$, and at the right-end $u(2, t)$ is set to be transmissive. The flux function for the Burgers equation here is $F(x, u) = k(x)f(u)$ where $k(x) = 1/(1+x^2)$ and $f(u) = u^2/2$.

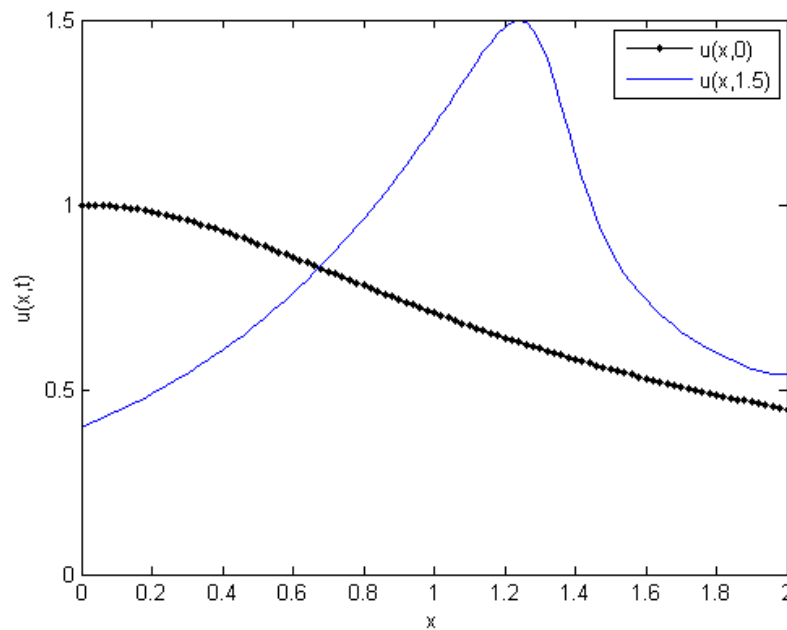


Figure 1. Numerical solutions produced using the Lax–Friedrichs finite volume method at $t = 1.5$.

We obtain that the Lax–Friedrichs finite volume method is fast in computation. The Lax–Friedrichs finite volume method only needs 21.4 seconds to compute numerical solutions, even though this running time can still be optimised. Numerical solution of the Lax–Friedrichs finite volume method is shown in Figure 1.

4.2. Results of the Jin–Xin relaxation method

Using the Jin–Xin relaxation method, we compute from the starting point $x = 0$ to final point $x = 2$ with cell width $\Delta x = 0.05$. We compute the solution from the initial time $t = 0$ until the final time $t = 1.5$ with time step $\Delta t = 0.01\Delta x$. The initial and boundary conditions are the same as the previous simulation. The small positive constant (relaxation rate) that we used here is $\varepsilon = 10^{-3}$.

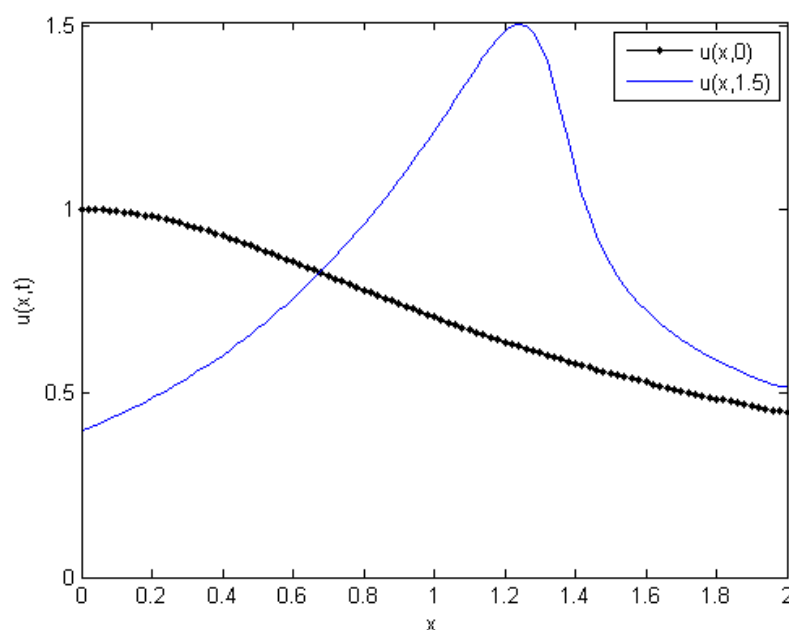


Figure 2. Numerical solutions produced using the Jin–Xin relaxation method at $t = 1.5$.

The Jin–Xin relaxation method needs 300.0 seconds to compute numerical solutions. But if we choose Δx too big, then the method becomes unstable. Numerical results of the Jin–Xin relaxation method are shown in Figure 2 giving similar behaviour to that of the Lax–Friedrichs results.

5. Conclusion

According to our research, the Lax–Friedrichs finite volume method is faster than the Jin–Xin relaxation method in the computation. However, the Jin–Xin relaxation method is easy to compute due to its linear approach. With these results, we suggest that researchers should be aware of these trade-off when they want to take one of these two methods to solve the spatially-varying Burgers equation.

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