

Analysis of wave equation in electromagnetic field by Proca equation

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Abstract. This research is aimed to analyze wave equation for the electric and magnetic field, vector and scalar potential, and continuity equation using Proca equation. Then, also analyze comparison of the solution on Maxwell and Proca equation for scalar potential and electric field, both as a function of distance and constant wave number.

1. Introduction

At the beginning of the 19th century was found a variety of elementary particles such as electrons, protons, neutrons, etc. which had a mass of relatively small[12]. Quantum mechanics describes the statistical behavior of the particles by analyzing the formula of energy and wave function[8]. Energy and wave function of particles can be obtained by solving Schrödinger equation directly[13]. Quantum field theory describes the behavior of particles which have relatively high speed and small mass[11]. Proca equation can also be combined with Maxwell equation to describe behavior of photons[2] and massive photons[21][4]. Bose-Einstein gravity condensate of particles which have spin 1 also can be explained by modifications of Proca equation[3]. Electromagnetic consists of the electric and magnetic fields that can spread in the vacuum in the universe[9].

Proca equation was used in various of teoritical studies, such as; non-Abelian supersymmetric N=1[10], solutions of Dirac-Proca equation[20], local gauge invariance test in electrodynamics[22], to explain the phenomena in the universe, namely the Tachyonic Cherenkov spectrum radiation[19] and Tachyonic synchrotron radiation[18] from relativistic thermal electrons in Jupiter's belt radiations[16]. Radiation was also detected at the Crab Nebula or explosion of supernova[15] and detection of Tachyonic gamma ray at the galactic core[17]. Proca equation is also used for basic research to obtained *GravitoElectroMagnetic* (GEM) equation[6] and London-Proca-Hirsch equation[5] for superconducting case in electrodynamics field[14].

2. Tensor Fields and Quantum Field Theory

Mathematical equation in tensor field using Lorentz transformations explains the case of relativistic particles[7]. According to Griffiths (1999) the relativistic case explain that the electric and magnetic fields are part of the electromagnetic field tensor $F^{\mu\nu}$ and $G^{\mu\nu}$ that is shown in equation (1) and (2) below:



$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix} \quad (1)$$

$$G^{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{pmatrix} \quad (2)$$

Equation (1) and (2) can be written in a general form, namely as a form 4-vectors of Maxwell's equation shown by equation (3) and (4) below:

$$\frac{\partial F^{\mu\nu}}{\partial x^\nu} = \mu_0 J^\mu \quad (3)$$

$$\frac{\partial G^{\mu\nu}}{\partial x^\nu} = 0 \quad (4)$$

Equation (3) is called inhomogeneous Maxwell's equation, whereas equation (4) is called homogeneous Maxwell's equation. Inhomogeneous and homogeneous Maxwell's equation can be reduced to Ampere, Gauss, and Faraday equations by means of varying the value of μ . If the value of μ in equation (3) is zero thus obtained form of the Gauss equation that describes the divergence of the electric field caused by the density of charge at a point. Gauss equation can be shown by equation (5) as:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \quad (5)$$

If the value of μ in equation (3) are 1, 2, 3 thus obtained equation (3) can be expressed in 3-vector form as an Ampere equation as: $\nabla \times \mathbf{B} - \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J}$. Ampere equation explains that the change of the electric field and current density will produce a magnetic field.

If the value of μ in equation (4) is zero, then we obtained Gauss equation for the magnetic field as:

$$\nabla \cdot \mathbf{B} = 0 \quad (6)$$

If the value of μ in equation (4) are 1, 2, 3 thus we obtained the formula which is expressed in 3-vector form of Faraday equation as: $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$. Faraday equation explain the relationship between the electric and magnetic fields, the change of the magnetic field will produce an electric field.

3. Proca Equation

Proca equation explain the behavior of particles with spin 1. A photon is one of particles that have spin 1[1]. The photon is also the electromagnetic waves that describes by Maxwell equation[8]. According Poenaru (2012) the electromagnetic field to massive photon is described by Proca equation shown in equation (7), (8), (9), and (10) as:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} - \mu_\gamma^2 \phi \quad (7)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (8)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (9)$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} - \mu_\gamma^2 \mathbf{A} \quad (10)$$

