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On stability of the solutions of inverse problem for determining the right-hand side of a degenerate parabolic equation with two independent variables

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Abstract. We prove the estimates of stability with respect to perturbations of input data for the solutions of inverse problems for degenerate parabolic equations with unbounded coefficients. An important feature of these estimates is that the constants in these estimates are written out explicitly by the input data of the problem.

1. Introduction. Unique solvability of the inverse problem

In the paper we obtain the estimate of stability with respect to perturbations of input data for the solution $\{u(t,x), p(t)\}$ of the inverse problem for nonuniformly parabolic equation

$$u_t - a(t,x)u_{xx} + b(t,x)u_x + d(t,x)u = p(t)g(t,x) + r(t,x), \quad (t,x) \in Q; \tag{1}$$

with initial and boundary conditions

$$u(0,x) = u_0(x), \quad x \in [0,l], \quad u(t,0) = u(t,l) = 0, \quad t \in [0,T]; \tag{2}$$

and the additional condition of integral observation

$$\int_0^l u(t,x)\omega(x)\,\mathrm{d}x = \varphi(t), \quad t \in [0,T]; \tag{3}$$

here $Q = [0,T] \times [0,l]$, where T and l are some numbers.

In the setting of the inverse problem (1)–(3) one allowed the unboundedness of all the coefficients of the equation (1) and the functions *g(t,x)*, *r(t,x)* in the right hand side of this equation as well as degeneration of the leading coefficient *a(t,x)*.

The questions of unique solvability of the problem (1)–(3) were considered in [1] (see also [2]).

Note that inverse problems for degenerate parabolic equation are important in applications, in particular, they arise in the study of models of price formation for options in financial markets (see, for example, [3-5], etc.).

In this paper we use Lebesgue and Sobolev spaces with corresponding norms in usual sense, and all equalities and inequalities are satisfied almost everywhere.

We also need the well-known the Poincare'–Steklov inequality which for n=1 is of the form

$$\|z\|^2_{L_2(0,l)} \le \frac{l^2}{2}\|z_x\|^2_{L_2(0,l)}, \quad \forall z \in \overset{0}{W}{}^1_2(0,l). \tag{4}$$

We assume that the functions occuring in the input data of the problem (1)–(3) are measurable and satisfy the following conditions:

$$0 \le a(t,x),\ (t,x)\in Q;\ a(t,x), \frac{1}{a(t,x)} \in L_q(Q),\ q>1,\ \|a\|_{L_q(Q)} \le a_1,\ \left\|\frac{1}{a}\right\|_{L_q(Q)} \le a_2; \tag{A}$$

$$\begin{gathered} b(t,x),\ d(t,x) \in L_\infty\left(0,T;L_2(0,l)\right),\ \frac{b^2(t,x)}{a(t,x)},\ \frac{d^2(t,x)}{a(t,x)} \in L_\infty(Q), \\ \|b\|_{L_\infty(0,T;L_2(0,l))} \le K_b,\ \|d\|_{L_\infty(0,T;L_2(0,l))} \le K_d,\ \left\|\frac{b^2}{a}\right\|_{L_\infty(Q)} \le K_b^*,\ \left\|\frac{d^2}{a}\right\|_{L_\infty(Q)} \le K_d^*; \end{gathered} \tag{B}$$

$$\begin{gathered} g(t,x),\ r(t,x) \in L_\infty\left(0,T;L_1(0,l)\right),\ \|r\|_{L_\infty(0,T;L_1(0,l))} \le K_r,\ \frac{g^2(t,x)}{a(t,x)} \in L_\infty\left(0,T;L_1(0,l)\right), \\ \left\|\frac{g^2}{a}\right\|_{L_\infty(0,T;L_1(0,l))} \le K_g^*,\ \frac{r^2(t,x)}{a(t,x)} \in L_1(Q),\ \left\|\frac{r^2}{a}\right\|_{L_1(Q)} \le K_r^*; \end{gathered} \tag{C}$$

$$u_0(x) \in \overset{0}{W}{}_2^1(0,l); \tag{D}$$

$$\varphi(t),\ \varphi'(t) \in L_\infty(0,T),\ \|\varphi'(t)\|_{L_\infty(0,T)} \le K_\varphi^*; \tag{E}$$

$$\begin{gathered} \omega(x) \in L_\infty(0,l),\ (a\omega)_x \in L_\infty\left(0,T;L_2(0,l)\right),\ a(t,x)\omega(x)\big|_{x=0,l} = 0, \\ \|\omega\|_{L_\infty(0,l)} \le K_\omega,\ \left\|(a\omega)_x\right\|_{L_\infty(0,T;L_2(0,l))} \le K_a; \end{gathered} \tag{F}$$

$$\left|\int_0^l g(t,x)\omega(x)\,\mathrm{d}x\right| \ge g_0 > 0; \tag{G}$$

$$\varphi(0) = \int_0^l u_0(x)\omega(x)\,\mathrm{d}x. \tag{H}$$

Here $a_1,\ a_2,\ K_g^*,\ K_\omega,\ K_a,\ g_0 = \text{const}>0,\ \ K_b,\ K_d, K_b^*,\ K_d^*, K_r,\ K_\varphi^* = \text{const} \ge 0.$

