

A method for correcting the structural instability of time-dependent atmospheric trajectory models under perturbations of the mass balance

M A Núñez and R Mendoza

Departamento de Física, Universidad Autónoma Metropolitana Iztapalapa,
A. P. 55-534, C. P. 09340, D. F.

E-mail: manp@xanum.uam.mx, oicor_192000@hotmail.com

Abstract. Lagrangian trajectory models of atmospheric fluid parcels have been used over a wide range of spatial scales to study the transport and dispersion of pollutants, radioactive materials, ash clouds produced by volcanic eruptions or modeling global carbon cycle. It was pointed out trajectory calculations have three error sources: error in the gridded data from measurements error or from approximations in Eulerian numerical models, error from the spatial and temporal resolution of data, and truncation error from numerical integration of the velocity field. In a recent work we showed that trajectory models can be structurally unstable under perturbations of the mass balance, e.g., the flow can go from hyperbolic to elliptic and vice versa. In this work we propose a mass-consistent approach to correct this structural instability. The orthogonal projection character of this approach guarantees the correction even for large perturbations of the mass-balance. This is illustrated by means of numerical examples. Mass-consistent models have been considered as diagnostic models of the wind field but the results of this work show that such a models can be used for four-dimensional data assimilation of nonstationary flows.

1. Introduction

Trajectory models of an atmospheric fluid particle are used for long-range atmospheric transport and dispersion calculations [1-3]. These models are obtained by interpolation and integration of velocity data given by meteorological networks and global numerical models. However, the sole interpolation does not guarantee a correct mass balance and in a previous work [4] it was shown that a small perturbation of the mass balance can change substantially the local structure of a flow. In regions with a complex topography an interpolated horizontal velocity field $\mathbf{v}_h = u\mathbf{i} + v\mathbf{j}$ is used to estimate the vertical velocity w by means of the equation $\nabla \cdot \mathbf{v} = 0$ with $\mathbf{v} = \mathbf{v}_h + w\mathbf{k}$. Although this procedure yields a mass-balanced flow \mathbf{v} , the horizontal flow may have an incorrect structure because of a mass imbalance. In this work we show that the structure of a time-dependent horizontal field \mathbf{v}_h can be corrected by means of the so-called variational mass consistent models (VMCM's) [5,6].



Among the diagnostic models of the wind field VMCM's have been used because of its simplicity and capacity to accept several measurements in a region with complex terrain [5,6], but these models have been ignored by a wide community devoted to the estimation of trajectory models [1-3]. In this work we show that VMCM's can be used to compute trajectory models from time-dependent data provided by operational networks and global numerical models.

2. Variational mass-consistent models as a projection method

An accepted approximation of the continuity equation that considers the vertical density variation is $\nabla \cdot (\rho_0 \mathbf{v}) = 0$ where ρ_0 is a reference density. In terms of the field $\mathbf{v} = \rho_0 \mathbf{\tilde{v}}$ we get the continuity equation $\nabla \cdot \mathbf{v} = 0$. In a previous work we show that a perturbation \mathbf{u}^δ with divergence of order $\nabla \cdot \mathbf{u}^\delta = 10^{-5}$ can yield a divergent field $\mathbf{v}^\delta = \mathbf{v} + \mathbf{u}^\delta$ with a substantially different local structure, e.g., if \mathbf{v} is elliptic then \mathbf{v}^δ can be hyperbolic. This poses the problem of recovering \mathbf{v} from \mathbf{v}^δ . A way is given by the following variational formulation. Consider the inner product between two vector fields

