

# Runge-Kutta and rational block methods for solving initial value problems

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**Abstract.** Three methods to solve initial value problems are considered. The methods are the first order Euler's, second order Heun's, and rational block methods. The Euler's and Heun's methods are of the Runge-Kutta type. Numerical results show that the rational block method is more robust than Runge-Kutta type methods in solving initial value problems.

## 1. Introduction

Mathematical models are very useful to solve real problems [1-4]. A class of mathematical models is ordinary differential equation (ODE). To solve ODEs in practice, we must have an initial condition and they form an initial value problem. There are a number of methods to solve initial value problems, such as Runge-Kutta methods and rational block methods.

In this paper, we solve initial value problems using the first order Runge-Kutta (Euler's) method, second order Runge-Kutta (Heun's) method and rational block method. Runge-Kutta methods are standard in the fields of numerical ODEs. The rational block method is a combination of rational methods [5]. In this work, we test the performance of the rational block method in comparison to Runge-Kutta methods. We note that the rational block method calculates approximate values of solution at two points in each iteration [6-9]. Runge-Kutta methods calculate approximate values of solution at one point in each iteration. A robust ODE solver is needed in the process of finding a more difficult problem, such as solving partial differential equations numerically [10-14].

The paper is structured as follows. We describe the problem formulation of the rational block method in Section 2. All three numerical methods to be tested are written in Section 3. Numerical results are presented in Section 4. The paper is concluded with some remarks in Section 5.

## 2. Problem formulation

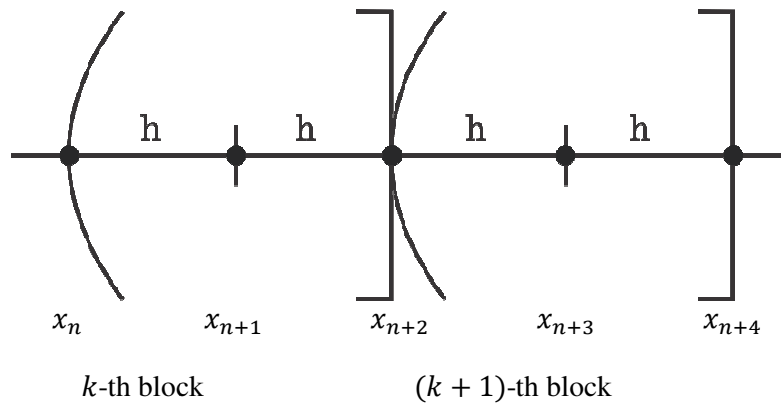
We recall the general formulation of initial value problems using the rational block method [6-8].

Given the initial value problem:

$$y'(x) = f(x, y), \quad y(a) = \eta \quad (1)$$

where  $f(x, y): \mathbb{R} \times \mathbb{R}^m \rightarrow \mathbb{R}^m$  and the initial value problem (1) has a unique solution. In this paper we want to solve the initial value problem from the starting point to the final point. Therefore, we have to form the interval of integration. Let  $x \in [x_0, x_b]$  and the interval  $[x_0, x_b]$  is discretised for the numerical integration as  $\{x_0, x_1, \dots, x_n, x_{n+1}, \dots, x_b\} \subset \mathbb{R}$ .





**Figure 1.** Discretisation of a 2-point explicit rational block method.

The discretisation in the rational block method makes a series of blocks with each point of these blocks is separated by a constant of step-size  $h$ , as shown in Figure 1 for a 2-point explicit rational block method. The  $k$ -th block contains three points, namely,  $x_n$ ,  $x_{n+1}$  and  $x_{n+2}$ . We use  $y_n$  at  $x_n$  to calculate the approximate value  $y_{n+1}$ . Then, we use  $y_n$  and  $y_{n+1}$  to calculate the approximate value  $y_{n+2}$ . On this block, the calculation process is run in one iteration. That is, the rational block method calculates the approximate values  $y_{n+1}$  and  $y_{n+2}$  simultaneously in one iteration. Likewise, the  $(k + 1)$ -th block contains three points, namely,  $x_{n+2}$ ,  $x_{n+3}$  and  $x_{n+4}$ . We use  $y_{n+2}$  at  $x_{n+2}$  to calculate the approximate value  $y_{n+3}$ . Then, we use  $y_{n+2}$  and  $y_{n+3}$  to calculate the approximate value  $y_{n+4}$ . Like in the  $k$ -th and  $(k + 1)$ -th blocks, we calculate the  $y$  value on the next blocks until at the point  $x = b$  is reached.

The discrete values of  $x$  are given by  $x_n$ ,  $x_{n+1}$  and  $x_{n+2}$  as

$$x_n = x_0 + nh \quad (2)$$

$$x_{n+1} = x_0 + (n + 1)h = x_n + h \quad (3)$$

$$x_{n+2} = x_0 + (n + 2)h = x_n + 2h \quad (4)$$

where  $h = \frac{x_b - x_0}{N}$  is a step-size or the distance from a point to the next one,  $N$  is the number of integration steps.

