

On complex roots of an equation arising in the oblique derivative problem

A B Kostin and V B Sherstyukov

National Research Nuclear University MEPhI (Moscow Engineering Physics Institute),
31 Kashirskoe shosse, 115409 Moscow, Russia

E-mail: abkosten@yandex.ru, shervb73@gmail.com

Abstract. The paper is concerned with the eigenvalue problem for the Laplace operator in a disc under the condition that the oblique derivative vanishes on the disc boundary. In a famous article by V. A. Il'in and E. I. Moiseev (Differential equations, 1994) it was found, in particular, that the root of any equation of the form

$$\mu J'_n(\mu) + i n \tan \alpha J_n(\mu) = 0, \quad n \in \mathbb{Z},$$

with the Bessel function $J_n(\mu)$ determines the eigenvalue $\lambda = \mu^2$ of the problem. In our work we correct the information about the location of eigenvalues. It is specified explicit view of the corner, containing all the eigenvalues. It is shown that all the nonzero roots of the equation are simple and given a refined description of the set of their localization on the complex plane. To prove these facts we use the partial differential equations methods and also methods of entire functions theory.

1. Statement of the problem

In the unit disc $D = \{(x, y) \mid x^2 + y^2 < 1\}$ we consider the problem

$$\Delta w + \lambda w = 0 \quad \text{in } D, \quad \frac{\partial w}{\partial l} = 0 \quad \text{on } \partial D. \quad (1)$$

Here, Δ is the Laplace operator, $\lambda \in \mathbb{C}$ is the spectral parameter, l is a direction at a fixed angle $\alpha \in (-\pi/2, \pi/2)$ with the outer normal vector n to the boundary $\partial D = \{(x, y) \mid x^2 + y^2 = 1\}$.

The problem with oblique derivative was extensively studied [1]–[5]. Our paper is closest to the papers [1], [5] and has been strongly influenced by these papers. Of great interest in problem (1) is to find nontrivial solutions $w = u + iv$ and the corresponding eigenvalues $\lambda = \mu^2 = \xi + i\eta$. It is known (see, for example, [1], [6, Ch. 8]), that any weak solution to problem (1) from the Sobolev class $W_2^2(D)$ is a classical solution. It is also known [7, Ch. 4, § 1, Theorem 3], [8, Ch. 5, § 19], that in the case $\alpha = 0$ (that is, when $\partial w / \partial l \equiv \partial w / \partial n$) all the eigenvalues of problem (1) are real and nonnegative, the eigenfunctions forming an orthogonal basis for $L_2(D)$. In [1] it was proved that for any $\varepsilon > 0$ outside the set $\{\lambda \in \mathbb{C} \mid |\arg \lambda| < \varepsilon\}$ there is at most finite number of eigenvalues of problem (1), and, hence, they all located in some angle on the complex plane. Besides, the system of eigenfunctions and associated functions was shown to be complete in L_2 in [1] for general elliptic problems (in particular, including problem (1)). The main result of [5] is that the system of root functions of problem (1) fails to have the basis property both in the



space $L_2(D)$ and in the space $L_p(D)$ for any $p > 1$. In [5] it was also shown that any root of an equation of the form

$$\mu J'_n(\mu) + i n \tan \alpha J_n(\mu) = 0, \quad n \in \mathbb{Z}, \quad (2)$$

with the Bessel function $J_n(\mu)$ defines an eigenvalue $\lambda = \mu^2$ of problem (1) with the corresponding eigenfunction $J_n(\mu r) \exp(in\varphi)$. Besides, there exists a subsequence of eigenvalues $\lambda = \mu^2$ for which the set $|\operatorname{Im} \mu|$ is unbounded. In other words, this subsequence does not lie “inside” the so-called Carleman parabola [9], [10]. In [11] an asymptotically exact set on the complex plane was found to contain all the eigenvalues of an elliptic operator with classical conditions on the boundary. Even though we failed to directly apply these result to problem (1), nevertheless the approaches of [11] were found to be instrumental in treating problem (1). The results of our paper extend the results of [1], [5] on the location of eigenvalues of problem (1). Our arguments depend on the methods of partial differential equations and methods of special considerations from the theory of entire functions.