$\langle \mathbf{a}, \mathbf{b} \rangle = \int_R \mathbf{a}(\mathbf{r}) \cdot \mathbf{b}(\mathbf{r}) dR$ in a region R . The norm is $\|\mathbf{a} - \mathbf{b}\| = \langle \mathbf{a}, \mathbf{a} \rangle^{1/2}$ and the distance between \mathbf{a} and \mathbf{b} is $d(\mathbf{a}, \mathbf{b}) \equiv \|\mathbf{a} - \mathbf{b}\|$. A variational mass-consistent model of the wind field is an (adjusted) velocity field \mathbf{v}^a that minimizes the functional $J(\mathbf{w}) = \|\mathbf{w} - \mathbf{v}^\delta\|^2$ subject to the condition $\nabla \cdot \mathbf{w} = 0$ [5,6]; that is, \mathbf{v}^a is the nondivergent field that is closer to the field \mathbf{v}^δ . The solution of this variational problem is

$$\mathbf{v}^a = \mathbf{v}^\delta + \nabla \lambda \quad (1)$$

where λ is determined by the equation

$$\nabla^2 \lambda = \nabla \cdot \mathbf{v}^\delta. \quad (2)$$

To get a unique solution λ we impose, by simplicity, the Dirichlet boundary conditions (DBC's)

$$\lambda = 0 \quad \text{on} \quad \Gamma \quad (3)$$

where Γ denotes the boundary of region R . A discussion of the improved boundary conditions in region with a complex topography is given in Ref. [7]. The power of VMCM's for trajectory calculations can be explained in terms of the projection properties of this variational approach. Let $\mathbf{L}^2(R)$ be the space of vector fields \mathbf{w} with components in $L^2(R)$ (the set of square-integrable functions in R). This space has the orthogonal decomposition $\mathbf{L}^2(R) = \mathbf{W} + \mathbf{A}$ where \mathbf{W} is a set of nondivergent fields \mathbf{w} and \mathbf{A} is the set of gradient fields $\nabla \Phi$ where Φ satisfies the DBC (3). The sets \mathbf{W} and \mathbf{A} are orthogonal, $\langle \mathbf{w}, \nabla \Phi \rangle = 0$ holds for $\mathbf{w} \in \mathbf{W}$ and $\nabla \Phi \in \mathbf{A}$, respectively, and each vector field \mathbf{u} in $\mathbf{L}^2(R)$ is the sum of a unique pair $\mathbf{w}^a, \nabla \Phi$: $\mathbf{u} = \mathbf{w}^a + \nabla \Phi$ [8]. The orthogonality and uniqueness of \mathbf{w}^a implies it is the orthogonal projection of \mathbf{u} on the set \mathbf{W} and the projection theorem for Hilbert spaces asserts that \mathbf{w}^a is the unique element in \mathbf{W} that minimizes the distance between \mathbf{u} and the set \mathbf{W} :

$d(\mathbf{u}, \mathbf{w}^a) < d(\mathbf{u}, \mathbf{w})$ for \mathbf{w} in \mathbf{W} . In the case of a divergent field $\mathbf{v}^\delta = \mathbf{v} + \mathbf{u}^\delta$ obtained by perturbation of a nondivergent one \mathbf{v} , we have the decomposition

$$\mathbf{v}^\delta = \mathbf{v} + \mathbf{u}^\delta = \mathbf{v}^a - \nabla \lambda$$

with \mathbf{v}^a in \mathbf{W} and λ is solution of the problem (2), (3). Thus the VMCM \mathbf{v}^a (1) is the orthogonal projection of \mathbf{v}^δ on the set of nondivergent fields \mathbf{W} . It should be emphasized that \mathbf{v}^δ is unique and independently of the size of a divergent perturbation \mathbf{u}^δ . The examples reported below show that \mathbf{v}^a is almost equal to \mathbf{v} even for large perturbations \mathbf{u}^δ , except in a vicinity of the boundary Γ because of the DBC (2) [7].

In this work we consider velocity fields in a rectangular region $R = [0, x_M] \times [0, y_M]$ and the problem (2), (3), is replaced by the approximated problem

$$\nabla^2 \lambda_{mn} = F_{mn}$$

with the Fourier series

$$\lambda_{mn} = \sum_{i=1}^{mn} \lambda_{ij} \varphi_i(x) \varphi_j(y), \quad F_{mn} = \sum_{i=1}^{mn} F_{ij} \varphi_i(x) \varphi_j(y),$$

where $F_{ij} = \int_R \varphi_j \varphi_i F dR$, $F = \nabla \cdot \mathbf{v}^\delta$, $\varphi_i = \sqrt{2/x_M} \sin \omega_i x$, $\varphi_j = \sqrt{2/y_M} \sin \omega_j y$, $\omega_i = i\pi/x_M$, $\omega_j = j\pi/y_M$.

