

A Sufficient Condition for the Harmonicity of Functions of Two Variables That Satisfy a Difference Laplace Equation

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Abstract. In this paper we weaken the sufficient conditions of harmonicity for functions of two variables. I.I. Privalov had shown that a continuous function that satisfies the Laplace equation in each point of the domain is harmonic. For function of two variables the Privalov's condition on continuity can be weakened. G.P. Tolstov replaced the continuity condition by the boundness condition, later the author had shown that summability is sufficient. At the same time summability condition can not be weakened substantially.

In this paper, while we keep the summability condition, we provide the sufficient condition for harmonicity of the functions, that satisfy less restricted condition than the Laplace equation in all points of the domain. We assume that arbitrary close to any point ζ there exists a collection of four nodes for which a difference relation of Schwartz type for the Laplace equation can be made arbitrary small by the absolute value. Nodes are the ends of two mutually perpendicular segments, that intersect at point ζ . We need to impose a certain weakened continuous assumption on function itself, in case of function that satisfy the traditional Laplace condition this continuity condition follows from the existence of the partial derivatives.

Analytical function is called harmonic in the domain of the function, if it satisfies the Laplace equation in each point of the domain. If function satisfies the Laplace equation, then to obtain that it is harmonic we do not need to assume its analyticity, we can impose substantially less restrictive conditions on the function. At the same time there are examples that show that if a function only satisfies the Laplace condition or some of kind its generalization, at all points of the domain, it is not sufficient for the harmonicity of the function. Some additional conditions are required.

I.I. Privalov (1925, [1]) have shown that the continuous functions that satisfy in each point of the domain the Laplace equation are harmonic. In Privalov's theorem the number of variables is arbitrary, only those partial derivatives assumed to exist that are included in the Laplace equation, and the derivatives can be considered in the sense of Schwartz. The continuity condition can be substantially weakened in case of the functions of two variables.

Let function $u(z) = u(x, y)$, $z \in G \subset \mathbb{R}^2$, satisfies the Laplace equation in each point of the domain G . G.P. Tolstov (1951, [2]) have shown that for harmonicity of $u(z)$ in G boundness of $u(z)$ is sufficient; next the author (1986, [3]) have shown that it is sufficient to assume the summability of $u(z)$ with respect to the planar Lebesgue measure.



Function

$$u(z) := \Re\left(\frac{i}{z^2}\right) = \frac{2xy}{(x^2 + y^2)^2}, \quad z \neq 0, \quad u(0) := 0,$$

satisfies the Laplace equation at every point of the unit disk D , it is harmonic inside of D except for $z = 0$ and the absolute value of $|u(z)|$ is summable for all positive powers less than 1. From this example we can see that in the sufficient condition for the harmonicity we can not relax the summability assumption of $|u(z)|$ in any substantial way. That's why while keeping the summability condition for $|u(z)|$, in order to relax the sufficient condition for harmonicity we replace the requirement that the Laplace equation holds at every point of the domain by a less strict condition.

Thus this author (2009, [4]) considered functions that satisfy the generalized Laplace equation in region G , where instead of the traditional second derivatives we take the second derivatives in the sense of Peano. Expression $\Delta^\times u$ at point $\zeta \in G$ equals to the sum of the second Peano derivatives of function $u(z)$ in point ζ , that are taken along some mutually perpendicular lines that intersect at point ζ . The value of $\Delta^\times u$ in general can depend on choice of two lines along which the derivatives are taken. In [4] it was shown that if the function $u(z)$ is summable in region G and for every point $\zeta \in G$ there exists a pair of perpendicular to each other lines that intersect in ζ for which $\Delta^\times u = 0$, then function $u(z)$ is harmonic in G . The condition of mutual perpendicularity of lines in the pair can not be removed, while the directions of lines for different points of the region are not related to each other.