2. Statement of the main results

In what follows, the phrase “ μ is a root of equation of the form (2)” will mean that μ satisfies at least one equation from the family of equations (2).

We have the following results.

Theorem 1. *Let $\lambda = \xi + i\eta \in \mathbb{C}$ be an arbitrary eigenvalue of problem (1). Then*

$$|\eta| \leq \xi |\tan \alpha|, \quad (3)$$

which shows that all the eigenvalues lie in the right half-plane.

Thus, for the angle appearing in [1] we give an explicit description by (3).

Theorem 2. *Each root $\mu \neq 0$ of an arbitrarily chosen equation from (2) is simple, $\operatorname{Im} \mu \neq 0$ with $n \neq 0$.*

Theorem 3. *For a fixed $n \in \mathbb{Z}$, the corresponding equation from (2) has no roots $\mu \neq 0$ on the set $\{\mu \in \mathbb{C} \mid \operatorname{Re} \mu^2 \leq n^2\}$.*

Theorem 4. $\lambda = \mu^2$ *is an eigenvalue of problem (1) if and only if when μ is a root of an equation of the form (2).*

Theorem 5. *All the eigenfunctions of problem (1) are given in polar coordinates by formulas of the form*

$$w(r, \varphi) = J_n(\mu r) \exp(in\varphi), \quad n \in \mathbb{Z}, \quad (4)$$

where μ assumes all such values from the set of roots of equation of the form (2) with given n for which $w(r, \varphi) \not\equiv 0$. There are no associated functions in problem (1).

Remark 2.1. The assertion $\operatorname{Im} \mu \neq 0$ with $n \neq 0$ in Theorem 2, the sufficiency in Theorem 4, as well as the fact from Theorem 5 that there are no associated functions in problem (1) were proved in [5]; these results are shown for the sake of completeness of exposition. We also note that Theorem 5 is an immediate consequence of Theorem 4 and the results of [5].

Remark 2.2. For each $n \in \mathbb{Z}$ a fixed equation from (2) has an infinite countable set of roots.

For later purposes, it will be convenient to split the family of equations (2) into two subfamilies:

$$\mu J'_n(\mu) + i n \tan \alpha J_n(\mu) = 0, \quad n \in \mathbb{N}, \quad (5)$$

$$\mu J'_n(\mu) - i n \tan \alpha J_n(\mu) = 0, \quad n \in \mathbb{N} \cup \{0\}. \quad (6)$$

The following properties of subfamilies (5), (6) are immediate.

- (i) If some $\mu = \mu(n)$ is a root of an equation from (5), then its complex conjugate $\bar{\mu}$ is a root of an equation in (6) with the same n , and vice versa.
- (ii) For each $n \neq 0$ the change α by $-\alpha$ carries over an equation from (5) into an equation from (6), and vice versa.
- (iii) If μ is a root of equation of the form (2), then $-\mu$ is a root of the same equation.

Thus, in the study of complex roots of the family of equations (2) it suffices to consider only equations (5) and only in the right half-plane, assuming that $\alpha \in (0, \pi/2)$, because the case $\alpha = 0$ has received a good deal of attention (see, for example, [8, Ch. 5, § 19]).

3. Proofs of the main results

We give only sketches of the proofs; detailed arguments will be given in a forthcoming paper.

3.1. Proof of Theorem 1.

Let $\mathbf{n}$, $\boldsymbol{\tau}$ be, respectively, the unit normal vector and the unit tangent vector at a point $(x, y) \in \partial D$. We assume that the shortest rotation from $\boldsymbol{\tau}$ to $\mathbf{n}$ forms a counterclockwise rotation. We fix an eigenvalue $\lambda = \xi + i\eta$ and the corresponding eigenfunction $w = u + iv$, $w \not\equiv 0$. On ∂D we have

$$\frac{\partial w}{\partial \mathbf{l}} = (\mathbf{n} \cos \alpha + \boldsymbol{\tau} \sin \alpha, \nabla w) \equiv \cos \alpha \frac{\partial w}{\partial \mathbf{n}} + \sin \alpha \frac{\partial w}{\partial \boldsymbol{\tau}} \equiv (\mathbf{n} \cos \alpha + \boldsymbol{\tau} \sin \alpha) \nabla w^T,$$