Example 1. Consider the nondivergent field $\mathbf{v} = u\mathbf{i} + v\mathbf{j}$ with $u = ax + (b + \omega)y$, $v = (b + \omega)x - ay$, $a = -10^{-5}$, $b = -a$, $\omega = a$ (s^{-1}). This flow has a hyperbolic structure and is perturbed by $\mathbf{u}^\delta = \delta(x\mathbf{i} + y\mathbf{j})$ with $\delta = -a$ which yields $\mathbf{v}^\delta = \mathbf{v} + \mathbf{u}^\delta$ with an elliptic structure [4]. The VMCM \mathbf{v}^a is computed with $m=n=200$. Figures 1a, 1b, 1c, show the fields \mathbf{v} , \mathbf{v}^δ , \mathbf{v}^a . We see that \mathbf{v}^a is almost equal to the desired nondivergent field \mathbf{v} and, mainly, \mathbf{v}^a recovers the hyperbolic structure of \mathbf{v} . The main difference between \mathbf{v} and \mathbf{v}^a is in a vicinity of the boundary Γ because of the DBC (3) [7].

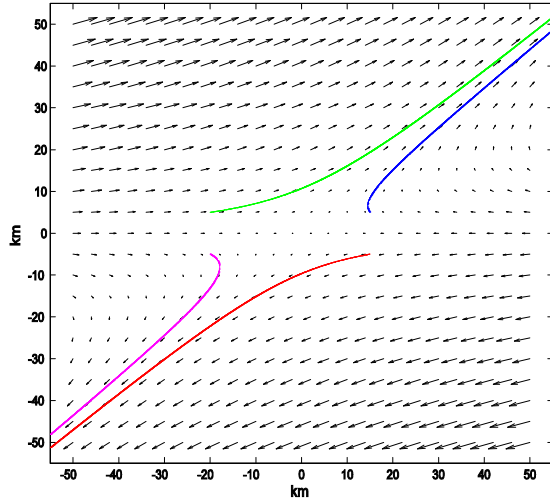


Figure 1a. Nondivergent hyperbolic field \mathbf{v} and some trajectories.

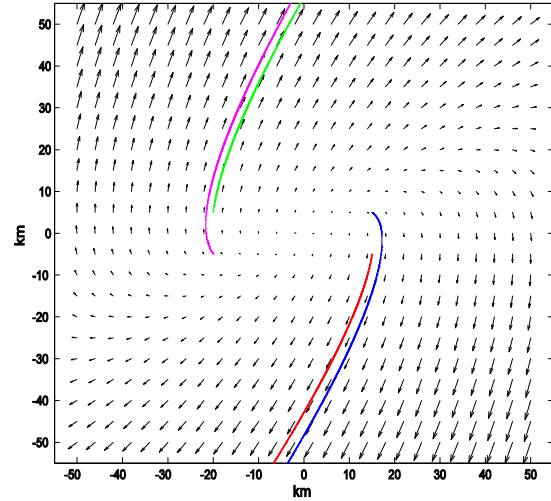


Figure 1b. Perturbed field $\mathbf{v}^\delta = \mathbf{v} + \mathbf{u}^\delta$ with an elliptic structure.

