

On Finite Strains in Continua with Complex Rheology

E Murashkin^{1,2,3a} and V Klindukhov¹

¹National Research Nuclear University MEPhI (Moscow Engineering Physics Institute), 31 Kashirskoe shosse, 115409 Moscow, Russia

²Institute for Problems in Mechanics of the Russian Academy of Sciences, Vernadski Ave. 101 Bldg. 1, Moscow, 119526, Russia

³Bauman Moscow State Technical University, 2nd Baumanskaya Str., 5/1, Moscow, 105005, Russia

E-mail: ^aevmurashkin@gmail.com

Abstract. The present study deals with the multiphysics modelling frameworks for the elastic-creep-plastic material. The proposed theory of large deformations is based on the classical formalism of nonequilibrium thermodynamics. Reversible and irreversible components of total deformations are derived by the constitutive differential equations. The development of this theory involves statements and solutions of the correct boundary value problems. The problem of deformation of the material in the vicinity of microdefect was considered. Resulting integro-differential equations system was derived.

1. Preliminary Remarks

Modern additive manufacturing technology [1, 2] states the problem of creating and developing of new mathematical models of growth and deformation of structural materials. The calculation of stress-strain state parameters is needed during the forming processes of material [3]. In the design of ship hulls and aircraft are widely used panels and profiles of hardly-deformed aluminum alloys [4]. Traditional methods of formation of such structural elements often leads to the appearance of the plastic breaks, cracks and other damages. Thus, the effective way of this fabrication is metal forming under creep and low strain rates. These processes ensures the production of construction with high accuracy, which reduces the complexity of assembly and welding, and increase the residual life and the quality of construction.

The mathematical description of the processes of thermomechanical treatment of contracture materials is faced with the necessity to consider the elastic properties of materials at all stages of the producing and utilization. Consideration of the problems in the classical models of small deformations is impossible when the relative shape of the body is strongly changed. Such problem is the process of deformation metals in vicinity of micropore under the intense pressure action. In this case, we are faced to assumption of large strains. Well known experiments [3, 4] when the long-term strength of metal products after their treatment under hydrostatic pressure is significantly increased. Attempts to simulate such process of the micropores "healing" in the metal have been made repeatedly [4-14]. In [5] such problems are considered due to the model of large elastic-plastic deformation. In this case, the effect of adaptability to periodic loading on the cycle "loading and unloading" was shown in [6].



Since classic work E. Lee [7], lots of plastic flow models with large reversible and irreversible deformations were developed [4-14]. Lot of them use a Lagrangian description [13, 14]. But in this case the results of mathematical modeling is physically difficult interpreted. If we want to build a framework of the flow in Eulerian descriptions, then we are faced with two fundamental problems. The first problem is identification irreversible and reversible components of the total strain tensor. The second problem is constitution of irreversible strain source. The mathematical model which was proposed in [5, 8, 12] is used throughout the paper. The problem of a spherically symmetric compression of the ball with micropore in the center is considered.

2. Constitutive Equation

Throughout the paper we use mathematical model of the finite elastocreep deformations (see details in [11]). The kinematic equation for parts of the Almansi total strain d_{ij} can be written in the Cartesian system (Eulerian coordinates) in the form

$$\begin{aligned} \frac{De_{ij}}{Dt} &= \varepsilon_{ij} - \gamma_{ij} - \frac{1}{2}((\varepsilon_{ik} - \gamma_{ik} + z_{ik})e_{kj} + e_{ik}(\gamma_{kj} - \varepsilon_{kj} - z_{kj})), \\ \frac{Dp_{ij}}{Dt} &= \gamma_{ij} - p_{ik}\gamma_{kj} - \gamma_{ik}p_{kj}, \end{aligned} \quad (1)$$

where $e_{ij}^e = e_{ij} - 0.5e_{ik}e_{kj}$ is the reversible part of the Almansi total strain tensor, p_{ij} is the irreversible part of the Almansi total strain tensor, D/Dt denotes the convective derivative with respect to time, γ_{ij} is the irreversible strain rate tensor, ε_{ij} is the strain rate tensor. The strain rate tensor can be computed by the equation

$$\varepsilon_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i}), \quad (2)$$

where v_i are the components of velocity vector, the index after comma denotes partial derivative with respect to the corresponding spatial coordinate.

