

On a Physical Field Theory of Micropolar Thermoelasticity

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Abstract. A non-linear mathematical model of thermoelastic micropolar continuum is developed. The model is presented in terms of 4-covariant field theoretical formalism. Lagrangian density for thermoelastic continuum with three micropolar directors is given and the least action principle is formulated. Corresponding field equations of micropolar thermoelasticity are obtained. Variational symmetries of the thermoelastic action are used to formulate covariant conservation laws. Following the usual procedure, micropolar thermoelastic Lagrangians are represented as functions of independent rotationally invariant arguments. The latter constitutes a complete system of objective finite strain measures of micropolar thermoelasticity. Constitutive equations of micropolar thermoelasticity are obtained and discussed.

1. Introduction

In recent years, metamaterials manufacturing and processing have attracted significant attention from the engineers community and researchers. These artificial metamaterials exhibit abnormal physical properties not usually found in nature. Examples include negative Poisson's ratio (auxetic materials), negative thermal expansion, negative electric permittivity and the magnetic permeability. These physical phenomena can not be described in terms of the theory of classical continuum mechanics. In this case, microstructure continuum theories based on the necessity of additional (extra) freedom degrees are required. The physically infinitesimal volume in such continua is not a material point, but a much more sophisticated object, with its intrinsic additional freedom degrees (microrotations, microoscillations). Therefore, the derivations of non-linear Lagrangian, Hamiltonian, extra-stress and extra-strain, valid in the most general case of finite deformations and microrotations, to continua with microstructure [1,2,3]. It stands today as one of the most important problems of the continuum mechanics. That is why a development of complex continuum theories seems to be actual.

The thermomechanical behavior of metamaterials can be described in terms of field formalism and frameworks of micropolar thermoelasticity. In the present study the Green–Naghdi (GN-theory) [4] thermoelastic model is employed. Application of the field formalism principles to continuum mechanics leads to a natural forms of constitutive equations [5,6]. Rotationally



invariant Lagrangian forms along with the requirement of the Galilean translational invariance provide determination of a complete system of strain measures.

After the preliminary remarks of Sec. 1 a brief outline of basic principles and requisite field equations are given (Sec. 2). It includes variational formulation of the principle of least action, Euler–Lagrange equations and divergent forms of conservation laws.

In Sec. 3 the thermoelastic Lagrangian and least action integral are defined. The differential field equations corresponding to the action integral are derived. The energy-momentum tensor for micropolar thermoelastic continuum is obtained and the corresponding conservation laws are formulated. Its components allow to derive the Eshelby stress tensor, Umov–Poynting vector, pseudomomentum vector and Hamiltonian (total energy) for thermoelastic micropolar media.

In Sec. 4 the objective strain measures for micropolar thermoelastic continuum are obtained from the action invariance under coordinate frame rotations, translations of spatial coordinates and time translations.

The relative strain measures are discussed in Sec.5. The rotationally invariant form of the Helmholtz free energy is given.

In Sec. 6 the objective forms of constitutive equations of micropolar thermoelastic continuum are obtained.

2. Covariant Fields. Field Equations. Action. Conservation Laws

Theory of micropolar thermoelasticity can be developed in terms of the field theory formalism by using the action integral. Following considerations shall be restricted to a plane spacetime. Thus a general form of action furnishes

$$\mathcal{I} = \int \mathcal{L}(X^\beta, \varphi^k, \partial_\alpha \varphi^k) d^4X, \quad (1)$$

where φ^k are covariant physical fields; $\mathcal{L}$ is the Lagrangian density; X^β are the referential coordinates; $d^4X = dX^1 dX^2 dX^3 dX^4$ is spacetime elementary volume.