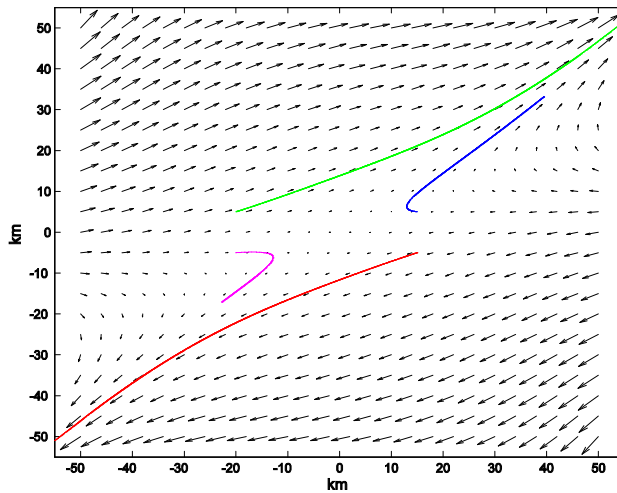


Figure 1c. Adjusted field \mathbf{v}_{mn}^a obtained by projection of the perturbed field in Figure 1b.

Variational mass-consistent models have been used mainly as diagnostic wind models at a given time t or for climatological studies [5,6] but the next example shows the capability of these models for data assimilation of time-dependent flows and for estimating trajectory models.

Example 2. Consider the nondivergent and nonstationary flow

$$u^0 = V_0 \pi \cos \pi y (\sin 2\pi x + \alpha \sin \pi x \cos \omega t), \quad v^0 = -V_0 \pi \sin \pi y (2 \cos 2\pi x + \alpha \cos \pi x \cos \omega t)$$

where $V_0=10\text{km/h}$, $\alpha = 5$, $\omega = 6 \text{ 1/h}$, which is modified with the time dependent perturbation $\mathbf{u}^\delta = (1 - \delta x \cos \omega t)\mathbf{i} + (1 - \delta y \cos \omega t)\mathbf{j}$, $\delta=5/\text{h}$. The perturbed flow $\mathbf{v}^\delta = \mathbf{v} + \mathbf{u}^\delta$ has the divergence $\nabla \cdot \mathbf{v}^\delta = 2\delta \cos \omega t$. The adjusted field \mathbf{v}_{mn} at a given time t is computed by solving problem (2), (3), in a sequence of time steps $t_0 = 0, \dots, t_n$, used to compute some trajectories with the fields \mathbf{v} , \mathbf{v}^δ , and a fourth-order Runge-Kutta Routine. Figures 2a 2b, 2c, shows the fields \mathbf{v} , \mathbf{v}^δ , \mathbf{v}_{mn} , at $t=6 \text{ hr}$, and some trajectories. The perturbation of \mathbf{v} yields \mathbf{v}^δ with substantially different trajectories because of the magnitude and time dependence of \mathbf{u}^δ . Nevertheless, Figure 2c shows that the adjusted field \mathbf{v}_{mn} recovers the structure of \mathbf{v} with high accuracy since the trajectories of \mathbf{v}_{mn} are very similar to those of \mathbf{v} . To best of our knowledge these time-dependent results have not been reported by the literature devoted to the estimation of trajectory models and mass-consistent models.

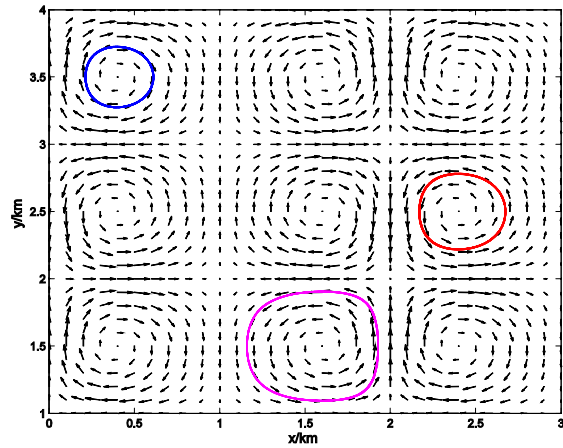


Figure 2a. Nondivergent time-dependent field \mathbf{v} and some of its trajectories at $t=6$ hr.

