IOP Conf. Series: Journal of Physics: Conf. Series **792** (2017) 012054 doi:10.1088/1742-6596/792/1/012054

Some real, periodic stationary solutions of the one-dimensional nonlinear Schrödinger equation for constant potentials

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Abstract. We introduce a nonsymmetric version of Jacobian elliptic functions. With these functions, we are able to obtain linear superposition solutions of the non-linear Schrödinger equation for a free condensate. These functions are more general than that of Jacobi and contain them as a special case. The eigenfunctions for a condensate in a box are considered.

1. Introduction
Since the Gross-Pitaevskii equation (GPE)[1,2] is widely accepted as a valid model for the dynamics of the Bose-Einstein condensate at $T{\sim}0$$ K, as well as for other systems, the properties of this equation are of interest, in particular the finding of its eigenstates.

For a free condensate, besides the ground state[3] and Zakharov's solutions[4], the solutions are known in terms of Jacobi's elliptic functions[5,6] and of hyperbolic functions and they have been used in order to find the eigenstates of the particle in free space, inside a box and in a ring[7,8].

It was also found a way to linearly superpose Jacobi's elliptic functions by adding constant terms to their arguments[9].

In this paper, we introduce a three parameters elliptic-type functions which allows us to deal with the linear superposition of the usual Jacobi's elliptic functions, and then find additional steady solutions for nonlinear quantum systems.

In Jacobi's elliptic function sn(u), the nonlinearity is found as the assignment sn(u)=sin(θ) with the relation between the variables u and θ given as $u = \int_0^\theta dx/\sqrt{1 - m\, sin^2(x)}$. In this paper we explore the use of other change of variable. In section 2, we use another relationship between u and θ and obtain additional solutions to the nonlinear Schrödinger equation, real, periodic ones and which contain the usual Jacobi's functions as special cases.

With these functions, one can find additional eigenstates for some quantum systems. In Sec. 3, we use the functions introduced in this paper in order to find some eigenstates for the nonlinear quantum condensate in the infinite line.

2. A set of nonlinear functions
Let us consider the change of variable from θ to u defined by the Jacobian

$$dna(u) := \frac{d\theta}{du} = \sqrt{1 + \frac{\alpha}{2}(A^2 - B^2)\cos(2\theta) + \alpha\, A\, B \sin(2\theta)}, \qquad (1)$$

Where $\alpha, A, B \in$ are real, $|\alpha| < 4|AB|/(A^2 + B^2)^2$. Thus, the relationship between θ and u is

$$u = \int_0^{\theta} \frac{d\theta}{\sqrt{1+\frac{\alpha}{2}(A^2-B^2)\cos(2\theta)+\alpha\, A\, B\sin(2\theta)}}, \tag{2}$$

and we define the nonlinear functions

$$sna(u) := A\sin(\theta) - B\, cos(\theta), \quad cna(u) := A\cos(\theta) + B\, sin(\theta). \tag{3}$$

As with the trigonometric and elliptic functions, let us also consider the ratios between the above functions,

$$osa(u) := \frac{1}{sna(u)}, \quad oca(u) := \frac{1}{cna(u)}, \quad oda(u) := \frac{1}{dna(u)}, \tag{4}$$

$$csa(u) := \frac{cna(u)}{sna(u)}, \quad sca(u) := \frac{sna(u)}{cna(u)}, \quad dsa(u) := \frac{dna(u)}{sna(u)}, \tag{5}$$

$$dca(u) := \frac{dna(u)}{cna(u)}, \quad sda(u) := \frac{sna(u)}{dna(u)}, \quad cda(u) := \frac{cna(u)}{dna(u)}, \tag{6}$$

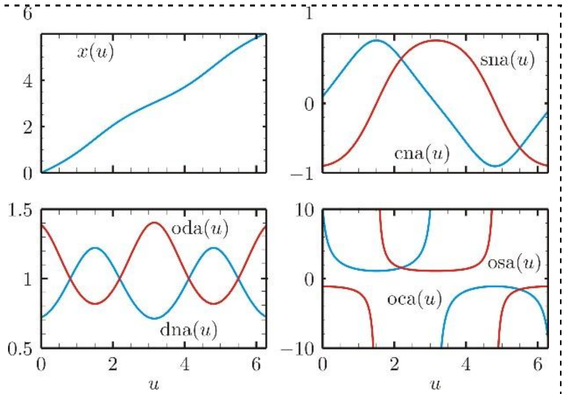

Figure 1. Plots of the nonlinear functions for A=0.1, B=0.9 and α=1.2. Note that the functions cna and sna have different shapes and, thus, they are not just the other shifted by some amount.