The least action principle states that the actual field is realized in the spacetime in a way that the action integral (1) is minimum, i.e. for any admissible variations of physical fields $\delta\varphi^k$ and $\delta X^\beta = 0$ the following equation is valid

$$\delta\mathcal{I} = 0. \quad (2)$$

The Euler–Lagrange field equations corresponding to action (1) are

$$\mathcal{E}_k(\mathcal{L}) = \frac{\partial \mathcal{L}}{\partial \varphi^k} - \partial_\alpha \frac{\partial \mathcal{L}}{\partial (\partial_\alpha \varphi^k)} = 0.$$

In field theory a conservation law can be formulated as

$$\partial_\beta J^\beta = 0, \quad (3)$$

where the vector $J^\beta = J^\beta(X^\beta, \varphi^k, \partial_\alpha \varphi^k)$ is called the 4-current. The 4-current can be obtained in the form

$$J^\beta = \frac{\partial \mathcal{L}}{\partial (\partial_\beta \varphi^k)} \delta^\nabla \varphi^k + \left(\mathcal{L} \delta_\alpha^\beta - (\partial_\alpha \varphi^k) \frac{\partial \mathcal{L}}{\partial (\partial_\beta \varphi^k)} \right) \delta^\nabla X^\alpha, \quad (4)$$

wherein $\delta^\nabla = \delta/\varepsilon$ denote finite variations corresponding to symmetry transformations of X^β and φ^k .

In fact the current (4) can be determined if variational symmetries of action (1) are known.

3. Thermoelastic Langrangian. Energy-Momentum Tensor for Micropolar Thermoelastic Continuum

Thermoelastic action for micropolar continuum is taken in the form

$$\mathcal{I} = \int \mathcal{L}(X^\alpha, x^j, d_\alpha^j, \vartheta, \partial_4 x^j, \partial_4 d_\alpha^j, \partial_4 \vartheta, \partial_\beta x^j, \partial_\beta d_\alpha^j, \partial_\beta \vartheta) dX^1 dX^2 dX^3 dX^4. \quad (5)$$

In (5) X^α ($\alpha = 1, 2, 3$) are referential coordinates; x^j ($j = 1, 2, 3$) are spatial coordinates; d_α^j are micropolar directors; ϑ is temperature displacement. Equations $x^j = x^j(X^\alpha)$ determine the deformation of continuum.

We assume the action density (5) in following form

$$\mathcal{L} = \frac{1}{2}(\partial_4 x^k) \rho_{kj} (\partial_4 x^j) + \frac{1}{2}(\partial_4 d_\alpha^i) \overset{\text{ab}}{\mathcal{I}}_{ij} (\partial_4 d_\alpha^j) - \psi(X^\alpha, x^j, d_\alpha^j, \vartheta, \partial_4 \vartheta, \partial_\alpha x^j, \partial_\alpha d_\alpha^j, \partial_\alpha \vartheta). \quad (6)$$

Hereafter $\overset{\text{ab}}{\mathcal{I}}_{ij}$ denotes the microinertia tensor; ρ_{ij} is mass density tensor; ψ is referential volume density of the Helmholtz's free energy. For tensors $\overset{\text{ab}}{\mathcal{I}}_{ij}$ and ρ_{ij} the symmetry conditions $\rho_{ij} = \rho_{ji}$, $\overset{\text{ab}}{\mathcal{I}}_{ij} = \overset{\text{ab}}{\mathcal{I}}_{ji}$ are to be valid.

The differential field equations corresponding to action integral (5) read

$$\begin{aligned} \partial_\alpha S_{\cdot j}^\alpha - \partial_4 P_j &= -\frac{\partial \mathcal{L}}{\partial x^j} & (\alpha = 1, 2, 3; j = 1, 2, 3), \\ \partial_\alpha \mathcal{M}_{\cdot j}^\alpha + \mathcal{A}_j - \partial_4 (\mathcal{Q}_j) &= 0 & (\alpha = 1, 2, 3; j = 1, 2, 3), \\ \partial_\alpha j_R^\alpha + \partial_4 s &= \frac{\partial \mathcal{L}}{\partial \vartheta} & (\alpha = 1, 2, 3), \end{aligned} \quad (7)$$

and supplemented by the following equations:

$$\begin{aligned} S_{\cdot j}^\alpha &= -\frac{\partial \mathcal{L}}{\partial (\partial_\alpha x^j)}, & \mathcal{M}_{\cdot j}^\alpha &= -\frac{\partial \mathcal{L}}{\partial (\partial_\alpha d^j)}, & \mathcal{A}_j &= \frac{\partial \mathcal{L}}{\partial d^j}, \\ P_j &= \frac{\partial \mathcal{L}}{\partial (\partial_4 x^j)}, & \mathcal{Q}_j &= \frac{\partial \mathcal{L}}{\partial (\partial_4 d^j)}, \\ s &= \frac{\partial \mathcal{L}}{\partial (\partial_4 \vartheta)}, & j_R^\alpha &= \frac{\partial \mathcal{L}}{\partial (\partial_\alpha \vartheta)}, \end{aligned} \quad (8)$$

