

Decaying steady states of the one-dimensional nonlinear Schrödinger equation with a constant potential

Torres-Vega G

Physics Department, Cinvestav, Apdo. postal 14-740, 07000 Mexico city, Mexico
gabino@fis.cinvestav.mx

Abstract. We introduce a set of non-linear functions of real, exponential type that can be useful for dealing with some physical problems. We use these functions to find steady states of the one-dimensional non-linear Schrödinger equation for a condensate between an infinite barrier and a step potential in the case where the height of the step is larger than the chemical potential of the condensate.

1. Introduction

The present work is concerned with the solutions to the nonlinear Schrödinger equation in one dimension[1,2] of the real, exponential type. The nonlinear Schrödinger equation models wave propagation phenomena in Bose-Einstein condensates as well as many other systems.[3-7] A large amount of early work was performed on the steady states of the nonlinear Schrödinger equation for a free condensate in several dimensions (see Ref. [8] and references there in), such as a spherically symmetric, positive function that decays for large distances and is related to the existence, uniqueness and stability of the ground state.

A set of real, periodic, steady states of the one-dimensional nonlinear Schrödinger equation is given in terms of Abel's and elliptic functions[9-19] and has been used to find the stationary states of a Bose-Einstein condensate in a box[17,18] and in a double square well[20] when the chemical potential of the Bose condensate is larger than the height of the potential barrier encountered by the condensate. These solutions include bright, gray and dark soliton solutions as special cases. A linear superposition of Jacobi's elliptic functions with constant terms added to their arguments is one useful result of this approach.[11]

However, non-periodic, decaying solutions can also be useful. These functions would be nonlinear versions of the functions $e^{\pm x}$ and are also similar to the elliptic functions. These functions can be useful for situations such as when the condensate encounters step potentials and potential barriers.

In section 2, we introduce non-linear exponential-type functions and determine some of their properties. Later, in section 3, we use these functions to find the steady states for a Bose-Einstein condensate between an infinite barrier and a step potential. In the final section, we offer some concluding remarks.

2. Non-linear exponential-type functions

Let us introduce the following nonlinear functions of exponential type:



$$pn(u)=e^x, \quad mn(u)=e^{-x}, \quad fn(u)=ae^x+be^{-x}, \quad (1)$$

$$gn(u)=ae^x-be^{-x}, \quad rn(x)=\sqrt{1+m(ae^x-be^{-x})^2}, \quad nf(u)=\frac{1}{fn(u)}, \quad (2)$$

$$ng(u)=\frac{1}{gn(u)}, \quad nr(u)=\frac{1}{rn(u)}, \quad (3)$$

with u and x related as

$$u=\int_0^x \frac{dt}{\sqrt{1+m(ae^t-be^{-t})^2}}, \quad (4)$$

where $m>0$ to avoid branching points.

We note that $rn(u) \neq rn(-u)$ and then that $mn(u)$ is not the mirror image of $pn(u)$, i.e., $mn(u) \neq pn(-u)$ unless $a=b$. A plot of these functions is found in Fig. 1 for a set of values of the parameters a , b and m . The values of a and b are related to the mirror symmetry between the functions $pn(u)$ and $mn(u)$, with $a=b$ as the more symmetric case (which would be the case of Jacobi's elliptic functions with complex arguments). The value of m causes these functions to decay or increase more rapidly than the regular exponential functions. The domain of these functions is finite unless $m=0$; in fact, increasing the magnitude of x beyond, for instance, $\ln(10^4/2a\sqrt{m})$ does not significantly increase the magnitude of u . One can extend the domain of these functions by setting the value of the function to zero or infinity for larger u , making them non-periodic functions on the real axes. We also note that some of these functions are actually bounded.

We can easily verify the following properties that are similar to those for the elliptic functions. The squares of these functions are related as

$$4ab = fn^2(u) - gn^2(u), \quad (5)$$

$$rn^2(u) - 1 = mgn^2(u) = m[rn^2(u) - 4ab], \quad (6)$$

$$fn(u)gn(u) = a^2pn^2(u) - b^2mn^2(u), \quad (7)$$

$$fn^2(u) + gn^2(u) = 2[b^2mn^2(u) + a^2pn^2(u)], \quad (8)$$

The derivatives of these functions are

$$pn'(u) = pn(u)rn(u), \quad mn'(u) = mn(u)rn(u), \quad fn'(u) = gn(u)rn(u), \quad (9)$$

$$gn'(u) = fn(u)rn(u), \quad rn'(u) = mfn(u)gn(u), \quad nf'(u) = -gn(u)nf^2(u)rn(u), \quad (10)$$

$$ng'(u) = -fn(u)ng^2(u)rn(u), \quad nr'(u) = -mfn(u)gn(u)nr^2(u), \quad (11)$$

As we can see from these derivatives, the rate of increase or decrease of the functions is not constant along u and would be constant only for the case of $m=0$.

