The heat equation source determination for the case of non-smooth boundary and initial conditions

V V Solovi'ev and D S Tkachenko

National Research Nuclear University MEPhI (Moscow Engineering Physics Institute), 31 Kashirskoe shosse, 115409 Moscow, Russia

E-mail: DSTkachenko@mephi.ru

Abstract. An inverse problem of reconstructing the source of a special kind for parabolic equations in a bounded region with smooth boundary is considered. Solutions are sought in the Hölder classes. We prove an uniqueness criterion for the solution and sufficient conditions of Fredholm property of the task at hand. As a consequence of the sufficient conditions for existence and uniqueness of solution of the inhomogeneous inverse problems are found.

1. The problem statement

Let $T > 0$, $0 < \alpha < 1$ is a fixed constant, $\Omega \subset \mathbb{R}^n$ is a bounded region with boundary of class $C^{2,\alpha}$. Let us consider in the cylinder $\Omega_T = \Omega \times (0, \mathrm{T}]$ the inverse problem of determining a pair of functions (u, f) satisfying the conditions

$$u_t(x,t) = \Delta u(x,t) + f(x)h(x,t) + g(x,t), \quad (x,t) \in \Omega_T \tag{1}$$

$$u(x,0) = \varphi(x), \quad x \in \bar{\Omega}, \tag{2}$$

$$u(x,t) = \mu(x,t), \quad (x,t) \in \Gamma_T = \partial\Omega \times [0,T], \tag{3}$$

$$u(x,T) = \chi(x), \quad x \in \bar{\Omega}. \tag{4}$$

The inverse problem (1) – (4) has been studied in various functional spaces. The history of the study of such problems are presented in [1], [2], [3], [4]. The solvability of problem (1) – (4) results have been obtained for the case of Hölder spaces $u \in C^{2+\alpha,1+\frac{\alpha}{2}}(\bar{\Omega}_T), f \in C^{\alpha}(\bar{\Omega})$ (see [5]) or in the Sobolev space $u \in W_2^{1,0}(\Omega_T), f \in L_2(\Omega)$ (see [6]). In all previous results it is assumed that μ, φ are smooth functions. In the first case it is assumed that $\mu \in C^{2+\alpha,1+\frac{\alpha}{2}}(\Gamma_T)$,$\varphi \in C^{2+\alpha}(\bar{\Omega})$, in the second one $\mu \in W_2^{2,1}(\Gamma_T)$, $\varphi \in W_2^1(\Omega)$. For the first case, in addition, matching conditions of the first order for functions φ, χ, μ, h and their derivatives on $\partial\Omega$, at t=0 and t=T are required. This paper is studying the problem (1) – (4) assuming that the function $\mu \in C(\Gamma_T)$, $\varphi \in C(\bar{\Omega})$ and the function μ is derived only if $t \in (\bar{t}, T], 0 < \bar{t} < T$.

The direct problem (1) – (3) is studied on a wider class of functions and does not imply smoothness of its solution up to $t = 0$.

As we'll see, this will allow to refuse requirements for the matching conditions of the first order to hold, and thus significantly expands the class of functions φ, g, μ, χ, for which the problem (1) – (4) has a unique solution. Let us give a precise definition.

Content from this work may be used under the terms of the Creative Commons Attribution 3.0 licence. Any further distribution of this work must maintain attribution to the author(s) and the title of the work, journal citation and DOI.
Published under licence by IOP Publishing Ltd

2. Main results

Denote as $\Omega_{\bar{t}} = \overline{\Omega} \times (\bar{t}, T]$ and $\Gamma_{\bar{t}} = \partial\Omega \times (\bar{t}, T]$. The solution of the problem (1) – (4) is a pair of functions

$$(u, f) \in \left(C^{2,1}_{x,t}(\Omega_T) \cap C(\overline{\Omega}_T) \cap C^{2+\alpha, 1+\frac{\alpha}{2}}(\Omega_{\bar{t}}) \right) \times C^{\alpha}(\overline{\Omega}) = \mathrm{U}(\Omega_T) \times C^{\alpha}(\overline{\Omega}),$$

satisfying conditions (1) – (4).

