

Closed solutions for model problems in generalized thermoelasticity

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Abstract. Closed solutions for model problems in non-dissipative thermoelasticity are obtained. The solutions are in the form of spectral expansion over biorthogonal system of eigenfunctions corresponded to mutual conjugate pair of operator pencils.

1. Introduction

Thermal disturbances in some materials (in particular, liquid and crystalline helium) are known to propagate at low temperatures as progressive waves (Peshkov [1]). This phenomenon is usually referred to as “second sound.” The wave nature of heat propagation substantially changes the medium heat conductivity, and this change should be taken into account when designing cryogenic devices. Various theories have been devised to explain this phenomenon (Tisza–Landau [2], Atkin et al. [3], etc.). Among them are Green–Naghdi theories of thermoelasticity [4], in which the time primitive of the temperature is chosen as an independent constitutive variable. Such theories results in very special nonself-adjoint boundary value problems, for which classical approaches apparently fail. Closed solution for the initial boundary value problems in the framework of Green-Naghdi type II thermoelasticity were obtained in [9]. The aim of this paper is to obtain closed solutions for more general problems arising in Green-Naghdi type III thermoelasticity. The solutions are obtained in the form of spectral expansions in complete biorthogonal systems of root functions (eigenfunctions and associated functions) of the corresponding nonself-adjoint operator pencils.

2. Differential operators

Consider an isotropic homogenous body \mathcal{B} . The equations of motion and generalized heat conduction (Green-Naghdi type III) have the form:

$$\begin{aligned}\mu\nabla^2\mathbf{u} + (\lambda + \mu)\nabla\nabla\cdot\mathbf{u} - \alpha\nabla\theta - \rho\ddot{\mathbf{u}} + \rho\mathbf{b} &= \mathbf{0}, \\ \Lambda\nabla^2\theta + \Lambda_*\nabla^2\dot{\theta} - \kappa\ddot{\theta} - \varpi\nabla\cdot\dot{\mathbf{u}} + \rho s &= 0,\end{aligned}$$

where \mathbf{u} is the displacement field, θ is the temperature, \mathbf{b} is the density of mass force, s is the power of heat supply, λ and μ are the Lamé elastic moduli and Λ , Λ_* , κ , and ϖ are the thermomechanical moduli. Parameters Λ , Λ_* are specific for Green-Naghdi type III



thermoelasticity. The limit case $\Lambda \rightarrow 0$ corresponds to conventional (Fourier) thermoelasticity, whereas the limit case $\Lambda_* \rightarrow 0$ corresponds to Green-Naghdi type II thermoelasticity. These equations, together with the prescribed boundary and initial conditions

$$\mathbf{u}|_{t=0} = \mathbf{u}_0, \quad \theta|_{t=0} = \Theta_0, \quad \dot{\mathbf{u}}|_{t=0} = \dot{\mathbf{u}}_0, \quad \dot{\theta}|_{t=0} = \dot{\theta}_0,$$

form a well-posed statement of the initial–boundary value problem.

Consider the space of square integrable four-component complex-valued vector functions (\mathbf{u}, θ) with the inner product

$$\langle (\mathbf{u}_1, \theta_1), (\mathbf{u}_2, \theta_2) \rangle = \int_{\mathcal{B}} (\mathbf{u}_1 \cdot \bar{\mathbf{u}}_2 + \theta_1 \bar{\theta}_2) dV.$$

The initial–boundary value problem can be stated as the following Cauchy problem with operator coefficients:

$$\begin{aligned} \mathcal{L}(\mathbf{u}, \theta) &= \mathbf{f}, \quad \mathcal{L}(\mathbf{u}, \theta) = \mathcal{A}_0(\mathbf{u}, \theta) + \mathcal{A}_1 \frac{\partial}{\partial t}(\mathbf{u}, \theta) + \mathcal{A}_2 \frac{\partial^2}{\partial t^2}(\mathbf{u}, \theta), \quad \mathbf{f} = -(\rho \mathbf{b}, \rho s), \\ \mathcal{A}_0 &= \begin{pmatrix} \mu \nabla^2 + (\lambda + \mu) \nabla \nabla \cdot & -\alpha \nabla \\ 0 & \Lambda \nabla^2 \end{pmatrix}, \quad \mathcal{A}_1 = \begin{pmatrix} 0 & 0 \\ 0 & \Lambda_* \nabla^2 \end{pmatrix}, \quad \mathcal{A}_2 = - \begin{pmatrix} \rho & 0 \\ \varpi \nabla & \kappa \end{pmatrix}. \end{aligned}$$