In a Proca equation, $\mu_\gamma^{-1} = \hbar / (m_\gamma c)$ describes the Compton's wavelength of the photon with mass m_γ [12]. According to Aquino (2011), a photon has imaginary mass which is expressed by the equation (11) below:

$$m_\gamma = \frac{2}{\sqrt{3}} \left(\frac{\hbar f}{c^2} \right) i \quad (11)$$

Equation (11) states that photon has imaginary mass which expressed by m_γ . Quantitation of imaginary mass can be explained by a factor of Planck's constant h . Equation (11) is identical to the

de-Broglie equation which expressed $\lambda = h/p$ that describes the correlation between mass and wavelength. De-Broglie equation describes the correlation between the mass of material and wavelength. Linear momentum p is the value of massive particles that moving at a certain speed. The linear momentum which is described by $p = mv$ in classical mechanics. Proca equation have additional correction factor magnetic field potential A and mass factor μ_γ^2 . The value of $\mu_\gamma = \frac{m_\gamma c}{\hbar}$ describes the imaginary mass of the photon. The real mass of the photon will be explained in equation (12) below:

$$\mu_\gamma^2 = \frac{4}{3} \left(\frac{2\pi}{\lambda} \right)^2 \quad (12)$$

Equation (12) is explain is a constant value of wave propagation that described by $\frac{2\pi}{\lambda}$. Wave propagation is defined by $k = \frac{2\pi}{\lambda}$, so that equation (12) becomes:

$$\mu_\gamma^2 = \frac{4}{3} k^2 \quad (13)$$

Substituting equation (13) into Proca equation, so that equation (7), (8), (9), and (10) becomes:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} - \frac{4}{3} k^2 \phi \quad (14)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (15)$$

$$\nabla \times \mathbf{E} = - \frac{\partial \mathbf{B}}{\partial t} \quad (16)$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} - \frac{4}{3} k^2 \mathbf{A} \quad (17)$$

Proca equation does not contradict with the classical theory of Maxwell in electrodynamics. If the value of imaginary mass was taken zero ($m_\gamma = k = 0$) then Proca equation will return to common form of Maxwell equation. Generally form of 4-dimensional form of Proca equation can be expressed in equation (18) below:

$$\left(\nabla^2 - \frac{1}{c} \frac{\partial^2}{\partial t^2} - \frac{4}{3} k^2 \right) \mathbf{A}_\mu = -\mu_0 \mathbf{J}_\mu \quad (18)$$

Vector \mathbf{A}_μ and \mathbf{J}_μ is 4-vector magnetic potential that expressed by $(\mathbf{A}, i \frac{\phi}{c})$ and current density that expressed as $(\mathbf{J}, i \frac{\rho}{c})$. If equation (18) applied in vacuum that is assumed there is no charge ($\mathbf{J}_\mu = 0$) then equation (18) as:

$$\left(\nabla^2 - \frac{1}{c} \frac{\partial^2}{\partial t^2} - \frac{4}{3} k^2 \right) \mathbf{A}_\mu = 0 \quad (19)$$

Equation (19) looks like to wave equation in general form. If it is considered existence of photons is in point of charge and with the use of separation variable then equation (19) can be written as:

$$\left(\nabla^2 - \frac{1}{c} \frac{\partial^2}{\partial t^2} - \frac{4}{3} k^2 \right) \phi = -\frac{\rho}{\epsilon_0} \quad (20)$$

From equation (20) we get solution of scalar potential as space function in Proca's case given as:

$$\phi(r) = \frac{1}{4\pi\epsilon_0} \frac{q}{r} e^{-\frac{2}{\sqrt{3}}kr} \quad (21)$$

Equation (21) looks like scalar potential named Coulomb's potential $\phi(r) = \frac{1}{4\pi\epsilon_0} \frac{q}{r}$ in Maxwell's case.

The different between Proca and Maxwell about scalar potential will be explained using graph shown in Fig. 4.1:

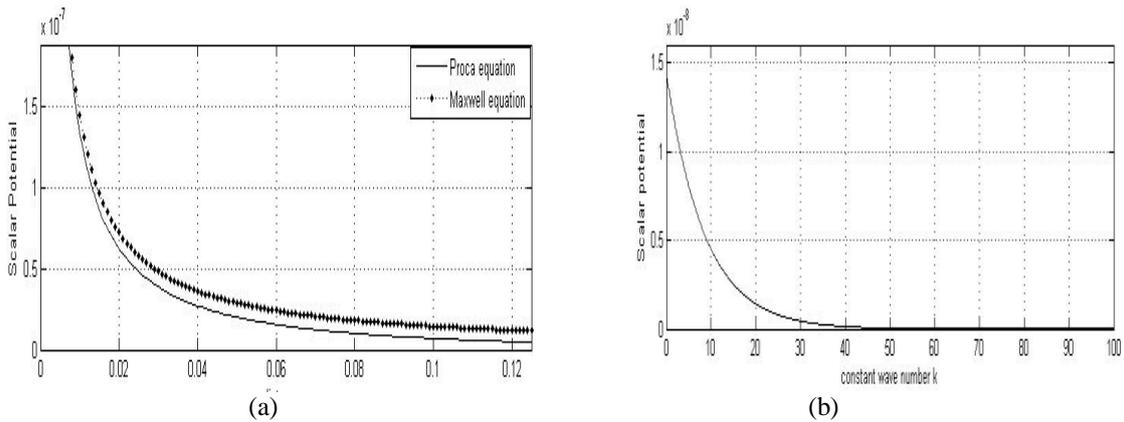


Figure 4.1 Graph of scalar potential as function: (a) distance, (b) constant wave number in Proca equation