Following solvability theorem for the direct problem (1) – (3) in this not quite conventional class of functions $\mathrm{U}(\Omega_T)$ holds.

Theorem 1. Let the functions φ, g, μ, f, h lie in classes $g \in C^{\alpha, \frac{\alpha}{2}}(\overline{\Omega}_T), \mu \in C^{2+\alpha, 1+\frac{\alpha}{2}}(\Gamma_{\bar{t}}) \cap C(\Gamma_T), \varphi \in C(\overline{\Omega}),\ f \in C^{\alpha}(\overline{\Omega})$, the matching condition of order zero holds:

$$\varphi(x) = \mu(x, 0), \quad x \in \partial\Omega.$$

Then there exists a unique solution of problem (1) – (3) a function $u \in \mathrm{U}(\Omega_T)$.

The proof is directly obtained by successive application of the results on the solvability of the direct problem from [7], 469 p., [8], 95, [7] p. 260.

Let us first consider the most important for inverse problems topic of the solution of problem (1) – (4) uniqueness in the considered class of functions.

Theorem 2. Let $h, h_t \in C^{\alpha, \frac{\alpha}{2}}(\overline{\Omega}_T)$ and $h(x,t)h_t(x,t) \geq 0$ for $(x,t) \in \overline{\Omega}_T$.

Then the problem (1) – (4) cannot have two different solutions in the class when, and then only when $\operatorname{supp} h(x,T) = \overline{\Omega}$.

The proof of the theorem is based on some modification of the reasoning made in the proof of the solution uniqueness for the inverse problem of the in [5].

Now consider the question of existence of solutions of problem (1) – (4). Let us assume the following definition: we say that matching condition of order zero holds for the problem (1) – (4) if the functions φ, χ, μ satisfy the conditions

$$\varphi(x) = \mu(x, 0), \ \ \chi(x) = \mu(x, T), \ \ x \in \partial\Omega\,. \tag{5}$$

Remark. It is clear that the matching conditions are necessary for the existence of solutions of problem (1) – (4) in the class $u \in C(\overline{\Omega}_{\mathrm{T}})$.

We will prove the following statement clarifying the nature of solvability of problem (1) – (4).

Theorem 3. Let $h, h_t \in C^{\alpha, \frac{\alpha}{2}}(\overline{\Omega}_T), |h(x,T)| \geq h_T > 0,\ x \in \overline{\Omega}$. Then the Fredholm Alternative holds for the inverse problem (1) – (4), which assumes equivalence of the two statements:

1) problem (1) – (4) when $g \equiv 0,\ \mu \equiv 0,\ \chi \equiv 0$ has only trivial solution, i.e. the solution of problem (1) – (4) is unique;
2) problem (1) – (4) has a unique solution $(u, f) \in \mathrm{U}(\Omega_T) \times C^{\alpha}(\overline{\Omega})$ for given functions $(g, \mu, \chi, \varphi) \in C^{\alpha, \frac{\alpha}{2}}(\overline{\Omega}_T) \times \left(C^{2+\alpha, 1+\frac{\alpha}{2}}(\Gamma_T) \cap C(\Gamma_T) \right) \times C^{2+\alpha}(\overline{\Omega}) \times C(\overline{\Omega})$, that satisfy matching conditions of order zero.