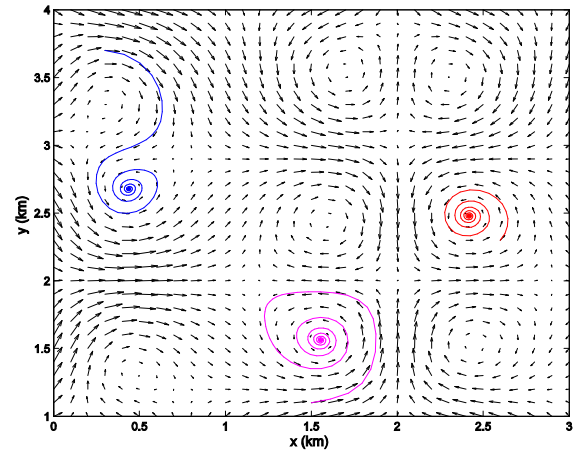


Figure 2b. Perturbed divergent $\mathbf{v}^\delta = \mathbf{v} + \mathbf{u}^\delta$ and trajectories with the initial conditions of those in Figure 2a.

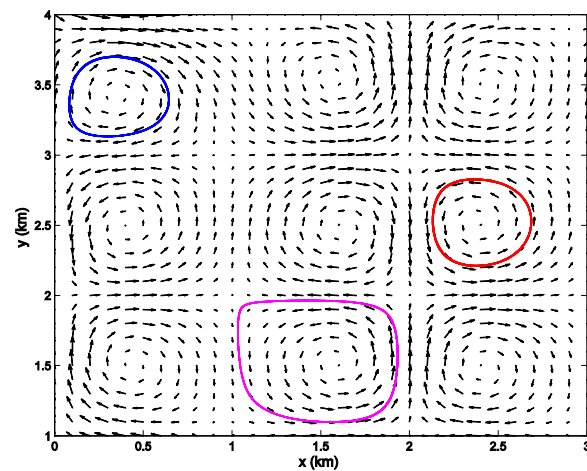


Figure 2c. Adjusted field \mathbf{v}^a_{mn} obtained by projection of the perturbed field in Figure 2b at $t=6$ hr and some trajectories with the initial conditions of those in Figure 2a.

3. Conclusions

The results of this work show that VMCM's can be used to improve significantly trajectories models, which are structurally unstable under perturbations of the mass balance. The projection character of VMCM's imply that such models can yield a mass-consistent wind field even when the perturbation of the mass balance is large and time-dependent. Almost all the literature devoted to compute trajectories models has ignored VMCM's [1,3], in part, because such models have been considered as diagnostic models [2,5,6]. However, the Example 2 of this work shows that VMCM's can yield reliable trajectories of real time-dependent atmospheric flows, in other words, VMCM's can be used for four-dimensional data assimilation of velocity data.

Acknowledgments

One of us (R. M.) wants to thank to the Universidad Autónoma Metropolitana for a scholarship.

4. References

- [1] Stohl A 1998 Computation, accuracy and applications of trajectories-A review and bibliography *Atmos. Environ.* **32** 947.
- [2] Atmospheric Dispersion Modelling Liaison Committee. Annual Report 2005-2006. ISBN 0-85951-587-7.
- [3] Bowman K P, Lin J C, Stohl A, Draxler R, Konopka P, Andrews A and Brunner D. 2013 *Input data requirements for Lagrangian trajectory models. Bull. Am. Meteorol. Soc.* **94**, 1051–1058.
- [4] Núñez M A and Mendoza R 2015 *Journal of Physics: Conference Series* **582** 012015.
- [5] Ratto C F, Festa R, Romeo C, Frumento O A and Galluzzi 1994, Mass-consistent models for wind fields over complex terrain: The state of the art *Environ. Software* **9** 247-268.
- [6] Homicz G F 2002 *Three-Dimensional Wind Field Modeling: A Review. Sandia National Laboratories*, Albuquerque, New Mexico.
- [7] Núñez M A 2102, *Improving variational mass-consistent models of hydrodynamic flows via boundary conditions, Eur. Phys. J. Plus* **127** 40.
- [8] Girault V and Raviart P A 1986 *Finite Element Methods for Navier-Stokes Equation, Theory and Algorithms* Springer, Berlin