where $S_{\cdot j}^\alpha$ is the first Piola–Kirchhoff tensor; $\mathcal{M}_{\cdot j}^\alpha$ is the extra-stress tensor; $\mathcal{A}_j$ are generalized moments; P_j and $\mathcal{Q}_j$ are generalized momenta; s is entropy density; j_R^α is the referential entropy flux. 1st eq. of (7) is the motion balance equation. 2nd eq. of (7) is the momentum balance equation. If $\frac{\partial \mathcal{L}}{\partial (\partial_\alpha \vartheta)} = 0$ then 3rd eq. of (7) constitutes the entropy balance equation.

We proceed to conservation laws for the system of field eqs. (7). It is known that action integral $\mathcal{L}$ is invariant under translations of all coordinates X^α if $\mathcal{L}$ does not depends on X^α explicitly. Therefore, the 4-covariant energy-momentum tensor can be easily obtained and the corresponding conservation laws are formulated. There are four groups of equations for the components of the canonical energy-momentum tensor T_{λ}^{μ} ($\lambda, \mu = 1, 2, 3, 4$):

$$T_{\lambda}^{\mu} = \mathcal{L} \delta_{\lambda}^{\mu} + S_{\cdot l}^{\mu} (\partial_{\lambda} x^l) + \overset{\text{a}}{\mathcal{M}}_{\cdot l}^{\mu} (\partial_{\lambda} d_{\alpha}^l) - j_R^{\mu} (\partial_{\lambda} \vartheta) \quad (\lambda, \mu = 1, 2, 3), \quad (9)$$

$$T_{\cdot 4}^{\mu} = S_{\cdot l}^{\mu} \dot{x}^l + \overset{\text{a}}{\mathcal{M}}_{\cdot l}^{\mu} \dot{d}_{\alpha}^l - j_R^{\mu} \dot{\vartheta} \quad (\mu = 1, 2, 3), \quad (10)$$

$$T_{\lambda}^{4\cdot} = -(\partial_{\lambda} x^l) P_l - (\partial_{\lambda} d_a^l) \dot{\mathcal{Q}}_l - s(\partial_{\lambda} \vartheta) \quad (\lambda = 1, 2, 3), \quad (11)$$

$$T_{\cdot 4}^{4\cdot} = \mathcal{L} - \dot{x}^l P_l - \dot{d}_a^l \dot{\mathcal{Q}}_l - s\dot{\vartheta}. \quad (12)$$

It is clearly seen that the above components of the energy-momentum tensor for micropolar thermoelastic field can be treated as:

$T_{\cdot 4}^{4\cdot} = \mathcal{H}$ is the Hamiltonian,

$T_{\lambda}^{4\cdot} = \mathcal{P}_{\lambda}$ is Umov–Poynting vector,

$T_{\cdot 4}^{\mu\cdot} = \Gamma^{\mu}$ is pseudomomentum vector,

$T_{\cdot \lambda}^{\mu\cdot} = P_{\cdot \lambda}^{\mu\cdot}$ is the Eshelby stress tensor.

Conservation laws corresponding to the translational symmetries of action read

$$\partial_{\mu} T_{\cdot \lambda}^{\mu\cdot} = 0 \quad (\lambda, \mu = 1, 2, 3, 4). \quad (13)$$

Eqs. (13) are splitted into the following symmetric canonical equations

$$-\dot{\mathcal{H}} + \partial_{\mu} \Gamma^{\mu} = 0, \quad (\mu = 1, 2, 3) \quad (14)$$

$$-\dot{\mathcal{P}}_{\lambda} + \partial_{\mu} P_{\cdot \lambda}^{\mu\cdot} = 0. \quad (\lambda, \mu = 1, 2, 3) \quad (15)$$

Eqs. (14) and (15) are the energy balance equation and the pseudomomentum balance equation respectively.

