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Finite volume numerical solution to a blood flow problem in human artery

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Abstract. In this paper, we solve a one dimensional blood flow model in human artery. This model is of a non-linear hyperbolic partial differential equation system which can generate either continuous or discontinuous solution. We use the Lax–Friedrichs finite volume method to solve this model. Particularly, we investigate how a pulse propagates in human artery. For this simulation, we give a single sine wave with a small time period as an impluse input on the left boundary. The finite volume method is successful in simulating how the pulse propagates in the artery. It detects the positions of the pulse for the whole time period.

1. Introduction

Blood is one of body components which has important functions. One of blood functions is to distribute the nutritions to all human body tissues. In some cases, blood flow may be hindered because of some problems, such as plugging and artery cavity stiffening [1-2]. It is a dangerous condition that must be overcome. In this case, medical treatment can affect the blood flow pattern. Blood flow in human artery can be represented in a mathematical model. We can investigate blood flow patterns from the solution of its model [3-5]. Numerical methods are considered, as they are powerful [6-8] to solve mathematical models.

In this paper, our focusses are to find and simulate numerical solution of the one dimensional blood flow model in human artery. There are a number of methods which can be used to solve this model, but we use the Lax–Friedrichs finite volume method [9-12] because of its simplicity. Moreover, finite volume methods can be used to find either continuous or discontinuous solution [13-14].

The paper is organised as follows. Section 2 gives the problem that we want to solve. Research method is presented in Section 3. Numerical results are provided in Section 4. Conclusion is drawn in Section 5.

2. Problem formulation

In this section, we describe the problem of blood flow in human artery, which we want to solve.

We consider a straight cylindrical tube with circular cross section and z coordinate is the axis of cylinder (see Figure 1). The one dimensional blood flow model for human artery [2] is

$$\frac{\partial A}{\partial t}+\frac{\partial Q}{\partial z}=0\,, \tag{1}$$

$$\frac{\partial Q}{\partial t}+\frac{\partial}{\partial z}\left(\propto\frac{Q^2}{A}\right)+\frac{A}{\rho}\frac{\partial p}{\partial z}+K_R\frac{Q}{A}=0\,, \tag{2}$$

for $z \in (0, l)$ and $t > 0$, where A, Q, and p are the cross section area of artery, the blood discharge, and the blood pressure, respectively. In addition, $\propto$ is the velocity function in every cross section artery which is assumed as a constant (in this paper we set its value equals to one), ρ is the density of blood, K_R is the coefficient which relates to the blood viscosity, z is the space variable, and t is the time variable.

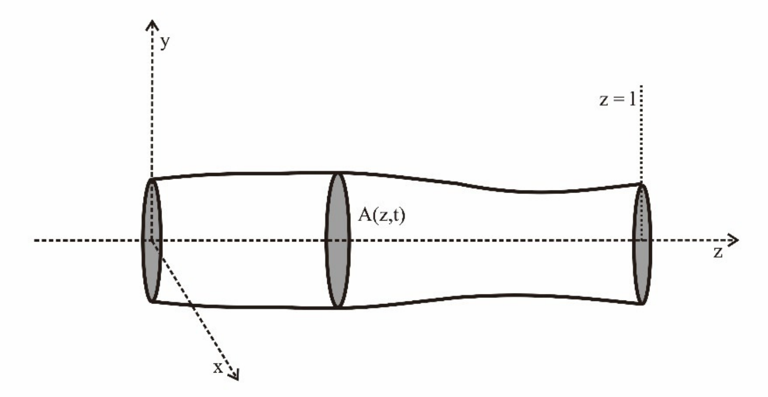

Figure 1. Ilustration of human artery.

In this model, there are three dependent variables (A, Q, dan p) and two equations. In order to have two equations with two unknowns, we define a relation that links the blood pressure with the cross section area of artery

$$p = p_{\text{ext}} + \beta\left(\sqrt{A} - \sqrt{A_0}\right). \tag{3}$$

Here p_{ext} is the external pressure and A_0 is the artery cross section area at initial time $t = 0$. In this paper, p_{ext} is assumed to be zero and A_0 is constant. Furthermore, β is a parameter relating to the artery wall elastic properties:

$$\beta(z) = \frac{4\sqrt{\pi} h_0 E(z)}{3A_0}, \tag{4}$$

where $E(z)$ is the elastic Young's modulus.