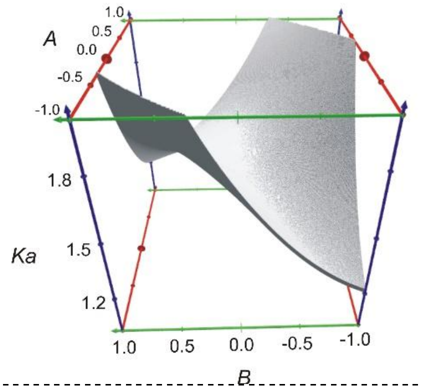

Figure 2. Some of the values of nonlinear quarter period $Ka(\alpha, A, B)$, for $\alpha = 1.2$.

Some plots of these functions are shown in Fig. 1. Note that the $cna(u)$ function is not just the shifed version of $sna(u)$, and viceversa, as is the case of the trigonometric functions $cos(u)$ and $sin(u)$ functions which have the same shape but are shifted by $\pi/2$.

Quarter period of these functions is defined as

$$Ka(\alpha, A, B) = \int_0^{\pi/2} \frac{dt}{\sqrt{1+\frac{\alpha}{2}(A^2-B^2)\cos(2t)+\alpha\, A\, B\sin(2t)}}, \tag{7}$$

A plot of $Ka(\alpha, A, B)$ can be found in Fig. 2 for $\alpha = 1.2$.

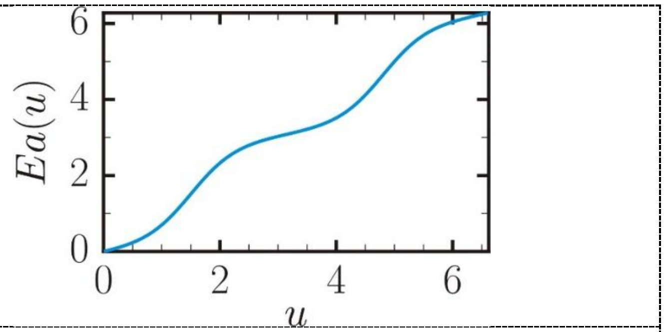

Figure 3. Plot of $Ea(u)$ for A=0.1, B=0.9 and α=1.2.

We also introduce the integral

$$Ea(u) = \int_0^u dw\, dna^2(w)$$

which resembles Jacobi's Elliptic integral of the second kind. This function is shown in Fig. 3 for a set of values of the parameters.

In next subsections, some algebraic properties of these functions are derived.

2.1. Algebraic properties

Some algebraic relationships between the above functions are

$$A^2 + B^2 = sna^2(u) + cna^2(u), \quad dna^2(u) = 1 - \frac{\alpha}{2}(sna^2(u) - cna^2(u)), \tag{8}$$
$$dna^2(u) = 1 - \frac{\alpha}{2}(A^2 + B^2) + \alpha\, cna^2(u) = 1 + \frac{\alpha}{2}(A^2 + B^2) - \alpha\, sna^2(u), \tag{9}$$
$$sda^2(u) = (A^2 + B^2)\, oda^2(u) - cda^2(u), \tag{10}$$
$$1 - oda^2(u) = \frac{\alpha}{2}[\, cda^2(u) - sda^2(u)], \tag{11}$$
$$1 + \alpha\, sda^2(u) = [1 + \frac{\alpha}{2}(A^2 + B^2)]oda^2(u), \tag{12}$$
$$1 - \alpha\, cda^2(u) = [1 - \frac{\alpha}{2}(A^2 + B^2)]oda^2(u), \tag{13}$$
$$sca^2(u) = (A^2 + B^2)\, oca^2(u) - 1, \quad dca^2(u) = \left[1 - \frac{\alpha}{2}(A^2 + B^2)\right] oca^2(u) + \alpha, \tag{14}$$
$$csa^2(u) = (A^2 + B^2)\, osa^2(u) - 1. \tag{15}$$