Proof. Show first that if condition 1) is true, then condition 2) holds. Let us build this solution. We seek a pair of functions (u, f) in the form

$$(u, f) = \left(u^{(1)}, 0\right) + \left(u^{(2)}, f\right),$$

where the function $u^{(1)} \in \mathrm{U}(\Omega_T)$ is the solution of the direct problem

$$\left(Lu^{(1)}\right)(x,t) = \ g(x,t), \qquad (x,t) \in \Omega_T, \tag{6}$$

$$u^{(1)}(x,0) = \varphi(x), \quad x \in \overline{\Omega}, u^{(1)}(x,t) = \mu(x,t), \quad (x,t) \in \Gamma_T. \tag{7}$$

The pair $(u^{(2)}, f) \in \mathrm{U}(\Omega_T) \times C^{\alpha}(\overline{\Omega})$ is a solution of the inverse problem:

$$\left(Lu^{(2)}\right)(x,t) = \ f(x)h(x,t), \qquad (x,t) \in \Omega_T, \tag{8}$$

$$u^{(2)}(x,0) = 0, \quad x \in \overline{\Omega}, \qquad u^{(2)}(x,t) = 0, \quad (x,t) \in \Gamma_T, \tag{9}$$

$$u^{(2)}(x,T) = \chi(x) - u^{(1)}(x,0) = \tilde{\chi}(x), \quad x \in \overline{\Omega}. \tag{10}$$

Because of the matching conditions and known results (see [7], 461, [7], p. 260) on the solvability of the problems for the heat equation the function $u^{(1)}(x,t)$ must lie in the class $\mathrm{U}(\Omega_T)$. Then $\tilde{\chi} \in$

$C^{2+\alpha}(\overline{\Omega})$. Thus to prove that assertion 1) implies assertion 2) it is sufficient to prove the existence of solutions of problem (8) – (10) under the assumption that $\tilde{\chi} \in C^{2+\alpha}(\overline{\Omega})$, $\tilde{\chi}(x) = 0,\ x \in \partial\Omega$. For this, we consider the problem of determining the three functions (u, v, f) from the conditions:

$$(Lu)(x,t) = f(x)h(x,t), \quad (x,t) \in \Omega_T \tag{11}$$
$$u(x,0) = 0, \quad x \in \overline{\Omega}, \qquad u(x,t) = 0, \quad (x,t) \in \Gamma_T, \tag{12}$$
$$(Lv)(x,t) = f(x)h_t(x,t), \quad (x,t) \in \Omega_T \tag{13}$$
$$v(x,0) = f(x)h(x,0), \ x \in \overline{\Omega}, \qquad v(x,t) = 0, \quad (x,t) \in \Gamma_T, \tag{14}$$
$$f(x) = \frac{1}{h(x,T)}v(x,T) - \frac{1}{h(x,T)}\Delta\tilde{\chi}, \quad x \in \overline{\Omega}. \tag{15}$$

Under the conditions (11) – (12) the function u will lie in the class $\mathrm{U}(\Omega)$, v is generalized solution (see [11]) of the direct problem (13) – (14), i.e., v is a bounded function from the class $v \in C^{2,1}_{x,t}(\Omega_T) \cap C\left(\overline{\Omega}_T \setminus (\partial\Omega \times \{\mathrm{t} = 0\})\right)$. The solution of this problem, as shown in [11], can be produced with the Green's function. Define a linear operator $A\colon\ C^{\alpha}(\Omega) \mapsto C^{\alpha}(\Omega)$, by the rule:

$$A(f)(x) = \frac{1}{h(x,T)}v(x,T), \quad x \in \overline{\Omega}. \tag{16}$$

Consider a function $v(x,t)$ in the form:

$$v(x,t) = v^{(1)}(x,t) + v^{(2)}(x,t),$$

where $v^{(1)} \in C^{2+\alpha,1+\alpha/2}_{x,t}(\overline{\Omega}_T)$ is the solution of the problem

$$\left(Lv^{(1)}\right)(x,t) = f(x)h_t(x,t), \quad (x,t) \in \Omega_T \tag{17}$$
$$v^{(1)}(x,0) = 0, \ x \in \overline{\Omega}, \qquad v^{(1)}(x,t) = 0, \quad (x,t) \in \Gamma_T, \tag{18}$$

function $v^{(2)}$ is generalized solution of the problem

$$\left(Lv^{(2)}\right)(x,t) = 0, \quad (x,t) \in \Omega_T, \tag{19}$$
$$v^{(2)}(x,0) = f(x)h(x,0), \ x \in \Omega, v^{(2)}(x,t) = 0, \ (x,t) \in \Gamma_T \setminus (\partial\Omega \times \{\mathrm{t} = 0\}). \tag{20}$$