Graph (a) in figure 4.1 showed the different of value of scalar potential between Proca and Maxwell for distance variation. It is caused by factor of mass from the photon which described by $e^{-\frac{2}{\sqrt{3}}kr}$ in equation (21). At the distance that is close to zero then the scalar potential of Proca or Maxwell shown the same value. But, by increasing the distance will cause the difference in value of scalar potential between Maxwell and Proca equation. This indicates a correction value of Proca equation for a relatively great distance. While the graph (b) in figure 4.1 showed the graph of a scalar potential for variation of constant wave number in equation (21). If the value of $k=0$ then the value of a scalar potential will be shown like Coloumb potential. Reduction of equation (21) against to function of space will produce the value of electric field E which is expressed by equation (22) below:

$$E(r) = \frac{q}{4\pi\epsilon_0 r^2} \left(1 + \frac{2}{\sqrt{3}}kr\right) e^{-\frac{2}{\sqrt{3}}kr} \tag{22}$$

If the value of $k=0$ then equation (22) will reduce to Maxwell equation for electric field as $(E(r) = \frac{q}{4\pi\epsilon_0 r^2})$. To describe visually of electric field in Proca and Maxwell equation (22) can be plotted graph as a function of distance and constant wave number shown in Fig. 4.2:

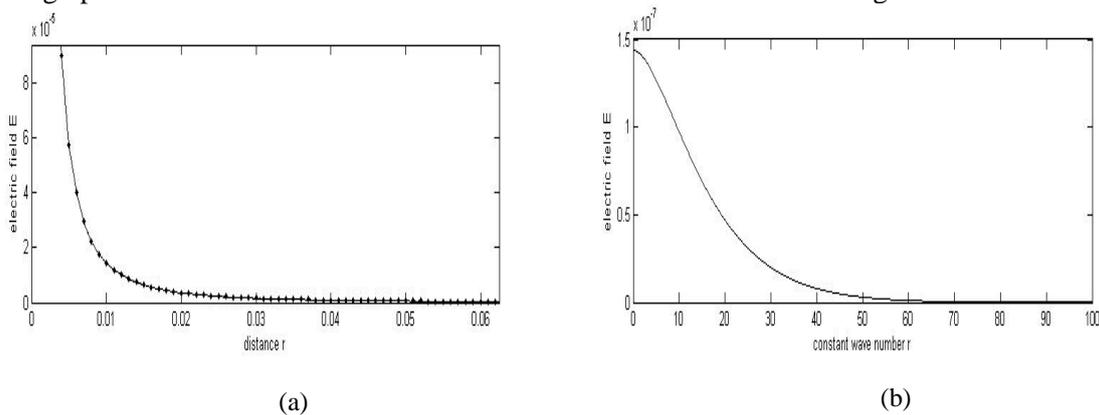


Figure 4.2 Graph electric field as function of: (a)distance and (b)constant wave number in Proca equation

The graph (a) in figure 4.2 is the graph of a electric field as function of space for Proca and Maxwell equation. The graph (a) explain that Maxwell or Proca equation have the same value of the electric field. This is due to the small effect of the value $\left(1 + \frac{2}{\sqrt{3}}kr\right) e^{-\frac{2}{\sqrt{3}}kr}$ in equation (22). The graph (b) in figure 4.2 is the graph of a electric field as function of the constant wave number for Proca equation. If equation (22) have value $k = 0$ then it would be reduced to Coloumb equation for the point charge that shown by $(\mathbf{E}(r) = \frac{q}{4\pi\epsilon_0 r^2})$. This indicates to ignore the presence of a photon. But if a presence of the photon is considered ($k \neq 0$) that describes of a effect of the massive photon on the value of a electric field.

4. Effect of Proca equation in Electromagnetic equation

Proca equation describes the massive photon in electrodynamics [1]. Therefore, Proca equation affects to the equations in electrodynamics cases. Here is an explanation of some of the equation in electrodynamics field that caused of Proca equation:

4.1. Wave equation for electric and magnetic fields

Wave equation for Proca equation can be traced from the electric field \mathbf{E} and magnetic field \mathbf{B} . Wave equation for electric field \mathbf{E} is derived from Proca equation, equations (9) and (10). By operating the *curl* operator to equation (9), we get:

$$\nabla^2 \mathbf{E} - \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} + \frac{4}{3} k^2 \frac{\partial \mathbf{A}}{\partial t} = \mu_0 \mathbf{J} \quad (23)$$