3. Numerical method

In this section, we explain the finite volume method for solving blood flow model (1)-(2). We use the flux formulation of Lax–Friedrichs [9].

To find numerical solutions of this blood flow model, let us consider the space domain discretisation (as shown in Figure 2) where $\Delta z = z_{i+1/2} - z_{i-1/2}$ or $\Delta z = z_i - z_{i-1}$, and the time domain discretisation $t^n = n\,\Delta t$ for integers n.

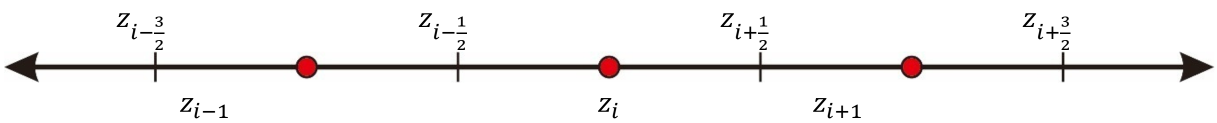

Figure 2. Ilustration of space domain discretisation.

From equation (3), we obtain

$$\begin{aligned}\frac{\partial p}{\partial z} &= \frac{\partial}{\partial z}\left(\beta\left(\sqrt{A} - \sqrt{A_0}\right)\right)\\ &= \frac{d\beta}{dz} A^{\frac{1}{2}} + \frac{\beta}{2} A^{-\frac{1}{2}} \frac{\partial A}{\partial z} - \frac{d\beta}{dz} A_0^{\frac{1}{2}}.\end{aligned}$$

If we multiply both sides of the previous equation with A/ρ , we get

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$$\frac{A}{\rho}\frac{\partial p}{\partial z}=\frac{d\beta}{dz}\frac{A^{\frac{3}{2}}}{\rho}+\frac{\beta}{2}\frac{A^{\frac{1}{2}}}{\rho}\frac{\partial A}{\partial z}-\frac{A}{\rho}\frac{d\beta}{dz}A_0^{\frac{1}{2}}. \tag{5}$$

We note that β is a function of z, and A is a function of z and t. Therefore, we have

$$\frac{\partial}{\partial z}\left(\frac{\beta A^{\frac{3}{2}}}{3\rho}\right)+\frac{A}{\rho}\frac{d\beta}{dz}\left(\frac{2}{3}\sqrt{A}-\sqrt{A_0}\right)=\frac{d\beta}{dz}\frac{A^{\frac{3}{2}}}{\rho}+\frac{\beta}{2}\frac{A^{\frac{1}{2}}}{\rho}\frac{\partial A}{\partial z}-\frac{A}{\rho}\frac{d\beta}{dz}A_0^{\frac{1}{2}}. \tag{6}$$

From equations (5) and (6), we get

$$\frac{A}{\rho}\frac{\partial p}{\partial z}=\frac{\partial}{\partial z}\left(\frac{\beta A^{\frac{3}{2}}}{3\rho}\right)+\frac{A}{\rho}\frac{d\beta}{dz}\left(\frac{2}{3}\sqrt{A}-\sqrt{A_0}\right). \tag{7}$$

Based on equation (7), equation (2) can be rewritten as

$$\frac{\partial Q}{\partial t}+\frac{\partial}{\partial z}\left(\frac{Q^2}{A}+\frac{\beta}{3\rho}A^{\frac{3}{2}}\right)=-K_R\frac{Q}{A}+\frac{A}{\rho}\frac{d\beta}{dz}\left(\sqrt{A_0}-\frac{2}{3}\sqrt{A}\right). \tag{8}$$

From the above derivation, blood flow model (1)-(2) are balance laws in the form of

$$\bar{v}_t+\bar{f}\,(\bar{v})_z=\bar{s}(\bar{v}) \tag{9}$$

where the conserved quantities, flux functions, and source terms are

$$\bar{v}=\begin{bmatrix}A\\Q\end{bmatrix},\ \bar{f}(\bar{v})=\begin{bmatrix}Q\\\frac{Q^2}{A}+\frac{\beta}{3\rho}A^{\frac{3}{2}}\end{bmatrix},\text{ and }\bar{s}(\bar{v})=\begin{bmatrix}0\\-K_R\frac{Q}{A}+\frac{A}{\rho}\frac{d\beta}{dz}\left(\sqrt{A_0}-\frac{2}{3}\sqrt{A}\right)\end{bmatrix}$$

respectively.