We also find that the derivatives of the inverse functions are given by

$$\frac{d p n^{-1}(y)}{dy} = \frac{1}{\sqrt{y^2 + m(a y^2 - b)^2}}, \quad \frac{d m n^{-1}(y)}{dy} = \frac{-1}{\sqrt{y^2 + m(a - b y^2)^2}}, \quad (12)$$

$$\frac{d g n^{-1}(y)}{dy} = \frac{1}{\sqrt{(y^2 + 4ab)(1 + m y^2)^2}}, \quad \frac{d n f^{-1}(y)}{dy} = \frac{-1}{\sqrt{(1 - 4ab y^2)(n_2 y^2 + m)^2}}, \quad (13)$$

$$\frac{d n g^{-1}(y)}{dy} = \frac{-1}{\sqrt{(1 + 4ab y^2)(y^2 + m)^2}}, \quad \frac{d n r^{-1}(y)}{dy} = \frac{-1}{\sqrt{(1 - y^2)(1 - n_2 y^2)^2}}, \quad (14)$$

where

$$n_1 = 1 - 8mab, \quad n_2 = 1 - 4mab, \quad n_3 = 1 - 2mab, \quad n_4 = 1 + 4mab. \quad (15)$$

As expected, from these derivatives, we can see that these functions also invert the same integral functions as Jacobi [9,12].

The second derivatives are

$$p n''(u) - p n(u)[n_3 + 2m a^2 p n^2(u)] = 0, \quad m n''(u) - m n(u)[n_3 + 2m b^2 m n^2(u)] = 0, \quad (16)$$

$$f n''(u) - f n(u)[n_1 + 2m f n^2(u)] = 0, \quad g n''(u) - g n(u)[n_4 + 2m g n^2(u)] = 0, \quad (17)$$

$$r n''(u) + 2r n(u)[n_3 - r n^2(u)] = 0, \quad f n''(u) - n f(u)[n_1 - 8ab n_2 n f^2(u)] = 0, \quad (18)$$

$$n g''(u) - n g(u)[n_4 + 8ab n g^2(u)] = 0, \quad n r''(u) - 2n r(u)[n_2 n r^2(u) - n_3] = 0, \quad (19)$$

Then, the functions that we have just introduced are solutions of the nonlinear second order differential equations with the one-dimensional Gross-Pitaevskii equation form for a constant potential and real functions.

Additionally, the corresponding energy, or the Liapunov functions are given by

$$[p n'(u)]^2 - p n^2(u)[n_3 + m a^2 p n^2(u)] = m b^2, \quad (20)$$

$$[m n'(u)]^2 - m n^2(u)[n_3 + m b^2 m n^2(u)] = m a^2, \quad (21)$$

$$[f n'(u)]^2 - f n^2(u)[n_1 + m f n^2(u)] = -4ab n_2, \quad (22)$$

$$[g n'(u)]^2 - g n^2(u)[n_4 + m g n^2(u)] = 4ab, \quad (23)$$

$$[r n'(u)]^2 + 2r n^2(u)[n_3 - r n^2(u)] = n_2, \quad (24)$$

$$[n f'(u)]^2 - n f^2(u)[n_1 - 4ab n_2 n f^2(u)] = m, \quad (25)$$

$$[n g'(u)]^2 - n g^2(u)[n_4 + 4ab n g^2(u)] = m, \quad (26)$$

$$[n r'(u)]^2 - n r^2(u)[n_2 n r^2(u) - 2n_3] = 1, \quad (27)$$

where we have made use of the relationships between the squares of the functions. We note that the functions $nf(u)$ and $ng(u)$ have the same energy, whereas the functions $pn(u)$ and $mn(u)$ would have the same energy if $b=a$.

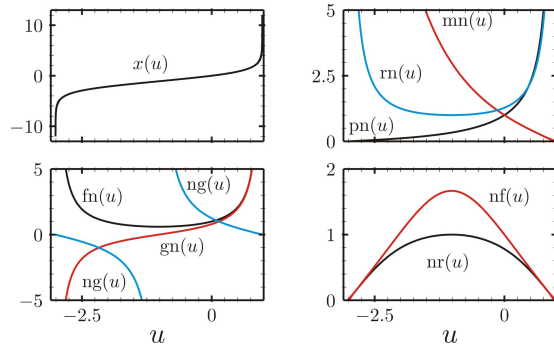


Fig. 1. Nonlinear exponential-type functions for $m=1$, $a=0.1$ and $b=0.9$.