The problem in question gives rise to the polynomial operator pencil

$$\mathcal{L}_\nu = \mathcal{A}_0 + \nu \mathcal{A}_1 + \nu^2 \mathcal{A}_2,$$

The domain \mathcal{D} of the pencil \mathcal{L}_ν is determined by the boundary operator \mathcal{B} as follows:

$$\mathcal{D} = \left\{ (\mathbf{u}, \theta) \mid \mathcal{B}(\mathbf{u}, \theta)|_{\partial \mathcal{D}} = \mathbf{0} \right\},$$

The adjoint pencil \mathcal{L}_ν^* can be obtained directly from the definition:

$$\forall (\mathbf{u}_1, \theta_1) \in \mathcal{D}, (\mathbf{u}_2, \theta_2) \in \mathcal{D}^* \quad \langle \mathcal{L}_\nu(\mathbf{u}_1, \theta_1), (\mathbf{u}_2, \theta_2) \rangle = \langle (\mathbf{u}_1, \theta_1), \mathcal{L}_\nu^*(\mathbf{u}_2, \theta_2) \rangle.$$

provided that

$$\mathcal{L}_\nu^* = \mathcal{A}_0^* + \bar{\nu} \mathcal{A}_1 + \bar{\nu}^2 \mathcal{A}_2^*, \quad \mathcal{A}_0^* = \begin{pmatrix} \mu \nabla^2 + (\lambda + \mu) \nabla \nabla \cdot & 0 \\ \alpha \nabla & \Lambda \nabla^2 \end{pmatrix}, \quad \mathcal{A}_2^* = \begin{pmatrix} -\rho & \varpi \nabla \\ 0 & -\beta \end{pmatrix}.$$

We define the domain \mathcal{D}^* of the adjoint pencil \mathcal{L}_ν^* as the set of vector functions $\mathbf{k}^* = (\mathbf{u}^*, \theta^*)$ such that the bilinear form

$$\langle \mathcal{L}_\nu(\mathbf{u}, \theta), (\mathbf{u}^*, \theta^*) \rangle - \langle (\mathbf{u}, \theta), \mathcal{L}_\nu^*(\mathbf{u}^*, \theta^*) \rangle = [(\mathbf{u}, \theta), (\mathbf{u}^*, \theta^*)]_{\partial \mathcal{B}}$$

is zero for arbitrary $(\mathbf{u}, \theta) \in \mathcal{D}$. Hence we obtain the adjoint boundary operator \mathcal{B}^* in closed form. It is well known that the system of root functions of the pencil \mathcal{L}_ν is biorthogonal to that of the adjoint pencil. Note that if $\mathcal{D} = \mathcal{D}^*$, then this system is a basis. This provides a theoretical background for the representation of the solution in the form of a spectral expansion.

3. Solution in the general case

The obtained solutions have the form of expansions based on the systems of eigenfunctions \mathbf{k}_i and \mathbf{k}_j^* that can be found as the solutions of the mutually adjoint boundary value problems

$$\mathcal{L}_\nu \mathbf{k}_i = \mathbf{0}, \quad \mathcal{L}_\nu^* \mathbf{k}_j = \mathbf{0}, \quad \mathbf{k}_i = (\mathbf{u}_i, \theta_i), \quad \mathbf{k}_j^* = (\mathbf{u}_j^*, \theta_j^*); \quad i = 1, \dots, \infty; \quad j = 1, \dots, \infty. \quad (1)$$

We assume that the body \mathcal{B} is compact. Hence the set of eigenfunctions and the corresponding set of eigenvalues are countable. According to [8], the following biorthogonal relations hold for \mathbf{k}_i and \mathbf{k}_j^* :

$$\langle \mathcal{A}_1 \mathbf{k}_i, \mathbf{k}_j^* \rangle + (\nu_i + \nu_j) \langle \mathcal{A}_2 \mathbf{k}_i, \mathbf{k}_j^* \rangle = 0, \quad \langle \mathcal{A}_0 \mathbf{k}_i, \mathbf{k}_j^* \rangle - \nu_i \nu_j \langle \mathcal{A}_2 \mathbf{k}_i, \mathbf{k}_j^* \rangle = 0.$$