where $\nabla w = \{w_x; w_y\}$ is the vector of the gradient, ∇w^T is the corresponding column. The vectors $\mathbf{n}$ and $\boldsymbol{\tau}$ depend on a point (x, y) , while the angle $\alpha \in (-\pi/2, \pi/2)$ between $\mathbf{l}$ and the outer normal vector $\mathbf{n}$ is fixed. A rotation for α in a clockwise direction is assumed to be positive. In polar coordinates, for $(x, y) \in \partial D$ we write $\mathbf{n} = \{\cos \varphi; \sin \varphi\}$, $\boldsymbol{\tau} = \{\sin \varphi; -\cos \varphi\}$. Consider the matrix

$$A = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix},$$

and calculate

$$\operatorname{div} (\nabla w A^T) = \operatorname{div} \{w_x \cos \alpha - w_y \sin \alpha; w_x \sin \alpha + w_y \cos \alpha\} \equiv \cos \alpha \Delta w.$$

This enables us to rewrite equation from the statement of problem (1) in the equivalent form

$$\operatorname{div} (\nabla w A^T) + w \lambda \cos \alpha = 0.$$

Multiplying the last equality in $L_2(D)$ by w , we find that

$$\int_D \operatorname{div} (\nabla w A^T) \bar{w} dx dy + \lambda \cos \alpha \int_D |w|^2 dx dy = 0. \quad (7)$$

Transforming the first integral in (7) we find that

$$\int_D \operatorname{div} (\nabla w A^T) \bar{w} dx dy = - \int_D [\cos \alpha |\nabla w|^2 + \sin \alpha (w_x \bar{w}_y - w_y \bar{w}_x)] dx dy.$$

We shall assume that the eigenfunction is normalized; that is, $\|w\|^2 \equiv \int_D |w|^2 dx dy = 1$. Equating to zero in succession the real and imaginary parts of (7), this gives

$$\xi = \int_D |\nabla w|^2 dx dy, \quad \eta \cos \alpha = \sin \alpha \operatorname{Im} \int_D (w_x \bar{w}_y - w_y \bar{w}_x) dx dy.$$

Hence,

$$\begin{aligned} |\eta| \cos \alpha &= |\sin \alpha| \left| \operatorname{Im} \int_D (w_x \bar{w}_y - w_y \bar{w}_x) dx dy \right| \leq |\sin \alpha| \int_D |w_x \bar{w}_y - w_y \bar{w}_x| dx dy \leq \\ &\leq |\sin \alpha| \int_D |\nabla w| |\nabla \bar{w}| dx dy = |\sin \alpha| \int_D |\nabla w|^2 dx dy = |\sin \alpha| \xi. \end{aligned}$$

This proves inequality (3). The proof of Theorem 1 is complete. $\square$

3.2. Proof of Theorem 2.

Let us prove that each root $\mu \neq 0$ of equation (2) with fixed $n \in \mathbb{Z}$ is simple. That for $n = 0$ the corresponding equation has simple roots is well-known (see, for example, [12, Ch. 15]). For definiteness, let us consider equation (5) with parameter $\alpha \in (0, \pi/2)$. Assume that $\mu \neq 0$ is a multiple root of (5) with fixed $n \in \mathbb{N}$. Then the following equalities hold simultaneously:

$$\begin{cases} \mu J'_n(\mu) + i n \tan \alpha J_n(\mu) &= 0, \\ i n \mu \tan \alpha J'_n(\mu) - (\mu^2 - n^2) J_n(\mu) &= 0. \end{cases} \quad (8)$$

System (8) is linear and homogeneous in $J'_n(\mu)$, $J_n(\mu)$. Hence, it has either only the zero solution or its determinant is zero. We claim that none of such cases is realized. If $J'_n(\mu) = J_n(\mu) = 0$, then the classical Bessel's equation with $\mu \neq 0$ shows that the derivatives $J_n^{(m)}(\mu)$ are zero for all $m \in \mathbb{N}$, which is impossible. If the determinant of system (8) is zero, then $\mu^2 = n^2(1 + \tan^2 \alpha) > 0$, and therefore, $\mu \in \mathbb{R} \setminus \{0\}$, which contradicts the first equation in (8). The property $\operatorname{Im} \mu \neq 0$ for each $\mu \neq 0$ was established in [5]. This proves Theorem 2. $\square$