In work [4] first was established the harmonicity of continuous functions that satisfy the equation $\Delta^\times u = 0$, next was proved the continuity of summable functions that satisfy this condition. The conditions on derivations that appear in the equation $\Delta^\times u = 0$ are only used to prove the harmonicity of the continuous functions. In the proof of continuity of summable functions in G we do not require the existence of derivatives along certain directions for function $u(z)$ in each point $\zeta \in G$, moreover the existence of the second derivatives, but we only need the Lipschitz condition to hold along those direction for point ζ . The fact that the Lipschitz condition holds, which in this case is used as a continuity condition, is clear from the existence of directional derivatives for $u(z)$ at point ζ .

In this paper the conditions of continuity and differentiability are imposed separately on the function. The differentiability condition is imposed not on the sum of second derivatives of the function along certain two mutually perpendicular directions, but on a sum of difference relations of Schwartz type for the second derivatives of function $u(z)$, that are taken along such directions in non-equally spaced nodes in general. We do not require the existence of any kind of derivatives. Also we impose some condition on continuity of the function by direction, this condition is less strict than the Lipschitz condition, but more strict than the ordinary continuity.

The second Schwartz partial derivatives of the function

$$u(x, y) := \begin{cases} 1 & \text{for } x > y, \\ 0 & \text{for } x = y, \\ -1 & \text{for } x < y, \end{cases}$$

are equal to zero in every point. This example shows, that if the derivatives appearing in Laplace equation are Schwartz derivatives, then without any continuity condition we can not obtain a sufficient condition for the harmonicity even for the bounded functions.

Let us introduce some definitions.

We define as a *right angle cross* with center at the point ζ the union of two mutually perpendicular intervals $a_1(\zeta)$ and $a_2(\zeta)$ that intersect in the middle at the point ζ .

Now we obtain the expression for difference relation of Schwartz type for the second derivative of the function of one variable for unequally spaced nodes.

Difference relation for the Schwartz second derivative of function $u(x)$ in point x for nodes $x-h$ and $x+h$ is equal to the twice the value of the coefficient in x^2 term in the equation for parabola $y = ax^2 + bx + c$ that passes through points $(x-h; u(x-h))$, $(x; u(x))$ and $(x+h; u(x+h))$. So analogously to it we call as a difference relation $\delta^{(2)}u(x_0; x_1, x_2)$ for the second Schwartz derivative of function $u(x)$ at point x_0 with nodes x_1 and x_2 twice the value of coefficient in x^2 term in the equation for parabola that passes through points $(x_1; u(x_1))$, $(x_0; u(x_0))$ and $(x_2; u(x_2))$. We write equation for this parabola using the Lagrange interpolation polynomial. To simplify the notation we denote as $u_1 := u(x_1)$, $u_0 := u(x_0)$, $u_2 := u(x_2)$. We have

$$ax^2 + bx + c = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}u_1 + \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}u_0 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}u_2.$$

From the above follows

$$\begin{aligned} \delta^{(2)}u(x_0; x_1, x_2) &:= 2a = 2 \left(\frac{u_1}{(x_1-x_0)(x_1-x_2)} + \frac{u_0}{(x_0-x_1)(x_0-x_2)} + \frac{u_2}{(x_2-x_0)(x_2-x_1)} \right) = \\ &= 2 \frac{u_1(x_2-x_0) - u_0(x_2-x_1) + u_2(x_0-x_1)}{(x_0-x_1)(x_2-x_1)(x_2-x_0)}. \end{aligned} \quad (1)$$

For equally spaced nodes $x_1 = x_0 - h$, x_0 and $x_2 = x_0 + h$ the expression $\delta^{(2)}u(x_0; x_1, x_2)$ is the difference relation for a regular Schwartz second derivative of the function $u(x)$ at point x_0 . In fact,

$$\delta^{(2)}u(x_0; x_1, x_2) = 2 \frac{u_1(x_2-x_0) - u_0(x_2-x_1) + u_2(x_0-x_1)}{(x_0-x_1)(x_2-x_1)(x_2-x_0)} = 2 \frac{hu_1 - 2hu_0 + hu_2}{2h^3} = \frac{u_1 - 2u_0 + u_2}{h^2}.$$