These properties permit one to represent the solution in the form [8]

$$\mathbf{y} = \sum_{i=1}^{\infty} \left[(\langle \mathcal{A}_1^* \mathbf{k}_i^*, \mathbf{y}_0 \rangle + \langle \mathcal{A}_2^* \mathbf{k}_i^*, \dot{\mathbf{y}}_0 \rangle) \exp(\nu_i t) - \int_0^t \langle \mathbf{f}(\tau), \mathbf{k}_i^* \rangle \exp(\nu_i(t-\tau)) d\tau \right] \mathbf{k}_i Q_i^{-1},$$

where

$$\mathbf{y} = (\mathbf{u}, \theta), \quad \mathbf{y}_0 = (\mathbf{u}_0, \theta_0), \quad \dot{\mathbf{y}}_0 = (\dot{\mathbf{u}}_0, \dot{\theta}_0), \quad \mathbf{f} = (\rho \mathbf{b}, \rho s), \quad Q_i = \langle \mathcal{A}_1 \mathbf{k}_i, \mathbf{k}_i^* \rangle + 2\nu_i \langle \mathcal{A}_2 \mathbf{k}_i, \mathbf{k}_i^* \rangle.$$

The spectral expansions in complete biorthogonal systems of root functions (eigenfunctions and associated functions) are of the same form and discussed in details in [8].

In the special case $\mathcal{A}_1 = 0$ and $\text{Re } \nu_i = 0$, this representation can be written as follows:

$$\mathbf{y} = \sum_{i=1}^{\infty} \left[(\langle \mathcal{A}_2^* \mathbf{k}_i^*, \mathbf{y}_0 \rangle \cos(\lambda_i t) + \frac{1}{\lambda_i} \langle \mathcal{A}_2^* \mathbf{k}_i^*, \dot{\mathbf{y}}_0 \rangle) \sin(\lambda_i t) - \frac{1}{\lambda_i} \int_0^t \langle \mathbf{f}(\tau), \mathbf{k}_i^* \rangle \sin(\lambda_i(t-\tau)) d\tau \right] \mathbf{k}_i N_i^{-1}.$$

Here $N_i = 2 \langle \mathcal{A}_2 \mathbf{k}_i, \mathbf{k}_i^* \rangle$ and $\lambda_i = \text{Im } \nu_i$.

4. Solution in a special case

The mutually adjoint systems (1) are reduced to

$$\begin{aligned} \mu \nabla^2 \mathbf{u}_i + (\lambda + \mu) \nabla \nabla \cdot \mathbf{u}_i - \alpha \nabla \theta_i - \nu_i^2 \rho \mathbf{u}_i &= \mathbf{0}, & (1 + \nu_i \epsilon) \nabla^2 \theta_i - \nu_i^2 \beta \theta_i - \nu_i^2 \gamma \nabla \cdot \mathbf{u}_i &= 0, \\ \mu \nabla^2 \mathbf{u}_i^* + (\lambda + \mu) \nabla \nabla \cdot \mathbf{u}_i^* + \bar{\nu}_i^2 \gamma \nabla \theta_i^* - \bar{\nu}_i^2 \rho \mathbf{u}_i^* &= \mathbf{0}, & (1 + \bar{\nu}_i \epsilon) \nabla^2 \theta_i^* - \bar{\nu}_i^2 \beta \theta_i^* + \alpha \nabla \cdot \mathbf{u}_i^* &= 0. \end{aligned} \quad (2)$$

Here $\beta = \varkappa/\Lambda$, $\gamma = \alpha/\Lambda$ and $\epsilon = \Lambda_*/\Lambda$. Now let us apply the Stokes–Helmholtz resolution

$$\mathbf{u}_i = \nabla \phi_i + \nabla \times \boldsymbol{\psi}_i, \quad \nabla \cdot \boldsymbol{\psi}_i = 0, \quad \mathbf{u}_i^* = \nabla \phi_i^* + \nabla \times \boldsymbol{\psi}_i^*, \quad \nabla \cdot \boldsymbol{\psi}_i^* = 0. \quad (3)$$

where ϕ_i , ϕ_i^* and $\boldsymbol{\psi}_i$, $\boldsymbol{\psi}_i^*$ are scalar and vector functions, respectively. By substituting (3) into (2), we obtain