4. Objective Strain Measures for Micropolar Thermoelastic Continuum

Action and action density should be satisfied some conditions of invariance with respect to arbitrary rotations of spatial coordinate system and time translations. Since the choice of spatial coordinates is rather arbitrary it should not in any way affect formulations of physical laws. For this reason the action invariance under rotations and translations of spatial coordinates and time translations is presumed.

The action, in particular, should be invariant under translations and rotations of the observer's coordinate system and time translations:

$$\begin{aligned} \tilde{x}^i &= R_j^i x^j + C^i, \\ \tilde{d}_a^i &= R_j^i d_a^j, \\ \tilde{t} &= t + C, \end{aligned} \quad (16)$$

wherein C^i , C are constants; R_j^i is an arbitrary orthogonal tensor.

The action invariance under spatial coordinates translations is known as the Galilean relativity principle. This principle is supplemented by the temperature displacement translations (C' is an arbitrary constant):

$$\tilde{\vartheta} = \vartheta + C', \quad (17)$$

that is provided by the following condition

$$\frac{\partial \mathcal{L}}{\partial \vartheta} = 0. \quad (18)$$

By the above reasons the Helmholtz free energy is a function of the variables

$$X^{\beta}, \quad \dot{\vartheta}, \quad \partial_{\alpha} \vartheta, \quad (19)$$

and following independent and invariant with respect to the rotations of the spatial coordinate system arguments [3]:

$$\begin{aligned} g_{\alpha\beta} &= g_{ij}(\partial_\alpha x^i)(\partial_\beta x^j), \\ \mathcal{R}_\alpha &= g_{ij}(\partial_\alpha x^i)d_\alpha^j, \\ \mathcal{T}_{\alpha\beta} &= g_{ij}(\partial_\alpha x^i)(\partial_\beta d_\alpha^j). \end{aligned} \quad (20)$$

Here g_{ij} is spatial metric tensor; $g_{\alpha\beta}$ is convective metric tensor.

It can be shown that (19) and (20) constitute a complete system of independent rotationally invariant argument of ψ .

The completeness of the rotationally invariant arguments system given by (20) can be proved by means of the algebraic invariants theory considering the contravariant vectors

$$\partial_\alpha x^i, d_\alpha^j, \partial_\beta d_\alpha^j. \quad (21)$$

A complete system of vectors invariants (21) consists of pairwise inner products, which leads to the Euler invariants (20).

This invariants system also has all kind of 3×3 -determinants, which are located in columns of various components of the Euler vector triples (21). It is clear that the determinants must contain at least one column of Euler component of strain gradient $\partial_\alpha x^i$. The determinants calculation can be carried out with the Gram–Schmidt process, i.e. through the determinants, which are all kinds of internal products of Euler vectors arranged in columns basic determinants, and the metric coefficients $g_{\alpha\beta}$.

Assuming that the continuum is homogeneous, i.e.,

$$\partial_\beta^{\text{expl}} \mathcal{L} = 0 \quad (\beta = 1, 2, 3), \quad (22)$$

and, consequently, all Lagrangian variables X^β are cyclic (neglected). We get the following rotationally-invariant form of the Helmholtz free energy, which satisfies the from indifference principle: ($\alpha=1,2,3$; $\beta=1,2,3$)

$$\psi = \psi(g_{\alpha\beta}, \mathcal{R}_\alpha, \mathcal{T}_{\alpha\beta}, \dot{\vartheta}, \partial_\alpha \vartheta). \quad (23)$$

We implicitly assume that the reduced form (23) should also depend on the reference metrics and referential position d -vectors d_α^j ($\alpha=1,2,3$).

5. Relative Strain Measures

In the equation (23) rotationally invariant argument $g_{\alpha\beta}$ can be replaced by the following relative strain tensor

$$\epsilon_{\alpha\beta} = \frac{1}{2}(g_{\alpha\beta} - g_{\alpha\beta}). \quad (24)$$

The tensor $\epsilon_{\alpha\beta}$ is called the Green's strain tensor.