2.2. Derivatives

We now find the derivatives of the functions introduced earlier.

$$sna'(u) = cna(u)\, dna(u), \quad cna'(u) = -sna(u)\, dna(u), \tag{16}$$
$$dna'(u) = -\alpha\, cna(u)\, sna(u), \quad osa'(u) = -cna(u)\, dna(u) osa^2(u), \tag{17}$$
$$oca'(u) = sna(u)\, dna(u) oca^2(u), \quad oda'(u) = \alpha\, cna(u)\, sna(u) oda^2(u). \tag{18}$$

2.3. Eliminant equations

The eliminant equations, also known as energy or Liapunov functions are

$$[sna'(u)]^2 + \left[1 + \frac{3}{2}\alpha(A^2 + B^2)\right] sna^2(u) - \alpha\, sna^4(u) = (A^2 + B^2)\left[1 + \frac{\alpha}{2}(A^2 + B^2)\right], \tag{19}$$
$$[cna'(u)]^2 + \left[1 - \frac{3}{2}\alpha(A^2 + B^2)\right] cna^2(u) + \alpha\, cna^4(u) = (A^2 + B^2)\left[1 - \frac{\alpha}{2}(A^2 + B^2)\right], \tag{20}$$
$$[dna'(u)]^2 - 2\, dna^2(u) + dna^4(u) = \frac{\alpha^2}{4}(A^2 + B^2)^2 - 1, \tag{21}$$
$$[osa'(u)]^2 + \left[1 + \frac{3}{2}\alpha(A^2 + B^2)\right] osa^2(u) - (A^2 + B^2)\left[1 + \frac{\alpha}{2}(A^2 + B^2)\right] osa^4(u) = \alpha, \tag{22}$$
$$[oca'(u)]^2 + \left[1 - \frac{3}{2}\alpha(A^2 + B^2)\right] oca^2(u) - (A^2 + B^2)\left[1 - \frac{\alpha}{2}(A^2 + B^2)\right] oca^4(u) = -\alpha, \tag{23}$$
$$[oda'(u)]^2 - 2\, oda^2(u) + \left[1 - \frac{\alpha^2}{4}(A^2 + B^2)^2\right] oda^4(u) = -1. \tag{24}$$

2.4. Differential equations

Second derivatives of the functions lead to the differential equations for them. For sna, cna and dna we have that

$$sna''(u) + \left[1 + \frac{3}{2}\alpha(A^2 + B^2)\right]\ sna(u) - 2\alpha\ sna^3(u) = 0, \tag{25}$$
$$cna''(u) + \left[1 - \frac{3}{2}\alpha(A^2 + B^2)\right]\ cna(u) + 2\alpha\ cna^3(u) = 0, \tag{26}$$
$$dna''(u) + 2\ dna(u)[dna^2(u) - 1] = 0, \tag{27}$$

The other functions, osa, oca and oda, are also solutions of similar differential equations, according to the following lemma,

Lemma. If $f(u)$ is a function which satisfy the nonlinear differential equation

$$f''(u) + E\ f(u) - 2C\ f^3(u) = 0, \tag{28}$$

and the derivative of $f(u)$ is such that

$$[f'(u)^2] - C\ f^4(u) + \frac{1}{2}(E - A)f^2(u) - \frac{B}{2} = 0, \tag{29}$$

with E and C real constants, then $1/f(u)$ also satisfy a similar nonlinear equation,

$$\frac{d^2}{du^2}\frac{1}{f(u)} - \frac{A}{f(u)} - \frac{B}{f^3(u)} = 0. \tag{30}$$