Note that for the function $v^{(1)}(x,t)$ Schauder estimation holds (see [8], p. 88):

$$\left\|v^{(1)}\right\|^{(2+\alpha,1+\alpha/2)}_{\Omega_T} \le c_1 \|f\ h_t\|^{(\alpha,\alpha/2)}_{\Omega_T}. \tag{21}$$

But then, by definition of the Hölder norm for the function $v^{(1)}$ (see, e.g., [7], p. 16), we obtain the estimate:

$$\left\|v^{(1)}_{x_i}(x,T)\right\| \le c_2 \|f\|^{(\alpha)}_{\Omega}, \tag{22}$$

the constant c_2 here does not depend on the function f.

Let us use the evaluation of the green's function (see [7], p. 469) for the function $v^{(2)}$:

$$\left|G_{x_i}(x,\xi;T,0)\right| \le c_3 T^{-1/2} \exp\left\{-c_4 \frac{|x-\xi|^2}{T}\right\}.$$

Then $v^{(2)}(x,T)$ well satisfy the following estimate:

$$\left|v^{(2)}_{x_i}(x,T)\right| \le c_4 \|fh(x,0)\| \le c_5 \|f\|^{(\alpha)}_{\Omega}.$$

From this, operator A can be estimated this way:

$$\left|A(f)_{x_i}(x)\right| \le c_6 \|f\|^{(\alpha)}_{\Omega}.$$

And now we get the final estimate:

$$\left\|A(f)_{x_i}\right\| = \max_{x\in\Omega}\left|A(f)_{x_i}(x)\right| \le c_6 \|f\|^{(\alpha)}_{\Omega}. \tag{23}$$

Since the space $C^1(\Omega)$ is compactly embedded into the space $C^{\alpha}(\Omega)$ (see, e.g., [11]), the estimate (23) implies that equation (15) is a Fredholm equation of the second kind with a compact operator. But for such equation from the Riesz theorem (see, e.g., [12], p. 212) we get that the uniqueness of the solution of the homogeneous equation (15) implies the existence of solutions of the equation (15) for any function $\frac{L\chi}{h(x,T)}$. But the uniqueness of the solution of the inverse problem (8) – (10) implies the uniqueness of the solution of the homogeneous equation (15). It is clear that the found function f and

the solution of the direct problem (11) – (12) satisfy all the equalities that define the inverse problem (1) – (4) (the proof is standard and is given in e.g. [5]). Thus, it is proven that the uniqueness of the solution of the inverse problem (5) – (7) implies that this solution exists.

Prove that, other way, if the inverse problem (1) – (4) has a unique solution for any four functions, then by setting $g \equiv 0,\ \mu \equiv 0,\ \varphi \equiv 0$, we obtain that the inverse problem (8) – (10) has a solution for any function $\chi \in C^{2+\alpha}(\bar{\Omega})$. But then (15) has a solution for any function $\Delta\tilde{\chi}$, as an equation of the second kind with a compact operator (16). From the Riesz theorem for these equations we find that the homogeneous equation (15) has only trivial solution. But then the inverse problem (1) – (4) has only trivial solution either.

Theorem 2 is completely proved.

3. The corollaries

Corollary 1. Let $h, h_t \in C^{\alpha,\frac{\alpha}{2}}(\bar{\Omega}_T), |h(x,T)| \geq h_T > 0,\ x \in \bar{\Omega},\ h(x,t)h_t(x,t) \geq 0, (x,t) \in \Omega_T$. Then for any four functions $g,\ \mu,\ \varphi,\ \chi$, such that $g \in C^{\alpha,\frac{\alpha}{2}}(\bar{\Omega}_T)$, $\mu \in C^{2+\alpha,1+\frac{\alpha}{2}}(\Gamma_{\bar{t}}) \cap C(\Gamma_T)$, $\varphi \in C(\bar{\Omega})$, $\chi \in C^{2+\alpha}(\bar{\Omega})$ satisfying the matching conditions:

$$\varphi(x) = \mu(x,0),\ \ \chi(x) = \mu(x,T),\ \ \ x \in \partial\Omega,$$

exists the solution of problem (1) – (4) in the class considered and it is unique.