The convective derivative with respect to time in (1) from an arbitrary tensor n_{ij} read:

$$\begin{aligned} \frac{Dn_{ij}}{Dt} &= \frac{dn_{ij}}{dt} - r_{ik}n_{kj} + n_{ik}r_{kj}, \\ r_{ij} &= w_{ij} + z_{ij}(e_{ij}, \varepsilon_{ij}), \quad w_{ij} = \frac{1}{2}(v_{i,j} - v_{j,i}), \end{aligned} \quad (3)$$

wherein w_{ij} is the angular rate tensor and $z_{ij}(e_{ij}, \varepsilon_{ij})$ is the nonlinear part of the rotation tensor r_{ij} (see in full in [5]). Thus, the components of the Almansi total strains d_{ij} in terms of its parts e_{ij} and p_{ij} taking account of equations (1) and (3) are presented as follows

$$d_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i} - u_{i,k}u_{k,j}) = e_{ij} + p_{ij} - \frac{1}{2}e_{ik}e_{kj} - e_{ik}p_{kj} - p_{ik}e_{kj} + e_{ik}p_{km}e_{mj}, \quad (4)$$

where u_i are the translational displacements. This assumption allows to obtain the constitutive equation like the Murnaghan equation of state well known in the non-linear elasticity [15]:

$$\sigma_{ij} = -p\delta_{ij} + \frac{\partial Q}{\partial e_{ik}}(\delta_{kj} - e_{kj}), \quad (5)$$

where Q is the strain-energy function, p denotes the hydrostatic pressure function as a Lagrangian multiplier to enforce the incompressibility constraint. For isotropic hyperelastic materials, the strain-energy function can be expressed in terms of the invariants of the reversible strain tensor. Let us expand Q into the Taylor series in the vicinity of the natural state $e_{ij} = 0$,

disregarding the terms of higher order than the second one. The following form of the expansion is obtained for isotropic, homogeneous and incompressible bodies

$$\begin{aligned} Q &= (\alpha - \mu)J_1 + \alpha J_2 + \beta J_1^2 - \kappa J_1 J_2 - \zeta J_1^2, \\ J_1 &= e_{jj}^e, \quad J_2 = e_{ij}^e e_{ji}^e, \end{aligned} \quad (6)$$

wherein $\alpha, \mu, \beta, \kappa, \zeta$ are elastic modules.

During process, anticipating plastic flow, and in the unloading, the irreversible strain rate tensor γ_{ij} is identified by the creep strain rate tensor $\gamma_{ij} = \varepsilon_{ij}^v$. The energy dissipation law is valid for creep stage of deforming. Let accept the dissipation potential in the power form like Norton-Bailey creep law [16, 17]:

$$\begin{aligned} \varepsilon_{ij}^v &= \frac{\partial V(\Sigma)}{\partial \sigma_{ij}}, \\ V(\Sigma) &= B \Sigma^n (\sigma_{ij}), \\ \Sigma &= \sqrt{\frac{3}{2} ((\sigma_1 - \sigma)^2 + (\sigma_2 - \sigma)^2 + (\sigma_3 - \sigma)^2)}. \end{aligned} \quad (7)$$

Here $\sigma_1, \sigma_2, \sigma_3$ are principal values of Cauchy stress tensor and B, n are the creep constants.

3. Boundary Value Problem

Let examine the changes in the geometry of a single spherical microdefect (micropore) under hydrostatic compression and stress relaxation process during unloading of the material within the proposed model [5]. We consider the solid ball of the initial radius R_0 with a single spherical defect (micropore) of the initial radius s_0 in the center of the sphere. The deformation is given by the boundary conditions

$$\begin{aligned} \sigma_{rr}|_{r=R(t)} &= -P(t), \\ \sigma_{rr}|_{r=s(t)} &= 0, \end{aligned} \quad (8)$$