3.3. Proof of Theorem 3.

Let us consider equation (5) with fixed $n \in \mathbb{N}$ and $\alpha \in (0, \pi/2)$. Let $\mu \in \mathbb{C} \setminus \{0\}$ be a root of this equation. By Theorem 2 we have $\operatorname{Im} \mu \neq 0$. The complex conjugate $\bar{\mu}$ satisfies the equality

$$\bar{\mu} J'_n(\bar{\mu}) - i n \tan \alpha J_n(\bar{\mu}) = 0. \quad (9)$$

Consider the real function

$$\Phi(x) = |J_n(\mu x)|^2 \equiv J_n(\mu x) J_n(\bar{\mu} x), \quad x \geq 0. \quad (10)$$

The first and second derivatives of Φ are as follows:

$$\begin{aligned} \Phi'(x) &= (J_n(\mu x) J_n(\bar{\mu} x))' = \mu J'_n(\mu x) J_n(\bar{\mu} x) + \bar{\mu} J'_n(\bar{\mu} x) J_n(\mu x), \\ \Phi''(x) &= -\frac{1}{x} \Phi'(x) - 2 \left(\operatorname{Re} \mu^2 - \frac{n^2}{x^2} \right) \Phi(x) + 2|\mu|^2 |J'_n(\mu x)|^2, \quad x > 0. \end{aligned}$$

We have $\Phi(0) = \Phi'(0) = 0$. Besides, using (5), (9) this gives $\Phi'(1) = 0$. So, the function $\Phi(x)$, as defined by (10), is a solution to the problem

$$\Phi''(x) + \frac{1}{x} \Phi'(x) + 2 \left(\operatorname{Re} \mu^2 - \frac{n^2}{x^2} \right) \Phi(x) = 2|\mu|^2 |J'_n(\mu x)|^2, \quad x > 0, \quad (11)$$

$$\Phi(0) = \Phi'(0) = 0, \quad \Phi'(1) = 0. \quad (12)$$

Since $\Phi(x)$ is continuous and nonnegative on $[0, 1]$, there exists a point $x_0 \in (0, 1]$ such that

$$\max_{x \in [0, 1]} \Phi(x) = \Phi(x_0) \geq 0.$$

Assume that the so-chosen root $\mu \neq 0$ satisfies the condition $\operatorname{Re} \mu^2 \leq n^2$. Then in equation (11) the coefficient multiplying $\Phi(x)$ is nonpositive:

$$c(x) \equiv 2 \left(\operatorname{Re} \mu^2 - \frac{n^2}{x^2} \right) \leq 0, \quad x \in (0, 1].$$

We first consider the case $x_0 < 1$. Taking into account equation (11), we see that subelliptic function $\Phi(x)$ attains its nonnegative maximum inside the interval $[0, 1]$. Hence, this function is constant. But in this case $\Phi(x) \equiv 0$, which leads to a contradiction with (10). Assume now that $x_0 = 1$. By the normal derivative lemma (see, for example, [13, Ch. 1, § 1, Theorem 4]) we have $\Phi'(1) > 0$, contradicting (12). So, equation (5) cannot have roots $\mu \neq 0$ that satisfy the condition $\operatorname{Re} \mu^2 \leq n^2$. This proves Theorem 3. $\square$

3.4. Proof of Theorem 4.

If μ is a root of an equation of the form (2), then a direct calculation shows that the function $J_n(\mu r) \exp(in\varphi) \neq 0$ is a solution to problem (1); that is, $J_n(\mu r) \exp(in\varphi)$ is an eigenfunction corresponding to the eigenvalue $\lambda = \mu^2$. Why problem (1) has no other eigenvalues? Such results are usually verified using the completeness and orthogonality (the basis property) of eigenfunctions (see [8, Ch. 5, § 19], [14, Lecture 27]). But problem (1) has no such properties!