Then, the other functions, namely osa, oca and oda will also satisfy a similar differential equation,

$$osa''(u) + \left[1 + \frac{3}{2}\alpha(A^2 + B^2)\right]\ osa(u) - 2(A^2 + B^2)\left[1 + \frac{\alpha}{2}(A^2 + B^2)\right]\ osa^3(u) = 0, \tag{31}$$
$$oca''(u) + \left[1 - \frac{3}{2}\alpha(A^2 + B^2)\right]\ oca(u) - 2(A^2 + B^2)\left[1 - \frac{\alpha}{2}(A^2 + B^2)\right]\ oca^3(u) = 0, \tag{32}$$
$$oda''(u) - 2\ oda(u) + 2\left[1 - \frac{\alpha^2}{4}(A^2 + B^2)^2\right]\ oda^3(u) = 0, \tag{33}$$

The derivatives of the inverse functions are

$$\frac{d}{dy}sna^{-1}(y) = \frac{\pm 1}{\sqrt{(A^2+B^2-y^2)\left(1+\frac{\alpha}{2}(A^2+B^2)-\alpha y^2\right)}}, \tag{34}$$
$$\frac{d}{dy}cna^{-1}(y) = \frac{\pm 1}{\sqrt{(A^2+B^2-y^2)\left(1-\frac{\alpha}{2}(A^2+B^2)+\alpha y^2\right)}}, \tag{35}$$
$$\frac{d}{dy}dna^{-1}(y) = \frac{\pm 1}{\sqrt{\left[1+\frac{\alpha}{2}(A^2+B^2)-y^2\right]\left[y^2-1+\frac{\alpha}{2}(A^2+B^2)\right]}}, \tag{36}$$

$$\frac{d}{dy}osa^{-1}(y) = \frac{\pm 1}{\sqrt{[(A^2+B^2)y^2-1]\left[\left(1+\frac{\alpha}{2}(A^2+B^2)\right)y^2-\alpha\right]}}, \tag{37}$$
$$\frac{d}{dy}oca^{-1}(y) = \frac{\pm 1}{\sqrt{[(A^2+B^2)y^2-1]\left[\left(1-\frac{\alpha}{2}(A^2+B^2)\right)y^2+\alpha\right]}}, \tag{38}$$
$$\frac{d}{dy}oda^{-1}(y) = \frac{\pm 1}{\sqrt{\left[\left(1+\frac{\alpha}{2}(A^2+B^2)\right)y^2-1\right]\left[1-\left(1-\frac{\alpha}{2}(A^2+B^2)\right)y^2\right]}}, \tag{39}$$

Then, as expected, we can see that these functions also invert the same integrals that Jacobi's functions invert.

This is the minimum set of properties which will allow us to find some eigenstates of the one-dimensional nonlinear Schrödinger equation for a free condensate.

3. Condensate in an infinite or periodic medium

As an application of the use of the functions introduced in the previous section, we find some real, periodic solutions of the non-linear Schrödinger equation for a condensate in an infinite medium and a constant potential.

The one-dimensional nonlinear time-dependent Schrödinger equation for a condensate in a constant potential is written as

$$-2i\frac{\partial\,\psi(u;t)}{\partial u} = \frac{\partial^2\,\psi(u;t)}{\partial u^2} - 2(V + \alpha|\psi(u;t)|^2)\psi(u;t), \tag{40}$$

where t, u, $\psi(u;t)$ and $\psi(u;t)$ are dimensionless quantities, with scaling factors $t_s = ML^2/\hbar$ for time, $V_s = \hbar^2/ML^2$ for energy. The length is scaled by L, a characteristic length of the system of interest, and the normalization factor for the wave function is A_s. $\psi(u;t)$ is the wave function for the Bose-Einstein condensate (BEC), $\alpha = ML^2A^2NU_0/\hbar^2$, M is the mass of a single atom, N is the number of atoms in the condensate, $U_0 = 4\pi\hbar^2 a/M$ characterizes the atom-atom interaction, and a is the scattering length.