where σ_{rr} is the radial component of the stress tensor in the spherical coordinate system (r, θ, φ) , $R(t) \gg r_0$ is the radius of the spherical surface which is given by the external pressure $P(t)$, $s(t)$ is the current radius of the micropore. Reversible (elastic) e_{ij} and irreversible (creep) p_{ij} parts of the Almansi total strain tensor equation (4) are defined by the differential equations of change (transfer) (1). Stresses with reversible deformations are related by equations (5). The constraints of the incompressibility in the present case of the spherical symmetry leads to the continuity equation

$$(1 - u_{r,r}) \cdot \left(1 - \frac{u_r}{r}\right)^2 = 1, \quad (9)$$

where u_r is the only nonzero displacement. The solution of the equation (9) is obtained in form

$$\begin{aligned} u_r &= r - (r^3 + \varphi(t))^{\frac{1}{3}}, \\ \varphi(t) &= s_0^3 - s^3(t) = R_0^3 - R^3(t). \end{aligned} \quad (10)$$

Note that the kinematics is specifies with an accuracy of an unknown function $\varphi(t)$.

The equation of motion in considered case of spherical symmetry can be deduced in form

$$\sigma_{rr,r} + 2 \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = -\rho_0 \left(\frac{\ddot{\varphi}(t)}{3r^2} + \frac{2}{9} \frac{\dot{\varphi}^2(t)}{r^5} \right), \quad (11)$$

wherein $\sigma_{\theta\theta}$ denotes the angular stress, ρ_0 denotes the mass density.

Equation of motion (11) should be supplemented by equation for components of irreversible strain tensor

$$\begin{aligned}\frac{dp_{rr}}{dt} &= Bn(1 - 2p_{rr})\Phi^{n-1}(e_{rr}, e_{\theta\theta}), \\ p_{\theta\theta} &= \frac{1}{2} \left(1 - \frac{1}{\sqrt{1 - 2p_{rr}}} \right),\end{aligned}\quad (12)$$

wherein Φ is derived by formula

$$\begin{aligned}\Phi(a, b) &= 2\mu(a - b) - \lambda_1 a^2 + \lambda_2 b^2 + \lambda_3 a^3 - \lambda_4 b^3 + \\ &+ \lambda_5 \left(\frac{1}{2} ab^2 - ab \right) + \lambda_6 \left(a^2 b - \frac{1}{2} a^2 b^2 \right) - \lambda_7 a^4 + \lambda_8 b^4 + \\ &+ \lambda_9 \left(a^4 b + 2a^3 b^2 - \frac{1}{2} a^4 b^2 - 4a^3 b \right) + \lambda_{10} \left(\frac{1}{2} a^6 - 3b^5 \right), \\ \lambda_1 &= \mu + 4\alpha + 4\beta + 2\xi, \\ \lambda_2 &= \mu + 4\alpha + 8\beta + 4\xi, \\ \lambda_3 &= 2(2\alpha + 2\beta + 4\xi + 3\chi), \\ \lambda_4 &= 4(\alpha + 2\beta + 4\xi + 6\chi), \\ \lambda_5 &= 4\beta + 2\xi, \\ \lambda_6 &= 2\beta + 7\xi + 18\chi, \\ \lambda_7 &= \alpha + \beta + \frac{19}{2}\xi + 9\chi, \\ \lambda_8 &= \alpha + 2\beta + 19\xi + 36\chi, \\ \lambda_9 &= \frac{3}{4}(\xi + 3\chi), \\ \lambda_{10} &= \frac{1}{2}(5\xi + 11\chi).\end{aligned}\quad (13)$$

Resulting system of the integro-differential equations after integrating equation (11) under condition (8) is transformed into

$$\begin{aligned}P(t) &= 2 \int_{s(t)}^{R(t)} \frac{\Psi(r, p_{rr}(r, t), \varphi(t))}{r} dr - \rho_0 \left(\frac{\ddot{\varphi}(t)}{3} \left(\frac{1}{R(t)} - \frac{1}{s(t)} + \frac{\dot{\varphi}^2(t)}{18} \left(\frac{1}{R(t)^4} - \frac{1}{s(t)^4} \right) \right) \right), \\ \frac{dp_{rr}}{dt} &= (1 - 2p_{rr})Bn\Psi^{n-1}(r, p_{rr}(r, t), \varphi(t)),\end{aligned}\quad (14)$$