When $a=b=1$, the nonlinear functions are reduced to Jacobi's elliptic functions with a complex argument,

$$u = \int_0^x \frac{dt}{\sqrt{1+4m \sinh^2(t)}} = -iF(ix; 4m), \quad (28)$$

where F is elliptic integral of the first kind.

In the solutions of physical problems, these functions can be chosen when functions that are small in one region of the real axes and large in other region are required or when functions that are non-zero at the ends of some interval of the real axes are required. They can also be useful when functions that decay at both ends of an interval are necessary.

This is the minimum set of properties of the functions that are necessary to give exponential-type solutions of the one-dimensional nonlinear Schrödinger equation with a constant potential. In the next section, we apply these functions to this case.

3. Wall-step potential and the nonlinear Schrödinger equation

Similarly to the treatment of the linear Schrödinger equation, we can use the functions $pn(u)$ and $mn(u)$ (now together with $rn(u)$) to solve the nonlinear Schrödinger equation when the condensate encounters or leaves a step potential, whereas the functions $fn(u)$ and $gn(u)$ (and $nr(u)$) can be used when the condensate is crossing a potential barrier.

Now, let us use these functions to find the steady state for the one-dimensional Gross-Pitaevskii equation with a wall and a step potential, i.e., with the potential function given by

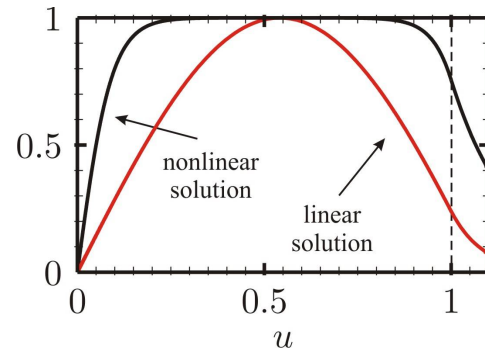


Fig. 2. Linear and nonlinear steady states (unnormalized), for the wall-step potential. The dashed vertical line indicates where the start of the step potential. Parameter values are $a=-0.3$, $m=0.9$, and $k_I=5.4$.

$$V(u) = \begin{cases} \infty, & x < 0, \\ 0, & 0 \leq x \leq 1, \\ V_0, & 1 < x. \end{cases} \quad (29)$$

Here u is a dimensionless length variable. The height of the step is larger than the chemical potential of the condensate, $\mu < V_0$.

We write the time independent Gross-Pitaevskii equation as

$$\frac{d^2 \psi(u)}{du^2} + \frac{2ML^2}{\hbar^2} \left(\mu - V_0 - \frac{NU_0}{A^2} |\psi(u)|^2 \right) \psi(u) = 0, \quad (30)$$

where $\psi(u)$ is the unnormalized steady state for the Bose-Einstein condensate (BEC), M is the mass of a single atom, N is the number of atoms in the condensate, $U_0 = 4\pi\hbar^2 a/M$ characterizes the atom-atom interaction, a is the scattering length, L is a scaling length corresponding to the width of the potential well, A^2 is the integral of the square of the wave function magnitude, μ is the chemical potential and V_0 is a constant external potential.

For $0 \leq u \leq 1$ (we call this region I), we can use the usual Jacobi elliptic function, $\psi_I(u) = \text{sn}(k_I u)$, with k_I and m_I constants to be determined later. This function satisfy the second order differential equation

$$\frac{d^2}{du^2} \text{sn}(k_I u) - k_I^2 [2m_I \text{sn}(k_I u)^3 - (1+m_I) \text{sn}(k_I u)] = 0. \quad (31)$$

Then, by comparison between Eq. (30) and Eq. (31), we find that

$$\mu = V_0 + \frac{\hbar^2 k_I^2}{2ML^2} + \frac{NU_0}{2A^2}, \quad m_I = \frac{ML^2 NU_0}{\hbar^2 k_I^2 A^2}. \quad (32)$$

Since it only depends on the characteristics of the condensate, the last term in the expression for μ is the self-energy of the condensate when the condensate is found in a region of constant potential. This result is in agreement with conjecture 1 of Ref. [21] that the self-energy $NU_0/2A^2$ is the difference between the chemical potential and the linear energy.

The relationship between u and x , in region I, is given by

$$u = \int_0^x \frac{dt}{\sqrt{1 - m_I \sin^2(k_I t)}}, \quad (33)$$

with $0 \leq u \leq 1$.

When $u > 1$ (we call this region II), we use the decaying function $mn(u)$ introduced in this work, with the origin at $u=1$, and $\psi_{II}(u) = T mn(k_{II}(u-1)) = T e^{-k_{II}y}$, where u and y are related by

$$u - 1 = \int_0^y \frac{dt}{\sqrt{1 + m_{II} \sinh^2(k_{II}t)}} , \quad (34)$$

Here, y is zero for the start of the step at $u=1$. Since we want to perform a comparison with the linear solution, we have set $b=a=1/2$.