In rational block method, the solution of initial value problem (1) is locally given in the interval  $[x_n, x_{n+1}]$  by the rational approximant

$$R(x) = \frac{a_0 + a_1 x}{b_0 + x} \quad (5)$$

where  $a_0$ ,  $a_1$  and  $b_0$  are unknown coefficients. In the calculation process, this rational approximant (5) passes through points  $(x_n, y_n)$  and  $(x_{n+1}, y_{n+1})$ . To obtain the  $y_{n+1}$  value, we assume that the derivative values at  $x_n$  is given by  $y'_n = f(x_n, y_n)$  and  $y''_n = f'(x_n, y_n)$ . Therefore, four equations must be satisfied as follows:

$$y_n = \frac{a_0 + a_1 x_n}{b_0 + x_n}, \quad (6)$$

$$y_{n+1} = \frac{a_0 + a_1 x_{n+1}}{b_0 + x_{n+1}}, \quad (7)$$

$$f_n = \frac{a_1 b_0 - a_0}{(b_0 + x_n)^2}, \quad (8)$$

$$f'_n = \frac{-2(a_1 b_0 - a_0)}{(b_0 + x_n)^3}. \quad (9)$$

Equations (6)-(9) contain three unknown coefficients  $a_0$ ,  $a_1$  dan  $b_0$ . Eliminating these three unknown coefficients, we have

$$y_{n+1} = y_n + \frac{2 h(f_n)^2}{2 f_n - h f'_n}. \quad (10)$$

Equation (10) is a one-step second order rational method (see Lambert [5] for details). This method is a formula to calculate the approximate value  $y_{n+1}$  by using previous information at point  $(x_n, y_n)$ .

Furthermore, to calculate the approximate value  $y_{n+2}$ , we once again assume that the approximate solution to problem (1) is locally given in the interval  $[x_n, x_{n+2}]$  by the rational approximant (5). In the calculation process, this rational approximant in equation (5) passes through points  $(x_n, y_n)$ ,  $(x_{n+1}, y_{n+1})$  and  $(x_{n+2}, y_{n+2})$ . The derivative values at  $x_n$  and  $x_{n+1}$  are given by  $y'_n = f(x_n, y_n)$  and  $y'_{n+1} = f(x_{n+1}, y_{n+1})$ . Therefore, five equations must be satisfied as follows:

$$y_n = \frac{a_0 + a_1 x_n}{b_0 + x_n} \quad (11)$$

$$y_{n+1} = \frac{a_0 + a_1 x_{n+1}}{b_0 + x_{n+1}} \quad (12)$$

$$y_{n+2} = \frac{a_0 + a_1 x_{n+2}}{b_0 + x_{n+2}} \quad (13)$$

$$f_n = \frac{a_1 b_0 - a_0}{(b_0 + x_n)^2} \quad (14)$$

$$f_{n+1} = \frac{a_1 b_0 - a_0}{(b_0 + x_{n+1})^2} \quad (15)$$

We notice that equations (11)-(15) contain four unknown coefficients  $a_0$ ,  $a_1$ ,  $b_0$  and  $f_n$ . Eliminating these four unknown coefficients, we have

$$y_{n+2} = y_{n+1} + \frac{h f_{n+1}(y_{n+1} - y_n)}{2(y_{n+1} - y_n) - h f_{n+1}}. \quad (16)$$

Equation (16) is a two-step third order rational method (see Lambert [5] for details). This method is a formula to calculate the approximate value  $y_{n+2}$  by using previous information at points  $(x_n, y_n)$  and  $(x_{n+1}, y_{n+1})$ .

The calculation of the rational block method is based on the rational approximant (5). By the elimination of unknown coefficients, we have formulas (10) and (16). These formulas are used to find

the approximate values  $y_{n+1}$  and  $y_{n+2}$ . If  $y_n$  value is known, the rational block method calculates the approximate value  $y_{n+1}$  using formula (10), and then it calculates the approximate value  $y_{n+2}$  using formula (16). This means that, in each block, the rational block method obtains the  $y_{n+1}$  and  $y_{n+2}$  values in one iteration.

### 3. Numerical method

We write three methods to solve the initial value problem (1). The methods are the first order Runge-Kutta (Euler's method), the second order Runge-Kutta (Heun's method) and the rational block method.

The first order Runge-Kutta (Euler's method) [1] has the iteration formula:

$$y_{n+1} = y_n + h f_n.$$

Here  $h$  is the step size.