Equation for the parabola that passes through three arbitrary points does not depend on the order in which these points are enumerated. We enumerate nodes in a more convenient way for us. Since the difference relation $\delta^{(2)}u(x_0; x_1, x_2)$ is considered as a difference relation for the second derivative at point x_0 , it is natural to assume that x_0 is located between x_1 and x_2 . If $x_1 < x_0 < x_2$, then all differences between coordinates of the nodes in formula (1) are positive and are equal to the distances between the corresponding nodes. So if we denote as $r_1 := |x_1 - x_0|$ and $r_2 := |x_2 - x_0|$, then the expression for $\delta^{(2)}u(x_0; x_1, x_2)$ can be rewritten as:

$$\delta^{(2)}u(x_0; x_1, x_2) = 2 \frac{u_1 r_2 - u_0(r_1 + r_2) + u_2 r_1}{r_1 r_2 (r_1 + r_2)}. \quad (2)$$

It is easy to show that the difference relation $\delta^{(2)}u(x_0; x_1, x_2)$ will give in the limit the second Peano derivative of function $u(x)$ at the point x_0 , if that derivative exists.

Now we obtain the expression for the difference Laplace equation that we analyze in this paper. Let function $u(z)$ be defined in region $G \subset \mathbb{R}^2$, take a point $z_0 \in G$ and a set $A_{z_0} = \{z_1, z_2, z_3, z_4\}$ of four points in the region G for which segments $[z_1; z_2]$ and $[z_3; z_4]$ are mutually perpendicular and intersect at the internal point z_0 . Denote as $r_j := |z_j - z_0|$, $u_j := u(z_j)$, $j = 1, \dots, 4$. Consider

$$\Delta^{(*)}u(z_0) = \Delta^{(*)}u(z_0; A_{z_0}) := \delta_{z_1 z_2}^{(2)}u(z_0; z_1, z_2) + \delta_{z_3 z_4}^{(2)}u(z_0; z_3, z_4),$$

where the quantities $\delta_{z_1 z_2}^{(2)}u(z_0; z_1, z_2)$ and $\delta_{z_3 z_4}^{(2)}u(z_0; z_3, z_4)$ are defined by formula (2). We obtain the expression

$$\Delta^{(*)}u(z_0) = 2 \frac{u_1 r_2 - u_0(r_1 + r_2) + u_2 r_1}{r_1 r_2 (r_1 + r_2)} + 2 \frac{u_3 r_4 - u_0(r_3 + r_4) + u_4 r_3}{r_3 r_4 (r_3 + r_4)}. \quad (3)$$

We say that the function $u(z)$ satisfies the *generalized difference Laplace equation* $\Delta^{(*)}u = 0$ at point z_0 , if the limit of $\Delta^{(*)}u(z_0; A_{z_0}^{(n)})$ is equal to zero at least for one sequence of sets of nodes $\{A_{z_0}^{(n)}\}$, that contract to z_0 . For different sequences of sets of nodes $\{A_{z_0}^{(n)}\}$, that contract to z_0 , the limits of $\Delta^{(*)}u(z_0; A_{z_0}^{(n)})$ could be different or not exist.

Now we define the continuity condition that we impose on function $u(z)$ in points of the region G .

Let $h(t)$, $t \geq 0$, be a modulus of continuity type function and let function $u(z)$ be defined on some set B_ζ , $z \in B_\zeta$, where point ζ is the limiting point. If for some value $L_\zeta > 0$ at every point $z \in B_\zeta$ the following inequality holds $|u(z) - u(\zeta)| \leq L_\zeta h(|z - \zeta|)$, then we say that function $u(z)$ is *h-regular at point ζ with respect to set B_ζ with coefficient L_ζ* , or simply *h-regular at ζ with respect to B_ζ* , if specific value of L_ζ is not important.

Clearly if function $u(z)$ is differentiable at point ζ along B_ζ as a function of two real arguments, then for some $L_\zeta > 0$ in all points $z \in B_\zeta$ the inequality $|u(z) - u(\zeta)| \leq L_\zeta |z - \zeta|$ holds. It means that function $u(z)$ is *h-regular at point ζ with respect to B_ζ for a linear function $h(t)$* .