The comparison between Eq. (30) and the differential equation for $\psi_{II}(u)$ (see Eq. (16))

$$\frac{d^2}{du^2} \psi_{II}(u) - k_{II}^2 \psi_{II}(u) \left[1 - \frac{m_{II}}{2} + \frac{m_{II}}{2T^2} \psi_{II}^2(u) \right] = 0, \quad (35)$$

indicates that

$$\mu = V_0 - \frac{\hbar^2 k_{II}^2}{2ML^2} + \frac{T^2 N U_0}{A^2} , \quad m_{II} = \frac{4ML^2 T^2 N U_0}{\hbar^2 k_{II}^2 A^2} . \quad (36)$$

Eqs. (32) and (36) should be consistent. This leads to

$$k_{II}^2 = \frac{2ML^2 N U_0}{\hbar^2 A^2} \left(T^2 - \frac{1}{2} \right) - k_I^2 . \quad (37)$$

The energy relations are given by

$$\left(\frac{d}{du} \psi_I(u) \right)^2 + k_I^2 \psi_I(u) \left[1 + m_I - m_I \psi_I^2(u) \right] = k_I^2 , \quad (38)$$

and

$$\left(\frac{d}{du} \psi_{II}(u) \right)^2 - k_{II}^2 \psi_{II}(u) \left[1 - \frac{m_{II}}{2} + \frac{m_{II}}{4T^2} \psi_{II}^2(u) \right] = T^2 k_{II}^2 \frac{m_{II}}{4} . \quad (39)$$

Thus, requiring the energy to be the same in both regions leads to $k_{II}^2 = 4k_I^2/m_{II}T^2$.

We now address the boundary conditions. With this choice of functions, $\psi(u)$ already satisfies the boundary condition at $u=0$, i.e., $\psi_I(u=0)=0$, and at $u=\infty$, $\psi_{II}(u=\infty)=0$. We therefore only should take care of the boundary conditions at $u=1$. Since the wave function should be continuous at $u=1$, we obtain $sn(k_I)=T mn(0)$, i.e., $\sin(k_I x_1)=T$, where x_1 is the value of x that corresponds to $u=1$, i.e.,

$$1 = \int_0^{x_1} \frac{dt}{\sqrt{1 - m_I \sin^2(k_I t)}}. \quad (40)$$

Now, equating the derivatives of $\psi_I(u)$ and $\psi_{II}(u)$ at $u=1$ leads to $k_I \operatorname{cn}(k_I) \operatorname{dn}(k_I) = -k_{II} T \operatorname{mn}(0) \operatorname{rn}(0)$, i.e., $k_I \cos(k_I x_1) \sqrt{1 - m_I \sin^2(k_I x_1)} = -k_{II} T$. The combination of the two conditions at $u=1$ gives the relationship between the parameters k_I and k_{II} that should be satisfied, $k_{II} = -k_I \cot(k_I x_1) \sqrt{1 - m_I \sin^2(k_I x_1)}$.

The normalization of the wave function is given by

$$A^2 = \int_0^\infty du |\psi(u)|^2 = \frac{1}{k_I m_I} (k_I - E(k_I; m_I)) + \frac{T^2}{k_{II}} \int_0^\infty dw m n^2(w), \quad (41)$$

where $E(u; m)$ is the elliptic integral of the second type. Then, the normalized wave function is obtained by dividing $\psi_I(u)$ and $\psi_{II}(u)$ by A .

Thus, a procedure for numerically finding the solutions starts with the setting of values for a, b, m_I and k_I , followed by the calculation of the value of x_1 that corresponds to $u=1$ using Eq. (40). We then calculate k_{II} using $k_{II} = -k_I \cot(k_I x_1) \sqrt{1 - m_I \sin^2(k_I x_1)}$, $T = \sin(k_I x_1)$ and $k_{II}^2 = 4k_I^2/m_{II} T^2$. This gives all the parameters necessary for defining the functions. Figure 2 shows a typical steady state together with the corresponding linear eigenfunction for comparison (calculated in such a way that their maxima are located at the same point). We found that the nonlinear wave function is less concentrated around its maximum than the linear state.

4. Remarks

The linear Schrödinger equation for a free particle has several sets of solutions, namely, trigonometric functions, real and complex exponential functions and their linear combinations. The functions introduced in this work will enable us to nonlinearly treat the same range of solutions as that for the linear equation. We are interested in these expression because they allow us to obtain physically appealing solutions in the presence of a decay of the wave function.

The nonlinear combination of other physical appealing functions will be explored in another paper.

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