$$\begin{aligned} (\lambda + 2\mu) \nabla^2 \phi_i - \alpha \theta_i - \nu_i^2 \rho \phi_i &= 0, & (1 + \nu_i \epsilon) \nabla^2 \theta_i - \nu_i^2 \beta \theta_i - \nu_i^2 \gamma \nabla^2 \phi_i &= 0, & \mu \nabla \boldsymbol{\psi}_i - \nu_i^2 \rho \boldsymbol{\psi}_i &= \mathbf{0}, \\ (\lambda + 2\mu) \nabla^2 \phi_i^* + \bar{\nu}_i^2 \gamma \theta_i^* - \bar{\nu}_i^2 \rho \phi_i^* &= 0, & (1 + \bar{\nu}_i \epsilon) \nabla^2 \theta_i^* - \bar{\nu}_i^2 \beta \theta_i^* + \alpha \nabla^2 \phi_i^* &= 0, & \mu \nabla \boldsymbol{\psi}_i^* - \bar{\nu}_i^2 \rho \boldsymbol{\psi}_i^* &= \mathbf{0}. \end{aligned}$$

In this section, we assume that the desired solutions are generated by scalar potentials alone¹ ($\boldsymbol{\psi}_i = \boldsymbol{\psi}_i^* = \mathbf{0}$). It follows in a standard way that

$$\begin{pmatrix} \phi_i \\ \theta_i \end{pmatrix} = \begin{pmatrix} \varrho \\ \varsigma \end{pmatrix} \Xi_i.$$

¹ This assumption, of course, restricts the problem under consideration to potential displacement fields, body forces, and initial data. However, the rotational components of the above-mentioned fields do not interact with the temperature field. Thus, the rotational components can be omitted in the case of purely thermal excitation.

Here ρ and ς are constants, Ξ_i is a solution of the auxiliary equation

$$\nabla^2 \Xi_i = \zeta_i \Xi_i$$

of the Helmholtz type, and ζ_i satisfies the algebraic equation ($\sigma = \lambda + 2\mu$):

$$\begin{vmatrix} \sigma\zeta_i - \rho\nu_i^2 & -\alpha \\ -\gamma\zeta_i\nu_i^2 & (1 + \nu_i\epsilon)\zeta_i - \beta\nu_i^2 \end{vmatrix} = 0.$$

By solving the latter for ν_i , we obtain

$$\begin{aligned} \nu_{i1} &= \frac{\zeta\epsilon}{4\beta} - \frac{a_{i1}}{2} - \frac{\sqrt{a_{i2} - a_{i3}}}{2}, & \nu_{i2} &= \frac{\zeta\epsilon}{4\beta} - \frac{a_{i1}}{2} + \frac{\sqrt{a_{i2} - a_{i3}}}{2}, \\ \nu_{i3} &= \frac{\zeta\epsilon}{4\beta} + \frac{a_{i1}}{2} - \frac{\sqrt{a_{i2} - a_{i3}}}{2}, & \nu_{i4} &= \frac{\zeta\epsilon}{4\beta} + \frac{a_{i1}}{2} + \frac{\sqrt{a_{i2} - a_{i3}}}{2}, \end{aligned}$$

where

$$a_{i1} = \sqrt{\frac{p_{i1}}{3\sqrt[3]{2}\beta\rho} + p_{i2} + p_{i3}}, \quad a_{i2} = -\frac{p_{i1}}{3\sqrt[3]{2}\beta\rho} - p_{i2} + 2p_{i3},$$

$$a_{i3} = \left(\frac{\zeta^3\epsilon^3}{\beta^3} - \frac{8\zeta^2\sigma\epsilon}{\beta\rho} - \frac{4\zeta p_{i0}\epsilon}{\beta^2\rho} \right) / (4a_{i1});$$

$$p_{i3} = \frac{\zeta^2\epsilon^2}{4\beta^2} - \frac{2p_{i0}}{3\beta\rho}, \quad p_{i2} = \frac{\sqrt[3]{2}q_{i1}}{3\beta p_{i1}\rho}, \quad p_{i1} = \sqrt[3]{\sqrt{q_{i0}^2 - 4q_{i1}^3} + q_{i0}}, \quad p_{i0} = -\zeta(\alpha\gamma + \beta\sigma + \rho);$$

$$q_{i1} = 3\zeta^2\rho\sigma(4\beta + \zeta\epsilon^2) + p_{i0}^2, \quad q_{i0} = 27\zeta^4\rho\sigma\epsilon^2(\beta\sigma + \rho) + 2p_{i0}^3 + 9\zeta^2 p_{i0}\rho\sigma(\zeta\epsilon^2 - 8\beta).$$