Using the finite volume framework, we assume that $\bar{V}_i^n\approx\bar{v}(z_i,t^n)$, $\bar{f}(\bar{V}_i^n)\approx\bar{f}\big(\bar{v}(z_i,t^n)\big)$, and $\bar{S}_i^n\approx\bar{s}(\bar{v}(z_i,t^n))$. We define the following vectors

$$\bar{V}_i^n=\begin{bmatrix}A_i^n\\Q_i^n\end{bmatrix},\ \bar{f}(\bar{V}_i^n)=\begin{bmatrix}Q_i^n\\\frac{(Q_i^n)^2}{A_i^n}+\frac{\beta}{3\rho}(A_i^n)^{\frac{3}{2}}\end{bmatrix}\text{ and }\bar{S}_i^n=\begin{bmatrix}0\\-K_R\frac{Q_i^n}{A_i^n}+\frac{A_i^n}{\rho}\frac{d\beta}{dz}\left(\sqrt{A_0}-\frac{2}{3}\sqrt{A_i^n}\right)\end{bmatrix}.$$

The fully discrete finite volume method [9] to get the numerical solution to balance laws (9) is

$$\bar{V}_i^{n+1}=\bar{V}_i^n-\frac{\Delta t}{\Delta z}\left(\bar{F}_{i+\frac{1}{2}}^n-\bar{F}_{i-\frac{1}{2}}^n\right)+\Delta t\,\bar{S}_i^n, \tag{10}$$

with the definition of the Lax–Friedrichs flux

$$\bar{F}_{i+\frac{1}{2}}^n=\frac{\bar{f}(\bar{V}_{i+1}^n)+\bar{f}(\bar{V}_i^n)}{2}-\frac{\Delta z}{2\Delta t}(\bar{V}_{i+1}^n-\bar{V}_i^n), \tag{11}$$

and

$$\bar{F}_{i-\frac{1}{2}}^n=\frac{\bar{f}(\bar{V}_i^n)+\bar{f}(\bar{V}_{i-1}^n)}{2}-\frac{\Delta z}{2\Delta t}(\bar{V}_i^n-\bar{V}_{i-1}^n). \tag{12}$$

From the finite volume scheme (10)-(12), we get the numerical scheme for blood flow model (1)-(2) as follows. The numerical scheme for equation (1) is

$$A_i^{n+1}=A_i^n-\frac{\Delta t}{\Delta z}\left(F_{i+\frac{1}{2}}^n-F_{i-\frac{1}{2}}^n\right), \tag{13}$$

with the definition of Lax–Friedrichs fluxes

$$F_{i+\frac{1}{2}}^n=\frac{1}{2}(Q_{i+1}^n+Q_i^n)-\frac{\Delta z}{2\Delta t}(A_{i+1}^n-A_i^n), \tag{14}$$

and

$$F_{i-\frac{1}{2}}^n=\frac{1}{2}(Q_i^n+Q_{i-1}^n)-\frac{\Delta z}{2\Delta t}(A_i^n-A_{i-1}^n). \tag{15}$$

In addition, the numerical scheme for equation (2) is

$$Q_i^{n+1}=Q_i^n-\frac{\Delta t}{\Delta z}\left(\mathcal{F}_{i+\frac{1}{2}}^n-\mathcal{F}_{i-\frac{1}{2}}^n\right)+\Delta t\left(-K_R\frac{Q_i^n}{A_i^n}+\frac{A_i^n}{\rho}\frac{d\beta}{dz}\left(\sqrt{A_0}-\frac{2}{3}\sqrt{A_i^n}\right)\right), \tag{16}$$

with the definition of Lax–Friedrichs fluxes

$$\mathcal{F}^n_{i+\frac{1}{2}} = \frac{1}{2}\left[\frac{(Q^n_{i+1})^2}{A^n_{i+1}} + \frac{\beta}{3\rho}(A^n_{i+1})^{\frac{3}{2}} + \frac{(Q^n_i)^2}{A^n_i} + \frac{\beta}{3\rho}(A^n_i)^{\frac{3}{2}}\right] - \frac{\Delta z}{2\Delta t}(Q^n_{i+1} - Q^n_i)\,, \tag{17}$$

and

$$\mathcal{F}^n_{i-\frac{1}{2}} = \frac{1}{2}\left[\frac{(Q^n_i)^2}{A^n_i} + \frac{\beta}{3\rho}(A^n_i)^{\frac{3}{2}} + \frac{(Q^n_{i-1})^2}{A^n_{i-1}} + \frac{\beta}{3\rho}(A^n_{i-1})^{\frac{3}{2}}\right] - \frac{\Delta z}{2\Delta t}(Q^n_i - Q^n_{i-1})\,. \tag{18}$$

We are now ready to present our results of numerical simulations of the blood flow model.