In points $\zeta \in G$ as a continuity condition we consider *h-regularity condition with respect to some right angle cross T_ζ or some star S_ζ* .

Theorem 1. *Let function $u(z)$ be locally summable in region $G \subset \mathbb{R}^2$, while $h(t)$, $t \geq 0$ is some function of modulus of continuity type. If for each point $\zeta \in G$ the function $u(z)$ satisfies the generalized difference Laplace equation and is h-regular with respect to some right-angle cross T_ζ , then function $u(z)$ is harmonic in region G .*

There is no connection between the locations of the nodes for different sets $A_\zeta^{(n)}$ for a given point $\zeta \in G$. Similarly we do not assume any connection between sets of nodes for different points of region G . The orientation and the length of the intervals that make up the crosses T_ζ for different points $\zeta \in G$ do not depend from each other.

In this paper we prove of a special case of the Theorem 1.

Theorem 2. *If function $u(z)$ is continuous in a region G and satisfies the generalized difference Laplace equation at each point $z \in G$, then function $u(z)$ is harmonic in the region G .*

Proof of Theorem 2 follows the approach similar to the Theorem 2, I.I. Privalov [1]. Proof of the Privalov's theorem is also provided in the footnote in the work of I.G. Petrovskii [5], pg. 107.

In the proof of the Theorem 2 we use the following Lemma. We do not give its proof since it amounts to the immediate check.

Lemma 1. *Assume that function $u(z)$ has the second Peano differential at point $z_0 = (x_0, y_0)$*

$$u(z) = u(z_0) + du + \frac{1}{2}d^2u + \alpha(z)|z - z_0|^2, \quad (4)$$

where $du = p(x - x_0) + q(y - y_0)$ and $d^2u = r(x - x_0)^2 + 2s(x - x_0)(y - y_0) + t(y - y_0)^2$, while $\alpha(z) \rightarrow 0$ when $z \rightarrow z_0$. Then for any sequence of sets of nodes $\{A_{z_0}^{(n)}\}$ that contract to z_0

$$\lim_{n \rightarrow \infty} \Delta^{(*)}u(z_0, A_{z_0}^{(n)}) = r + t.$$

Corollary 1.1. *If the function is harmonic in the region, then in every point of the region it satisfies the generalized difference Laplace equation for any sequence of sets of nodes that contract to that point.*

Corollary 1.2. *For the polynomial of the second degree $u(x) = ax^2 + bx + c$ for any collection of nodes $\{x_0, x_1, x_2\}$ the expression $\delta^{(2)}u(x_0; x_1, x_2)$ is exactly equal to $2a$. That why, if function $u(z) = a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + a_{11}x + a_{22}y + a_0$ is a polynomial of the second degree in two variables, then for any point z and any set of nodes A_z we have the identity $\Delta^{(*)}u(z, A_z) \equiv 2(a_{11} + a_{22})$.*

Proof of the Theorem 2. Proof of the Theorem 2. Assume that the statement is not true and that the function $u(z)$ is not harmonic everywhere in G . We take a disk $D \subset G$ inside of which $u(z)$ is not harmonic everywhere. Let $v(z)$ be a function that is harmonic inside of the disk D and coincides with the continuous function $u(z)$ on ∂D . Then from the Corollary 1.1 to Lemma follows that function $w(z) := u(z) - v(z)$ satisfies the generalized difference Laplace equation everywhere inside of D .