The second order Runge-Kutta (Heun's method) [1] has the iteration formula:

$$y_{n+1} = y_n + \frac{h}{2}(k_1 + k_2),$$

where

$$k_1 = f(x_n, y_n),$$

and

$$k_2 = f(x_n + h, y_n + h k_1).$$

The rational block method is given by the iteration formulas:

$$y_{n+1} = y_n + \frac{2 h (f_n)^2}{2 f_n - h f'_n},$$

$$y_{n+2} = y_{n+1} + \frac{h f_{n+1}(y_{n+1} - y_n)}{2 (y_{n+1} - y_n) - h f_{n+1}}.$$

We note that the rational block method is a two-step method.

### 4. Numerical results

In this section, we give some examples to assess the performance of the Runge-Kutta (Euler's and Heun's) and rational block methods. We want to see the maximum error of each method with the definition of the error is as follows

$$\text{error} = \max_{0 \leq n \leq N} \{|y_{\text{exact}} - y_{\text{numeric}}|\}$$

where  $N$  is the number of integration steps. For computation, we use the Matlab software for numerical programming and plotting the results. As follows, we have three problems considered by Ying *et al.* [7] to solve.

#### Problem 1

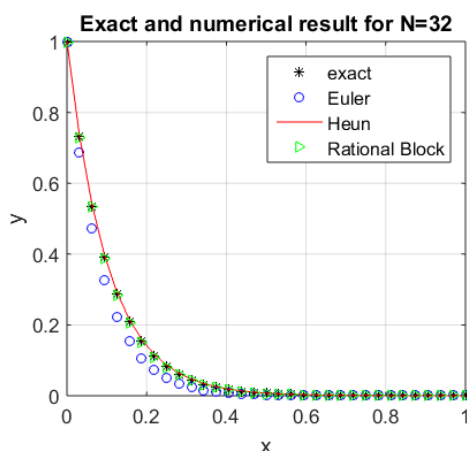
Consider the initial value problem

$$y'(x) = -10 y(x), \quad y(0) = 1, \quad x \in [0, 1].$$

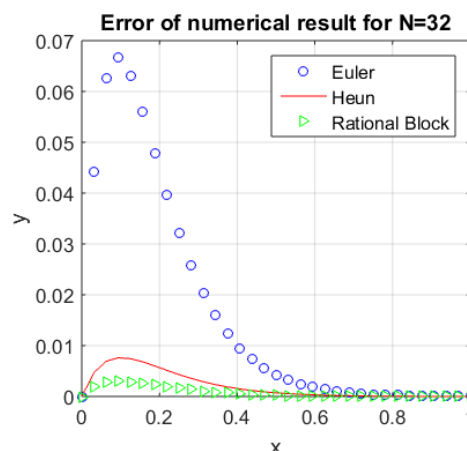
The exact solution is given by  $y(x) = e^{-10x}$ .

**Table 1.** Maximum errors for Problem 1.

$N$	Euler	Heun	Rational block
32	0.066654	0.007616	0.003021
64	0.030792	0.001683	0.000749
128	0.014851	0.000397	0.000187
256	0.007304	0.000096	0.000047



**Figure 2.** Exact and numerical results for Problem 1.



**Figure 3.** Errors of numerical results for Problem 1.

As given in Table 1 we obtain that the maximum error of the rational block method is smaller than the Euler and Heun methods. All three methods are convergent. That is, if the value of  $N$  is greater, the error of the numerical solution gets smaller. From Table 1, Figure 2, and Figure 3 we find that the numerical results are accurate, as numerical solutions are close to the exact solution. The error of each method is less than 0.07. We infer that all three numerical methods solve the problem quite well.

### Problem 2

Consider the initial value problem as follows:

$$y''(x) + 101 y'(x) + 100 y(x) = 0, \quad y(0) = 1.01, \quad y'(0) = -2, \quad x \in [0, 1].$$

This problem can be rewritten as a system of first order ordinary differential equations:

$$y'(x) = z(x), \quad y(0) = 1.01,$$

$$z'(x) = -100 y(x) - 101 z(x), \quad z(0) = -2,$$

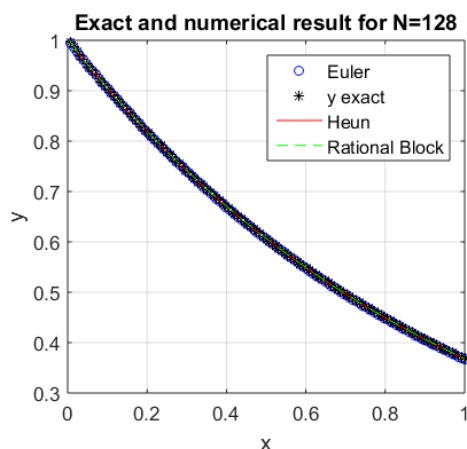
where  $x \in [0, 1]$ . The exact solutions to this system are given by:

$$y(x) = 0.01 e^{-100x} + e^{-x},$$

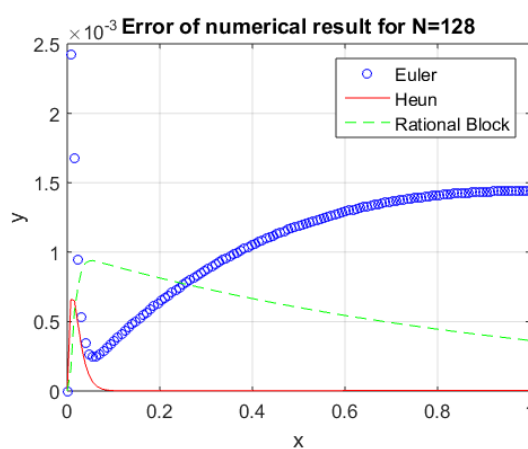
$$z(x) = y' = -e^{-100x} - e^{-x}.$$

**Table 2.** Maximum errors for Problem 2.

$N$	Euler	Heun	Rational block
32	298872461.45	$1.253348 \times 10^{12}$	0.017842
64	0.007843	0.004487	0.003982
128	0.002421	0.000661	0.000940
256	0.000880	0.000126	0.000233



**Figure 4.** Exact and numerical results for Problem 2.



**Figure 5.** Errors of numerical results for Problem 2.

From Table 2, we observe that if the number of discrete points is too small, such as  $N = 32$  (that is, the step size is too large) the Euler's method and Heun's method are not stable, so their numerical errors are very large. However, the Euler's error and Heun's error get smaller for larger number of discrete points. For this problem, the Heun's method performs best giving smallest error for large number of discrete points. All three methods are convergent. For this problem, illustration of exact and numerical results as well as their numerical errors are shown in Figure 4 and Figure 5, respectively.

### Problem 3

Consider the following initial value problem:

$$y'(x) = 1 + y(x)^2, \quad y(0) = 1, \quad x \in [0, 1].$$

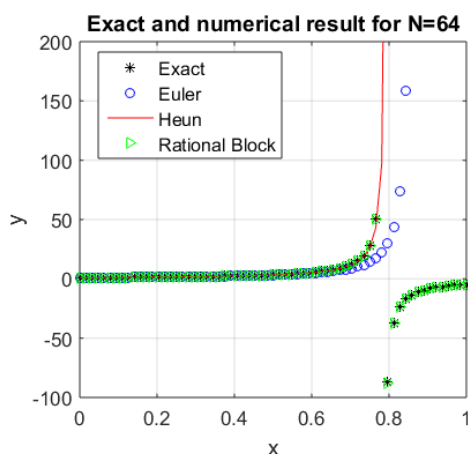
The exact solution to this problem is given by

$$y(x) = \tan \left( x + \frac{\pi}{4} \right).$$

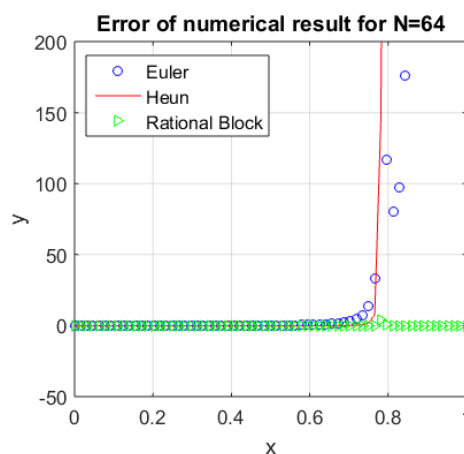
This exact solution has a singularity point.

**Table 3.** Maximum errors for Problem 3. Here inf stands for infinity.

$N$	Euler	Heun	Rational Block
32	186471279.48	inf	13.92
64	inf	inf	3.64
128	inf	inf	1.20
256	inf	inf	67.13



**Figure 6.** Exact and numerical results for Problem 3.



**Figure 7.** Errors of numerical results for Problem 3.

We have noticed that the solution has a singularity point at  $x = \pi/4 \approx 0.7854$ . Due to the singularity of the problem, Runge-Kutta methods are divergent, as indicated in Table 3. That is, the Euler's and Heun's methods are not able to solve this problem. In contrast, the rational block method is still able to solve this problem, even though its approximate value is very not accurate. For this problem, illustration of exact and numerical results as well as their numerical errors are shown in Figure 6 and Figure 7, respectively.

## 5. Conclusion

We have solved initial value problems using the first order Runge-Kutta, second order Runge-Kutta, and rational block methods. The rational block method is fast in computation, because this method can solve two points in one iteration. From numerical results, we find that rational block method is more robust than the other two methods. That is, the rational block method is able to solve a wider range of problems including those with singularity in their solutions.

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