4. Numerical simulation

In this section, we investigate how a pulse propagates in human artery. Numerical settings, initial and boundary conditions are taken as follow.

4.1. Numerical settings

For this numerical simulation, we take $l = 15$ cm and $t \in [0,0.035]$ second. Figure 3 shows the layout of numerical simulations. We set E as a constant, so it implies β is a constant. It means that the value of E and β do not change for all $z \in (0, l)$. So, the value of $d\beta/dz$ equals to zero. And then, we take $\Delta z = 0.005$, and $\Delta t = 0.002\Delta z$. As long as Δt satisfies the Courant–Friedrichs–Lewy (CFL)'s condition, the method will be stable.

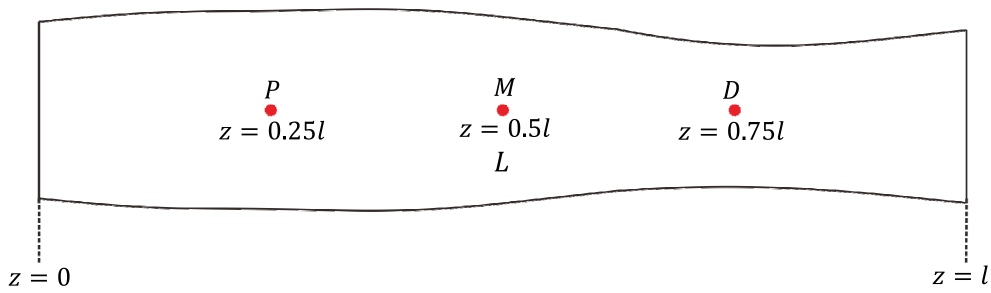

Figure 3. A layout of human artery for numerical simulation.

We investigate at the three monitoring points (*P*, *M*, and *D*) for the pressure variations, where points *P*, *M*, and *D* are proximal, medium, and distal points, respectively. Consider that point *P* is the nearest point from the heart and point *D* is the farthest point from the heart. Table 1 shows the values of coefficients that we use in all simulations.

Table 1. Coefficients value for numerical simulation.

Coefficient	Value
Blood density ρ	1 g/ cm^3
Young's modulus E_0	3x10^6 dyne/ cm^2
Wall artery thickness h	0.05 cm
Initial cross section area A_0	$\pi 0.5^2$ cm^2
Blood viscosity K_R	10^{-8} dyne s/ cm^2

4.2. Initial and boundary conditions

Given the initial values of $A(z,0) = A_0$, $Q(z,0) = 0$, and $p(z,0) = 0$, for every $z \in (0,15)$. The boundary values for this simulation is taken as follows. At the left boundary, we give an impluse input in the form of a single sine wave with a small time period:

$$p(0,t) = 10^3 \cdot \sin\left(\frac{\pi t}{0.0025}\right). \tag{19}$$

For the left boundary value of A and Q, let us consider the characteristic variables of model (1)-(2) which has been explained in [2]: W_1 and W_2 are given by

$$W_2 = \frac{Q}{A} - 2\sqrt{\frac{2\beta}{\rho}} A^{\frac{1}{4}}, \tag{20}$$

$$W_1 = W_2 + 4\sqrt{\frac{2}{\rho}}\left(\sqrt{p + \beta\sqrt{A_0}}\right), \tag{21}$$

$$A = \left(\frac{\rho}{\beta}\right)^2 \frac{(W_1 - W_2)^4}{4^5}, \tag{22}$$

$$Q = A\frac{W_1 + W_2}{2}. \tag{23}$$

We set W_2 as a constant and equals to its initial value. And the last, for each right boundary value of A, p and Q equals to the corresponding value of the nearest neighbour in the domain.