Definition 1. By generalized solution of the inverse problem (1)–(3) we mean a pair of functions $\{u(t,x), p(t)\}$ such that

1) $u(t,x) \in W_s^{1,2}(Q) \cap \overset{0}{W}{}_2^1(0,l),\ s>1,\ a(t,x)u_{xx}^2,\ \dfrac{u_t^2}{a(t,x)} \in L_1(Q),\ p(t) \in L_\infty(0,T);$

2) this pair satisfies equation (1) almost everywhere in Q;

3) $\lim\limits_{t\to 0^+} \displaystyle\int_0^l \left|u(t,x) - u_0(x)\right|\,\mathrm{d}x = 0;$

4) equality (3) is satisfied at each point $t \in [0,T]$.

Under the assumptions (A)–(H) in the paper [1] (see also [2]) we proved the following theorem of existence and uniqueness of the generalized solution of the problem (1)–(3).

Theorem 1. Let conditions (A)–(H) are satisfied. Set

$$q^* = \frac{2q}{q+1},\ \ \lambda = \frac{3}{2}\left(K_b^* + \frac{l^2}{2}K_d^*\right),\ \ \beta = \frac{\sqrt{3K_g^*}}{g_0}\mathrm{e}^{\lambda T}\left(K_a + K_b K_\omega + K_d K_\omega \frac{l}{\sqrt{2}}\right),\ \ \gamma = 4\beta^2. \tag{5}$$

Then there exists a unique generalized solution $\{\hat{u}(t,x), \hat{p}(t)\}$ of the inverse problem (1)–(3), $\hat{u}(t,x) \in W_{q^*}^{1,2}(Q)$ and the following estimates hold:

$$\|\hat{p}\|_{L_\infty(0,T)} \le \frac{2e^{\gamma T/2}}{g_0}\left\{ \mathrm{e}^{\lambda T}\left(K_a + K_b K_\omega + K_d K_\omega \frac{l}{\sqrt{2}} \right) \times \left(\|u_0'\|^2_{L_2(0,l)} + 3K_r^* \right)^{1/2} + K_\varphi^* + K_\omega K_r \right\}, \quad (6)$$

$$\sup_{0\le t\le T} \|\hat{u}_x(t,\cdot)\|^2_{L_2(0,l)} \le \mathrm{e}^{2\lambda T}\left(\|u_0'\|^2_{L_2(0,l)} + 6K_g^* T \|\hat{p}\|^2_{L_\infty(0,T)} + 6K_r^* \right), \quad (7)$$

$$\|\hat{u}_{xx}\|^2_{L_{q^*}(Q)} + \|\hat{u}_t\|^2_{L_{q^*}(Q)} \le \mathrm{e}^{2\lambda T}(a_1 + a_2)\left(\|u_0'\|^2_{L_2(0,l)} + 6K_g^* T \|\hat{p}\|^2_{L_\infty(0,T)} + 6K_r^* \right). \quad (8)$$

2. Main result

Now let us obtain the estimates of stability.

Theorem 2. Consider the following two inverse problems in Q:

$$u_t - a(t,x)u_{xx} + b(t,x)u_x + d(t,x)u = p(t)g^{(j)}(t,x) + r^{(j)}(t,x), \quad (9)$$

$$u(0,x) = u_0^{(j)}(x), \quad u(t,0) = u(t,l) = 0, \quad (10)$$

$$\int_0^l u(t,x)\omega(x)\,\mathrm{d}x = \varphi^{(j)}(t), \quad (11)$$

$j = 1,2.$

Suppose that for these problems conditions (A)−(H) hold. Let $\{u^{(j)}(t,x), p^{(j)}(t)\}$, $j = 1,2$ be the corresponding solutions of these problems. (These solutions exist and are unique by theorem 1.) Then the following estimates hold:

$$\left\|p^{(1)} - p^{(2)}\right\|_{L_\infty(0,T)} \le \frac{2\mathrm{e}^{\gamma T/2}}{g_0}\left\{ e^{\lambda T}\left(K_a + K_b K_\omega + K_d K_\omega \frac{l}{\sqrt{2}} \right) \times \right.$$