The wave equation for the electric field in vacuum $\mathbf{J} = 0$ then equation (23) becomes:

$$\nabla^2 \mathbf{E} - \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} + \frac{4}{3} k^2 \frac{\partial \mathbf{A}}{\partial t} = 0 \quad (24)$$

Equation (24) explain to wave equation with massive photon. Meanwhile, the wave equation for magnetic field \mathbf{B} can be derived from Proca equation by using *cross product* operator on equation (10) and substituting equation (9) into equation (10) it is obtained:

$$\nabla^2 \mathbf{B} - \frac{1}{c^2} \frac{\partial^2 \mathbf{B}}{\partial t^2} + \frac{4}{3} \mu_0 \mathbf{B} = \mu_0 (\nabla \times \mathbf{J}) - \mathbf{A} \times \nabla k^2 \quad (25)$$

Wave equation for vacuum condition $\mathbf{J} = 0$ and for $k=\text{constant}$, so equation (25) becomes to:

$$\nabla^2 \mathbf{B} - \frac{1}{c^2} \frac{\partial^2 \mathbf{B}}{\partial t^2} + \frac{4}{3} \mu_0 \mathbf{B} = 0 \quad (26)$$

If value of constant wave number $k=0$, then equation (24) and (26) will be returned to standard form of the wave equation. The value of $k=0$ indicates the absence of imaginary mass of a photon.

4.2. Scalar and vector potential

Scalar and vector potentials for Proca condition can be derived by substituting equation $\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}$ into equation (7), so that obtained *dot* operations are shown in equation (27):

$$\nabla^2 \phi - \mu_Y^2 \phi + \frac{\partial}{\partial t} (\nabla \cdot \mathbf{A}) = -\frac{\rho}{\epsilon_0} \quad (27)$$

Equation (27) describes scalar potential. It can be expressed in the form of wave propagation, thus becoming:

$$\nabla^2 \phi - \frac{4}{3} k^2 \phi + \frac{\partial}{\partial t} (\nabla \cdot \mathbf{A}) = -\frac{\rho}{\epsilon_0} \quad (28)$$

Equation (28) explain to wave equation with massive photon. Substituting equation $\mathbf{B} = \nabla \times \mathbf{A}$ and $\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}$ into equation (10), it will obtain the following form:

$$\left(\nabla^2 \mathbf{A} - \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2}\right) - \nabla \left(\nabla \cdot \mathbf{A} + \mu_0 \epsilon_0 \frac{\partial V}{\partial t}\right) = -\mu_0 \mathbf{J} + \mu_Y^2 \mathbf{A} \quad (29)$$

Equation (29) describes the relationship between current density with 4-vector potential. Equation (29), has factor correction of mass as expressed by vector potential. If a value of massive photon is ignored $\mu_Y^2 = \frac{4}{3} k^2 = 0$, then equation (29) will be reduced according to the theory of electrodynamics for scalar and vector potential.

4.3. Continuity equation

In the same way as Maxwell's equation, it will obtain a new equation that described about continuity equation by Proca. We have derived equation (7) and equation (10) with vector operation, so that results the formula that shown in equation (30):

$$\mu_0 \left(\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} \right) = \mu^2 \left(\nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial \phi}{\partial t} \right) \quad (30)$$

Equation (30) is relevant to continuity equation in Maxwell's condition $\left(\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0 \right)$ if values of $\mu^2 = 0$, which means effect imaginary mass of photon ignored.

Equation (30) can be written becomes to:

$$\left(\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} \right) = \frac{\mu^2}{\mu_0} \left(\nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial \phi}{\partial t} \right) \quad (31)$$

If equation (31) compared to continuity equation in Maxwell's condition, then the value of $\frac{\mu^2}{\mu_0} \left(\nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial \phi}{\partial t} \right) = 0$, so that continuity equation is fulfilled. It can be explained by equation (32) below:

$$\frac{\mu^2}{\mu_0} \left(\nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial \phi}{\partial t} \right) = 0 \quad (32)$$

From equation (32), we have two condition. First, the value of $\frac{\mu^2}{\mu_0} = 0$, this condition illustrates the absence of imaginary mass propagation of the photon. Second, the value of $\nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial \phi}{\partial t} = 0$, this is a form of the equation describing an eternity. The change from a scalar potential proportional to the divergence of vector potential.

5. Conclusion

Proca equation can be analyzed based on the electric or magnetic field. The result are shown by equation (24) for electric field and equation (26) for a magnetic field. Equation (24) and (26) are wave form equation, with parameters of massive photon. If the mass is ignored then the formula will be returned to being standard wave equation.

From analysis, it has been obtained of vector and scalar potential shown by equation (27) and (29), and also is obtained a new formula from continuity equation shown by equation (31). There equation are obtained as result of an application of Proca equation in electrodynamics. If the mass of photon is ignored, the equation will be reduced back to Maxwell's equation in classical electrodynamics.

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