In the case of simple spectrum the eigenfunctions can be obtained as

$$\begin{aligned} \mathbf{k}_{i,s} &= (\mathbf{u}_{i,s}, \theta_{i,s}) = ([1 - \beta\nu_{i,s}^2/\zeta + \nu_{i,s}\epsilon] \nabla\Xi_i, \gamma\nu_{i,s}^2\Xi_i), \\ \mathbf{k}_{i,s}^* &= (\mathbf{u}_{i,s}^*, \theta_{i,s}^*) = ([1 - \beta\bar{\nu}_{i,s}^2/\zeta + \bar{\nu}_{i,s}\epsilon] \nabla\Xi_i, -\alpha\Xi_i). \end{aligned}$$

Here $s = 1, \dots, 4$.

It is clear that the solution thus obtained depends on the boundary conditions for the auxiliary scalar function Ξ . Consider the Neumann type boundary conditions $\mathbf{n} \cdot \nabla\Xi|_{\partial\mathcal{B}} = 0$. This corresponds to the following boundary conditions in terms of the variables \mathbf{u} and θ (where \mathbf{I} is the unit tensor)

$$\mathbf{n} \cdot \mathbf{u} = 0, \quad \mathbf{n} \cdot \mathbf{T} \cdot (\mathbf{I} - \mathbf{n} \otimes \mathbf{n}) = \mathbf{0}, \quad \mathbf{n} \cdot \nabla\theta = 0.$$

Now let \mathcal{B} be a cube with edge length π , and let x, y, z be Cartesian coordinates. Assume that the body force and the entropy production are zero, and so are the initial displacements, velocities, and temperature rate; i.e.,

$$\mathbf{b} = \mathbf{0}, \quad s = 0, \quad \mathbf{u}|_{t=0} = \mathbf{0}, \quad \dot{\mathbf{u}}|_{t=0} = \mathbf{0}, \quad \dot{\theta}|_{t=0} = 0.$$

The solution of the auxiliary problem is

$$\Xi_k = \cos nx \cos my \cos lz, \quad \zeta_k = -n^2 - m^2 - l^2.$$

Let the initial temperature distribution be given by $\theta_0 = H(z) - H(z - h)$, $0 < h < \pi$, where $H(z)$ is the Heaviside step function. In this case only $\zeta_k = -l^2$ may be taken into account. The normalising factors $\mathcal{Q}_{i,s}$ are of the form

$$\mathcal{Q}_{i,s} = \pi^3 \nu_{i,s} \left(-\alpha\gamma n^4 (\nu_{i,s}\epsilon + 2) - 2\rho (\beta\nu_{i,s}^2 + n^2 (\nu_{i,s}\epsilon + 1))^2 \right) / (2n^2),$$

and non-zero expansion coefficients $\langle \mathcal{A}_1^* \mathbf{k}_i^* + \nu_i \mathcal{A}_2^* \mathbf{k}_i^*, \mathbf{y}_0 \rangle$ may be calculate as follows

$$P_{i,s} = (\pi^2 \alpha) \sin(hn) (\beta \nu_{i,s} + n^2 \epsilon) / n$$

Thus the solution of the corresponding initial–boundary value problem can be represented in the form of a spectral expansion:

$$\theta = \frac{h}{\pi} + \sum_{i=1}^{\infty} (S_{i,1} + S_{i,2} + S_{i,3} + S_{i,4}), \quad \mathbf{u} = \mathbf{k}w, \quad w = \sum_{i=1}^{\infty} (W_{i,1} + W_{i,2} + W_{i,3} + W_{i,4}),$$

where

$$S_{i,s} = \frac{\gamma \nu_{i,s}^2 P_{i,s} \exp(\nu_{i,s} t) \cos(nz)}{Q_{i,s}}, \quad W_{i,s} = (\beta \nu_{i,s}^2 / \zeta - \nu_{i,s} \epsilon - 1) \frac{n P_{i,s} \exp(\nu_{i,s} t) \sin(nz)}{Q_{i,s}}.$$

Obtained solution generalise the solution given in [9].

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