$$\times \left(\left\|u_{0x}^{(1)} - u_{0x}^{(2)}\right\|^2_{L_2(0,l)} + 6T \left\|p^{(2)}\right\|^2_{L_\infty(0,T)} \times \left\| \frac{\left(g^{(1)} - g^{(2)}\right)^2}{a} \right\|_{L_\infty(0,T;L_1(0,l))} + \left\| \frac{\left(h^{(1)} - h^{(2)}\right)^2}{a} \right\|_{L_1(Q)} \right)^{1/2} + \quad (12)$$

$$\left. + \left\|\varphi_t^{(1)} - \varphi_t^{(2)}\right\|_{L_\infty(0,T)} + K_\omega \left\|p^{(2)}\right\|_{L_\infty(0,T)} \left\|g^{(1)} - g^{(2)}\right\|_{L_\infty(0,T;L_1(0,l))} + K_\omega \left\|r^{(1)} - r^{(2)}\right\|_{L_\infty(0,T;L_1(0,l))} \right\},$$

$$\sup_{0\le t\le T} \left\|u_x^{(1)}(t,\cdot) - u_x^{(2)}(t,\cdot)\right\|^2_{L_2(0,l)} + \left\|u_{xx}^{(1)} - u_{xx}^{(2)}\right\|^2_{L_{q^*}(Q)} + \left\|u_t^{(1)} - u_t^{(2)}\right\|^2_{L_{q^*}(Q)} \le$$

$$\le \mathrm{e}^{2\lambda T}(1 + a_1 + a_2)\left(\left\|u_{0x}^{(1)} - u_{0x}^{(2)}\right\|^2_{L_2(0,l)} + 6K_g^* T \left\|p^{(1)} - p^{(2)}\right\|^2_{L_\infty(0,T)} + 12T \left\|p^{(2)}\right\|^2_{L_\infty(0,T)} \times \right. \quad (13)$$

$$\left. \times \left\| \frac{\left(g^{(1)} - g^{(2)}\right)^2}{a} \right\|_{L_\infty(0,T;L_1(0,l))} + 12 \left\| \frac{\left(r^{(1)} - r^{(2)}\right)^2}{a} \right\|_{L_1(Q)} \right).$$

Proof. Set

$$v^*(t,x)=u^{(1)}(t,x)-u^{(2)}(t,x),\ \delta^*(t)=p^{(1)}(t)-p^{(2)}(t),\ v_0(x)=u_0^{(1)}(x)-u_0^{(2)}(x),$$

$$\psi(t)=\varphi^{(1)}(t)-\varphi^{(2)}(t),\ \rho(t,x)=g^{(1)}(t,x)-g^{(2)}(t,x),\ \sigma(t,x)=r^{(1)}(t,x)-r^{(2)}(t,x),$$

$$G^{(j)}(t)=\int_0^l g^{(j)}(t,x)\omega(x)\,\mathrm{d}x,\qquad j=1,2.$$

Then the pair $\{v^*(t,x),\delta^*(t)\}$ is a solution in Q of the inverse problem

$$v_t-a(t,x)v_{xx}+b(t,x)v_x+d(t,x)v=\delta(t)g^{(1)}(t,x)+p^{(2)}(t)\rho(t,x)+\sigma(t,x), \tag{14}$$

$$v(0,x)=v_0(x),\quad v(t,0)=v(t,l)=0, \tag{15}$$

$$\int_0^l v(t,x)\omega(x)\,\mathrm{d}x=\psi(t). \tag{16}$$

Let us multiply equation (14) by $\omega(x)$ and integrate the resulting relation over the segment $[0,l]$. Taking into account conditions (15), (16) and integrating by parts using the assumptions (E)–(D) we obtain the relation

$$\delta^*(t)=\frac{1}{G^{(1)}(t)}\left[\int_0^l\left((a\omega)_x+b\omega\right)v_x^*\,\mathrm{d}x+\int_0^l d\omega v^*\mathrm{d}x+\psi'(t)-p^{(2)}(t)\int_0^l\rho\omega\,\mathrm{d}x-\int_0^l\sigma\omega\,\mathrm{d}x\right]. \tag{17}$$