The functions introduced in this work can be used to find stationary states for the infinite line or for a system which is periodic like a particle moving on a ring. For the ring, the condition is that the length of the ring should be a multiple of the period of the nonsolitonc functions which is $4Ka(\alpha, A, B)$ For a solitonic function we can have symmetry-breaking eigenstates.[4,5]

As with the other solutions,[7] there are states which can travel without distortion; the soliton-type states. Their time dependency is given by

$$\psi(u;t) = k\ e^{\pm iu\sqrt{-k^2(1+3\alpha(A^2+B^2)/2)-2V}} sna\left[k\left(u \mp t\sqrt{-k^2\left(1+\frac{3\alpha(A^2+B^2)}{2}\right) - 2V}\right)\right], \tag{41}$$

for $V < -k^2(1 + 3\alpha(A^2 + B^2)/2)/2$. Another soliton-type solution is

$$\psi(u;t) = \frac{k}{\sqrt{-\alpha}}\ e^{i\theta \pm iu\sqrt{2(k^2-V)}} dna\left[k\left(u \mp t\sqrt{2(k^2 - V)}\right)\right], \tag{42}$$

valid when $k^2 > V$. A third soliton-type solution is provided by the oca function,

$$\begin{aligned}\psi(u;t) = \frac{k}{\sqrt{\alpha}}\ &\sqrt{(A^2 + B^2)\left(1 - \frac{\alpha}{2}(A^2 + B^2)\right)}\ e^{\pm iu\sqrt{-k^2\left(1-\frac{3\alpha(A^2+B^2)}{2}\right)-2V}} \\ &\times\ oca\left[k\left(u \mp t\sqrt{-k^2\left(1 - \frac{3\alpha(A^2+B^2)}{2}\right) - 2V}\right)\right],\end{aligned} \tag{43}$$

with $\alpha > 0$, $2 > \alpha(A^2 + B^2)$, and $2V < -k^4(1 - 3\alpha(A^2 + B^2)/2)$. There is also a soliton type solution involving the function osa,

$$\begin{aligned}\psi(u;t) = \frac{k}{\sqrt{\alpha}}\ &\sqrt{(A^2 + B^2)\left(1 + \frac{\alpha}{2}(A^2 + B^2)\right)}\ e^{\pm iu\sqrt{-k^2(1+3\alpha(A^2+B^2)/2)-2V}} \\ &\times\ osa\left[k\left(u \mp t\sqrt{-k^2\left(1 + \frac{3\alpha(A^2+B^2)}{2}\right) - 2V}\right)\right],\end{aligned} \tag{44}$$

This solution is valid if $\alpha < -2(k^2 + 2V)/3k^2(A^2 + B^2)$. And there is a last soliton-type solution given by

$$\psi(u;t) = \pm\frac{k}{\sqrt{\alpha}}\sqrt{\frac{\alpha^2}{4}(A^2+B^2)^2-1}\, e^{\pm iu\sqrt{2(k^2-V)}}\, oda\left[k\left(u \mp t\sqrt{2(k^2-V)}\right)\right], \quad (45)$$

where $\sqrt{-k^2(1+3\alpha(A^2+B^2)/2)-2V}$, $\sqrt{2(k^2-V)}$, $\sqrt{-k^2(1-3\alpha(A^2+B^2)/2)-2V}$, $\sqrt{-k^2(1+3\alpha(A^2+B^2)/2)-2V}$ and $\sqrt{2(k^2-V)}$ are the corresponding velocities of these waves.

Remarks

We have found that one can obtain additional solutions of the nonlinear Schrödinger equation, with a constant potential, by using other change of coordinate than the one used in Jacobi's elliptic functions.

An advantaje of these functions is that they facilitate the solving of boundary problems for a free Bose-Einstein condensate. Then, we can have solutions for the same type of systems that are found in the linear case.

As is the case with Jacobi's functions, the set of functions we have introduced are interpolations functions between the linear solutions (trigonometric functions) and purely nonlinear (hyperbolic and trigonometric times a complex factor) functions.

An inconvenience of these solutions is that they do not carry any momentum at all. Another one is that they do not describe the decay of the wave function. These behaviours are found in the linear system and we expect that something similar can be found in the nonlinear case. We are currently looking for the appropriate changes of variables for those situations, as well as for solutions in higher dimensions.

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