By the definition $w(z) \equiv 0$ on ∂D , but this function is not identically equal to zero inside of D . Assume that $\max_D w(z) > 0$, if not then instead of $w(z)$ we consider the function $-w(z)$. Let $z_0 = (x_0, y_0)$ be a point of maximum $w(z)$ in disk D : $\max_D w(z) = w(z_0) > 0$, and this point z_0 belongs to interior of D . Denote as

$$\varphi(z) := w(z) + \frac{w(z_0)}{2 \text{diam}^2 D} ((x-x_0)^2 + (y-y_0)^2), \quad z \in D.$$

Since the function $\varphi(z)$ is equal to the sum of function $w(z)$ and polynomial of the second degree $\frac{w(z_0)}{2 \text{diam}^2 D} ((x-x_0)^2 + (y-y_0)^2)$, it means that from the Corollary 1.2 to Lemma, we obtain that for any point ζ from the interior of disk D and any set of nodes $A_\zeta \subset D$:

$$\Delta^* \varphi(\zeta, A_\zeta) = \Delta^{(*)} w(\zeta, A_\zeta) + \frac{w(z_0)}{\text{diam}^2 D}. \quad (5)$$

By the definition of $\varphi(z)$ everywhere inside of D we have $\varphi(z) \geq w(z)$, therefore

$$\max_D \varphi(z) \geq \max_D w(z) = w(z_0) > 0.$$

Since all the points of ∂D are closer to z_0 (the interior point of D) than the value of $\text{diam } D$ and on ∂D the function $w(z) \equiv 0$, it follows from the definition of $\varphi(z)$ that on ∂D we have

$$\varphi(z) = w(z) + \frac{w(z_0)}{2 \text{diam}^2 D} ((x-x_0)^2 + (y-y_0)^2) = \frac{w(z_0)}{2 \text{diam}^2 D} ((x-x_0)^2 + (y-y_0)^2) \leq \frac{w(z_0)}{2}.$$

This is the reason why $\varphi(z)$ takes the maximum value in the interior of disk D . Let ζ_0 be the point of maximum: $\max_D \varphi(z) = \varphi(\zeta_0)$.

For each interior point ζ of disk D there exists a set of nodes A_ζ for which the absolute value of $\Delta^{(*)} w(\zeta, A_\zeta)$ can be made arbitrary small. Let $A_{\zeta_0} = \{\zeta_1, \zeta_2, \zeta_3, \zeta_4\} \subset D$ be the set of nodes for which $\left| \Delta^{(*)} w(\zeta_0, A_{\zeta_0}) \right| < \frac{w(z_0)}{2 \text{diam}^2 D}$. According to the expression (5) for this set the following inequality holds

$$\Delta^{(*)} \varphi(\zeta_0, A_{\zeta_0}) = \Delta^{(*)} w(\zeta_0, A_{\zeta_0}) + \frac{w(z_0)}{\text{diam}^2 D} > \frac{w(z_0)}{2 \text{diam}^2 D} > 0.$$

Assuming that $\varphi_j := \varphi(\zeta_j)$ and $r_j := |\zeta_j - \zeta_0|$ for $j = 1, 2, 3, 4$, from formula (3) we obtain that

$$\Delta^{(*)} \varphi(\zeta_0, A_{\zeta_0}) = 2 \frac{\varphi_1 r_2 - \varphi_0(r_1 + r_2) + \varphi_2 r_1}{r_1 r_2 (r_1 + r_2)} + 2 \frac{\varphi_3 r_4 - \varphi_0(r_3 + r_4) + \varphi_4 r_3}{r_3 r_4 (r_3 + r_4)} > 0.$$

At least one of the terms in $\Delta^{(*)} \varphi(\zeta_0)$ is positive. For example, assume that it is the first term. We can rewrite the first term as:

$$\frac{\varphi_1 r_2 - \varphi_0(r_1 + r_2) + \varphi_2 r_1}{r_1 r_2 (r_1 + r_2)} = \frac{r_2(\varphi_1 - \varphi_0) + r_1(\varphi_2 - \varphi_0)}{r_1 r_2 (r_1 + r_2)} > 0.$$

Values of r_1 and r_2 are positive, since they are distances. So at least one of the differences $\varphi(\zeta_1) - \varphi(\zeta_0)$ or $\varphi(\zeta_2) - \varphi(\zeta_0)$ is positive, but it contradicts to the statement that at the point ζ_0 the function $\varphi(z)$ reaches the maximum value. Therefore our assumption, that the function $u(z)$ is not harmonic everywhere inside of the region G is not true. It finishes the proof of the Theorem 2.

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