Let us introduce the operator $B:L_\infty(0,T)\to L_\infty(0,T)$ by the formula

$$B(\delta)=\frac{1}{G^{(1)}(t)}\left[\int_0^l\left((a\omega)_x+b\omega\right)v_x\,\mathrm{d}x+\int_0^l d\omega v\,\mathrm{d}x\right], \tag{18}$$

where $v(t,x)$ is a solution of the direct problem (14), (15) with a given function $\delta(t)$ in the right–hand side of equation (14). Then we derive from (18) that $\delta^*(t)$ is a solution of operator equation

$$\delta(t)=B(\delta)(t)+\frac{1}{G^{(1)}(t)}\left[\psi'(t)-\int_0^l\left(p^{(2)}(t)\rho+\sigma\right)\omega\,\mathrm{d}x\right]. \tag{19}$$

Let us introduce in space $L_\infty(0,T)$ the norm

$$\|z\|_\gamma=\left\|\mathrm{e}^{-\gamma t/2}z\right\|_{L_\infty(0,T)},\quad z\in L_\infty(0,T),$$

where γ is defined in (5).

Then by analogy with the proof of theorem 3.1 from [2] we obtain that the operator B is a contraction operator in $L_\infty(0,T)$ with chosen norm $\|\cdot\|_\gamma$ and the following estimate

$$\left\|B\left(\delta^{(1)}\right)(t)-B\left(\delta^{(2)}\right)(t)\right\|_\gamma\le\frac{1}{2}\left\|\delta^{(1)}(t)-\delta^{(2)}(t)\right\|_\gamma, \tag{20}$$

$\delta^{(1)}(t)$, $\delta^{(2)}(t)\in L_\infty(0,T)$, hold.

Denote by $w_0(t,x)$ the solution of direct problem (14), (15) with $\delta(t)\equiv 0$ in the right-hand side of (14). Then for $w_0(t,x)$ we have the estimate of the form (7) which in our case looks as follows

$$\sup_{0\le t\le T}\left\|w_{0x}(t,\cdot)\right\|^2_{L_2(0,l)}\le\mathrm{e}^{2\lambda T}\left(\left\|v_0'\right\|^2_{L_2(0,l)}+6T\left\|p^{(2)}\right\|^2_{L_\infty(0,T)}\left\|\frac{\rho^2}{a}\right\|_{L_\infty(0,T;L_1(0,l))}+\left\|\frac{\sigma^2}{a}\right\|_{L_1(Q)}\right). \tag{21}$$

Let $\delta(t)\equiv 0$ be the zeroth approximation of the solution of equation (19) when applying the method of iterations. Then from formula (18) in view of conditions (B), (F), (G) and inequality (4) we obtain

$$\left|B(\delta)(t)\right| \leq \frac{1}{g_0}\left(K_a + K_b K_\omega + K_d K_\omega \frac{l}{\sqrt{2}}\right)\left\|w_{0x}(t,\cdot)\right\|_{L_2(0,l)}. \tag{22}$$

Let $\delta_1(t)$ be the first approximation of the solution of equation (19) when applying the method of iterations. Then using the well-known estimate of the n-th approximation of the solution in the iterative process (see, for example, [6], p.43) and also (20) we have

$$\left\|\delta_1(t)-\delta^*(t)\right\|_\gamma \leq \left\|B(0)(t)+\frac{\psi'}{G^{(1)}}+\frac{1}{G^{(1)}}\int_0^l\left(p^{(2)}(t)\rho+\sigma\right)\omega\,\mathrm{d}x\right\|_\gamma,$$

whence in view of (22) and (21) we obtain the inequality

$$\left\|\delta^*\right\|_\gamma \leq \frac{2}{g_0}\left\{\mathrm{e}^{\lambda T}\left(K_a + K_b K_\omega + K_d K_\omega \frac{l}{\sqrt{2}}\right)\times\right.$$

$$\times\left(\left\|v_0'\right\|^2_{L_2(0,l)} + 6T\left\|p^{(2)}\right\|^2_{L_\infty(0,T)}\times\left\|\frac{\rho^2}{a}\right\|_{L_\infty(0,T;L_1(0,l))}+6\left\|\frac{\sigma^2}{a}\right\|_{L_1(Q)}\right)^{1/2}+$$

$$\left.+\left\|\psi'\right\|_{L_\infty(0,T)}+\left\|p^{(2)}\right\|^2_{L_\infty(0,T)}K_\omega\left\|\rho\right\|_{L_\infty(0,T;L_1(0,l))}+K_\omega\left\|\sigma\right\|_{L_\infty(0,T;L_1(0,l))}\right\}.$$

This inequality obviously implies estimate (12). Note that the equation (14) is similar to the equation (1) with replacement of the function $r(t,x)$ on $p^{(2)}(t)\rho(t,x)+\sigma(t,x)$ (there are three terms in the right-hand side of (14), not two as in the equation (1)). So with this in mind, estimate (13) follows from estimates (7) and (8).

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