4.3. Numerical results

In this subsection, we summarise our numerical results.

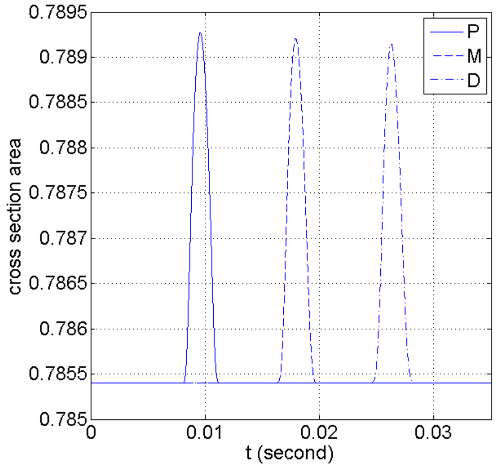

Figure 4. Graphics of the artery cross section area with respect to time.

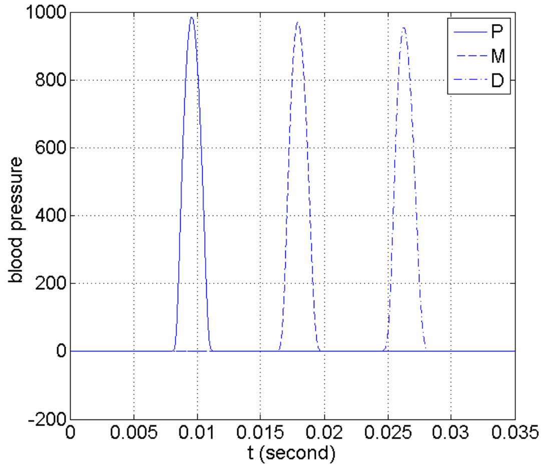

Figure 5. Graphics of the blood presure with respect to time.

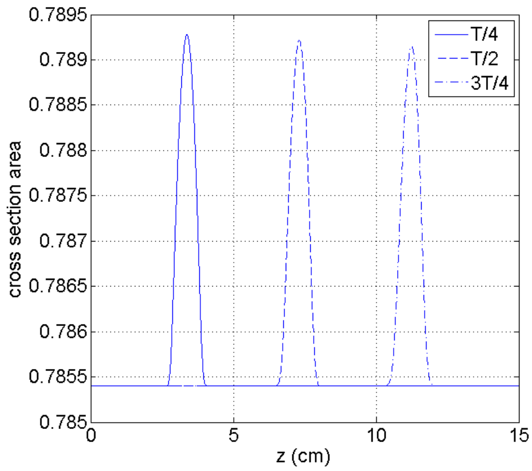

Figure 6. Graphics of the artery cross section area with respect to space.

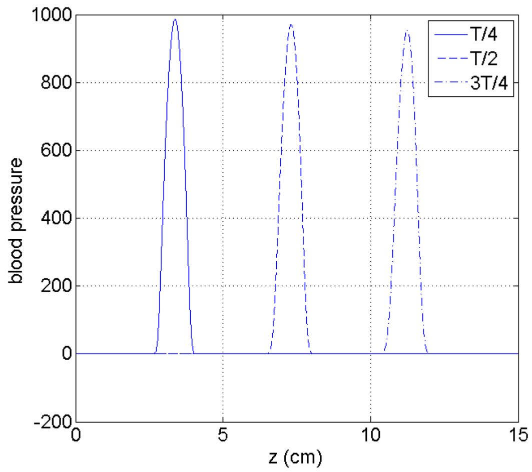

Figure 7. Graphics of the blood pressure with respect to space.

For this numerical simulation, where the elastic Young's modulus is constant, the blood pressure is directly propotional with the artery cross section area (see Figure 4 and Figure 6). Furthermore, the blood pressure amplitude decreases with respect to time and space (see Figure 5 and Figure 7). We infer that the decrease is influenced by the dissipation of the numerical method. We note that as the cell width is taken smaller, the amplitude can be maintained almost the same as the original one, as long as the solution is continuous.

5. Conclusion

Based on the numerical results, the form of pulse does not change but the amplitude decreases with respect to time and space due to numerical dissipation. The dissipation can be small if the numerical cell width is small. This research is limited to problems solved using the first order Lax–Friedrichs finite volume method. Future work could extend the method to higher order ones.

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