The proof follows from theorem 1 and theorem 2.

Corollary 2. Let $h, h_t \in C^{\alpha,\frac{\alpha}{2}}(\bar{\Omega}_T), |h(x,T)| \geq h_T > 0,\ x \in \bar{\Omega}$, $l = \operatorname{diam}\Omega$. Then there exists a number l_0 such that for any domains $\Omega, \operatorname{diam}\Omega < l_0$ inverse problem (1) – (4) has a unique solution for any four functions $g,\ \mu,\ \varphi,\ \chi$ from the class considered satisfying the matching conditions.

The proof of uniqueness of the solution of the inverse problem (1) – (4) follows from the estimates for Green function (see [7], p. 169). A detailed proof of some similar propositions see in [5]. The existence follows from theorem 2.

Choose one of the axes $x = x_i,\ \ i \in \{1, \cdots, n\}$, and define the parameters

$$l_i = \sup_{(x,y)\in\Omega}|x_i - y_i| = \operatorname{diam}_i\Omega.$$

Then the following statement is true.

Corollary 3. Let $h, h_t \in C^{\alpha,\frac{\alpha}{2}}(\bar{\Omega}_T), |h(x,T)| \geq h_T > 0,\ x \in \bar{\Omega}$. Then exists a number l_0^i that for any Ω of class $\partial\Omega \in C^{2,\alpha}$, $\operatorname{diam}_i\Omega < l_0^i$ inverse problem (1) – (4) has a unique solution for any four functions $g,\ \mu,\ \varphi,\ \chi$ from the class considered satisfying the matching conditions.

The proof of uniqueness of the solution of the inverse problem (1) – (4) in the conditions of the Corollary 3 is in detail carried out in [5].

Existence again follows from theorem 2.

Acknowledgements

In conclusion, the authors express deep gratitude to all participants of the workshop on inverse problems under the guidance of V. A. Sadovnichii and A. I. Prilepko, Mechanics and Mathematics department, MSU for their active and useful discussion the above results.

References

[1] Denisov A M 1994 *Introduction into the inverse problems theory* (Moscow: Moscow State University Press) p 207

[2] Lavrent'ev M M 1973 *Conditionally well-posed problems for differential equations* (Novosibirsk: Novosibirsk State University Press) p 71
[3] Isakov V 2006 *Inverse problems for partial differential equations* (New York: Springer) p 344
[4] Prilepko A I, Orlovsky D G, Vasin I A 2000 *Methods for solving inverse problems in mathematical physics* (New York – Basel: Marcel Dekker Inc.) p 723
[5] Soloviev V V 1989 *Differential equations* On solvability of source restoration problem with final overdetermination for a parabolic equation **25** 1577-83
[6] Prilepko A I, Kostin A B 1993 *Sib. Mathem. Journal* **34** 147 –162
[7] Ladyjenskaya O A, Solonnikov V A, Ural'ceva N N 1967 *Linear and quasilinear equations of parabolic kind* (Moscow: Nauka) p 736
[8] Fridman A 1968 *Equations with partial derivatives of parabolic kind* (Moscow: Mir) p 428
[9] Mihailov V P 1983 *Differential equations in partial derivatives* (Moscow: Nauka) p 391
[10] Gilbarg D, Trudinger N 1989 *Elliptic Differential equations with partial derivatives of the second order* (Moscow: Nauka) p 465
[11] Landis E M 1971 *Equations of the second order of elliptic and parabolic kinds* (Moscow: Nauka) p 288
[12] Trenogin V A 2002 *Functional analysis* (Moscow: Fizmatlit) p 488