wherein

$$\Psi(r, p_{rr}(r, t), \varphi(t)) = \Phi(e_{rr}, e_{\theta\theta}),$$

after substituting $e_{rr}, e_{\theta\theta}$ by the rules

$$\begin{aligned}e_{rr} &= 1 - H^{-\frac{2}{3}}(1 - 2p_{rr})^{-\frac{1}{2}}, \\ e_{\theta\theta} &= 1 - H^{\frac{1}{3}}(1 - 2p_{rr}), \\ H &= 1 + r^{-3}\varphi(t).\end{aligned}\quad (15)$$

For the calculations of the strain-stress state parameters, we define the following dimensionless parameters:

$$\begin{aligned}\alpha\mu^{-1} &= 0.9, \quad \beta\mu^{-1} = 4, \quad \xi\mu^{-1} = 20, \\ B_0 &= nB\rho_0 R_0 \mu^{n-2} = 3.5, \quad \chi\mu^{-1} = 80, \\ n &= 3, \quad k\mu^{-1} = 0.003, \quad s_0 R_0^{-1} = 0.03.\end{aligned}$$

The resulting system (14) is numerically analyzed by using symbolic computation algebra *Mathematica*.

4. Acknowledgments

The authors wish to thank Professor A. A. Burenin for the statement of the problem, scientific advice and necessary support. This work was financially supported by the Russian Science Foundation under Grant 14-19-01280.

5. Conclusions

The presented mathematical model of large elastocreep deformations is based on equations (1), (4), (5), (7), (11). One can use the presented approach for mechanical design creep fabricated parts from creep-elastic materials determining the strength and the shape of final products. Solved problems for a ball-shaped body with a single spherical defect are consistent with well-known microdefects welding process [3]. The proposed method of the loading pressure computing for a given displacements can be used to optimize the treatment metal process under creep conditions [18]. Moreover, on the basis of this mechanical analysis one can work out effective recommendations for refinements of the technological process.

6. References

- [1] Manzhairov A V and Parshin D A 2015 *Mechanics of Solids* **50:5** 559
- [2] Manzhairov A V and Parshin D A 2015 *Mechanics of Solids* **50:6** 657
- [3] Gorelov V I 1984 *Journal of Applied Mechanics and Technical Physics* **25** 813
- [4] Rogachev E E Polonik M V, Dudko O V and Murashkin E V 2014 *Advanced Materials Research* **1040** 898
- [5] Burenin A A and Kovtanyuk L V 2011 *On Elastic Features Under Intensive Irreversible Deformation* (Vladivostok: FEFU)
- [6] Burenin A A, Kovtanyuk L V and Polonik M V 2000 *Doklady Physics* **45** 694
- [7] Lee E H 1969 *Journal of Applied Mechanics* **36** 1
- [8] Shitikov A V 1995 *Journal of Applied Mathematics and Mechanics* **59:1** 147
- [9] Burenin A A, Kovtanyuk L V and Murashkin E V 2006 *Journal of Applied Mechanics and Technical Physics* **47** 241
- [10] Murashkin E and Polonik M 2013 *Applied Mechanics and Materials* **249-250** 833
- [11] Bazhin A A and Murashkin E V 2012 *Doklady Physics* **57(8)** 294
- [12] Shitikov A V and Bykovtsev G I 1990 *Dokl. Akad. Nauk SSSR*, **311:1** 32
- [13] Levitas V I 1996 *Large deformation of materials with complex rheological properties at normal and high pressure* (NY, Nova Science Publishers)
- [14] Bruhns O T 2007 *Key Engineering Materials* **340-341** 787
- [15] Murnaghan F D 1937 *American Journal of Mathematics* **59** 235
- [16] Norton F H 1929 *Creep of Steel at High Temperatures* (NY: McGraw Hill)
- [17] Bailey R W 1929 *Transactions of the World Power Conference* **3** 528
- [18] Anop M F, Polonik M V, Mikhlichuk V I and Murashkin E V 2016 *Key Engineering Materials* **685** 83