We claim that if $\lambda = \mu^2$ is an eigenvalue of problem (1), then μ is a root of at least one of the equations from family (2). Let $w = w(r, \varphi)$ be the eigenfunction corresponding to this eigenvalue. We write the equation for $w(r, \varphi)$:

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial w}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 w}{\partial \varphi^2} + \mu^2 w = 0. \quad (13)$$

For each fixed r the function $w(r, \varphi)$ is φ -smooth and 2π -periodic, and hence, it can be expanded into a Fourier series.

$$w(r, \varphi) = \sum_{m \in \mathbb{Z}} R_m(r) e^{im\varphi}, \quad R_m(r) = \frac{1}{2\pi} \int_0^{2\pi} w(r, \varphi) e^{-im\varphi} d\varphi, \quad m \in \mathbb{Z}. \quad (14)$$

Here, at least one of the Fourier coefficients does not vanish identically; that is, there exists an $n \in \mathbb{Z}$ such that $R_n(r) \neq 0$.

Multiplying equation (13) by $\exp(-im\varphi)$ and integrating in $\varphi \in [0, 2\pi]$, we see that for each $m \in \mathbb{Z}$

$$\frac{1}{r} \frac{d}{dr} \left(r \frac{dR_m(r)}{dr} \right) + \left(\mu^2 - \frac{n^2}{r^2} \right) R_m(r) = 0, \quad r \in (0, 1).$$

Since the functions $R_m(r)$ are bounded, it follows that $R_m(r) = C_m J_m(\mu r)$ for each Fourier coefficients with some constant C_m . Substituting the expression for $R_m(r)$ into the Fourier series in (14), we have

$$w(r, \varphi) = \sum_{m \in \mathbb{Z}} C_m J_m(\mu r) e^{im\varphi}.$$

The boundary condition of problem (1) implies that

$$\sum_{m \in \mathbb{Z}} C_m e^{im\varphi} [\mu J'_m(\mu) + i m \tan \alpha J_m(\mu)] = 0.$$

But in this case

$$C_m [\mu J'_m(\mu) + i m \tan \alpha J_m(\mu)] = 0, \quad m \in \mathbb{Z}.$$

We have $R_n(r) \not\equiv 0$, and hence $C_n \neq 0$. Therefore, for such $n \in \mathbb{Z}$ the number μ is a root of an equation of the form (2). This proves Theorem 4. $\square$

4. Concluding remarks

Consider an equation of the form (5) with fixed $\alpha \in (0, \pi/2)$, $n \in \mathbb{N}$. We expand the entire function $J_n(\mu)$ as an infinite product

$$J_n(\mu) = \frac{1}{n!} \left(\frac{\mu}{2}\right)^n \prod_{k=1}^{\infty} \left(1 - \frac{\mu^2}{j_{n,k}^2}\right), \quad \mu \in \mathbb{C},$$

where $j_{n,k}$ are its positive roots (see [12, Ch. 15, p. 498]). Taking the logarithmic derivative we conclude that, for the above α , n with $\mu \neq 0$ equation (5) is equivalent to the equation

$$\sum_{k=1}^{\infty} \frac{1}{\mu^2 - j_{n,k}^2} + \frac{n}{2\mu^2} (1 + i \tan \alpha) = 0.$$

Setting $\mu^2 = \xi + i\eta$, we have the following system for eigenvalues:

$$\begin{cases} \sum_{k=1}^{\infty} \frac{\xi - j_{n,k}^2}{(\xi - j_{n,k}^2)^2 + \eta^2} + \frac{n}{2} \frac{\xi + \eta \tan \alpha}{\xi^2 + \eta^2} = 0, \\ \sum_{k=1}^{\infty} \frac{\eta}{(\xi - j_{n,k}^2)^2 + \eta^2} - \frac{n}{2} \frac{\xi \tan \alpha - \eta}{\xi^2 + \eta^2} = 0. \end{cases} \quad (15)$$

System (15) provides additional tools for the study of the eigenvalues of problem (1). In particular, in this way we get a different proof of Theorem 1. Indeed, (15) may only have solutions that simultaneously satisfy the inequalities $\eta > 0$, $\xi \tan \alpha - \eta > 0$. Thus, for the roots of equation (5) with $\alpha \in (0, \pi/2)$ the eigenvalues $\lambda = \mu^2$ lie in the angle $0 < \eta < \xi \tan \alpha$. Note that this fact cannot be ‘felt’ by the methods of partial differential equations.

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