Instead of a vector measure of extra strain $\mathcal{R}_\alpha$ a relative extra strain vector is defined as

$$-\gamma_\alpha = \mathcal{R}_\alpha - g_{\alpha\beta} d_\alpha^\beta. \quad (25)$$

Hereafter vectors d_α^β denote the referential position of the d -vectors. x^i are the spatial positions in the reference configuration. Note the following equation:

$$d_\alpha^i = \frac{\partial x^i}{\partial X^\alpha} d_\alpha^\alpha.$$

Vector γ_{α} vanishes only if each of d -vectors rotates and elongates exactly so as prescribed by the deformation $X^{\alpha} \rightarrow x^i$. If this in effect then d -vectors and d -vectors are related by

$$d_{\alpha}^i - (\partial_{\alpha} x^i) d_{\alpha}^{\alpha} = 0.$$

Multiplying this equation by the deformation gradient $\partial_{\beta} x^j$ and contracting with the metric tensor g_{ij} lead to equation

$$\mathcal{R}_{\beta} - g_{\beta\alpha} d_{\alpha}^{\alpha} = 0,$$

i.e. relative vector of extra-strain equals to zero:

$$\gamma_{\beta} = 0.$$

Finally rotationally invariant form of the Helmholtz free energy is represented by

$$\psi = \psi(\epsilon_{\alpha\beta}, \gamma_{\alpha}, \mathcal{T}_{\alpha\beta}, \dot{\vartheta}, \partial_{\alpha} \vartheta) \quad (\alpha = 1, 2, 3; \alpha, \beta = 1, 2, 3). \quad (26)$$

The Helmholtz free energy depends on one scalar argument $\dot{\vartheta}$; four referential vector arguments $\partial_{\alpha} \vartheta$, γ_{α} ($\alpha = 1, 2, 3; \alpha = 1, 2, 3$); and four referential tensor arguments $\epsilon_{\alpha\beta}$, $\mathcal{T}_{\alpha\beta}$ ($\alpha = 1, 2, 3; \alpha, \beta = 1, 2, 3$); $\epsilon_{\alpha\beta}$ symmetric and $\mathcal{T}_{\alpha\beta}$ asymmetric second rank tensors.

6. Constitutive Equations of Micropolar Thermoelasticity

As shown in the previous sections, the objective basis for the thermodynamic micropolar thermoelastic continuum, in which the heat transport is not accompanied by the entropy production consists of the following functional independent variables

$$\epsilon_{\alpha\beta}, \gamma_{\alpha}, \mathcal{T}_{\alpha\beta}, \dot{\vartheta}, \partial_{\alpha} \vartheta \quad (\alpha = 1, 2, 3; \alpha, \beta = 1, 2, 3). \quad (27)$$

The following objective form of the constitutive equations in terms of the rotationally invariant basis can be deduced from (8):

$$\begin{aligned} -S_{\cdot j}^{\alpha} &= \frac{\partial \mathcal{L}}{\partial (\partial_{\alpha} x^j)} = \frac{\partial \mathcal{L}}{\partial \epsilon_{\mu\alpha}} g_{jk} \partial_{\mu} x^k - \frac{\partial \mathcal{L}}{\partial \gamma_{\beta}} g_{jk} d_{\alpha}^k \delta_{\beta}^{\alpha} + \\ &+ \frac{\partial \mathcal{L}}{\partial \gamma_{\beta}} g_{jk} (\partial_{\beta} x^k d_{\alpha}^{\alpha} + \partial_{\sigma} x^k \delta_{\beta}^{\alpha} d_{\alpha}^{\sigma}) + \frac{\partial \mathcal{L}}{\partial \mathcal{T}_{\mu\nu}} g_{jk} \partial_{\nu} d_{\alpha}^k, \end{aligned} \quad (28)$$

$$-\mathcal{M}_{\cdot j}^{\alpha} = \frac{\partial \mathcal{L}}{\partial (\partial_{\alpha} d^j)} = \frac{\partial \mathcal{L}}{\partial \mathcal{T}_{\mu\nu}} g_{jk} \partial_{\mu} x^k \delta_{\nu}^{\alpha}, \quad (29)$$

$$\mathcal{A}_{\cdot j} = \frac{\partial \mathcal{L}}{\partial d^j} = \frac{\partial \mathcal{L}}{\partial \gamma_{\beta}} g_{jk} \partial_{\beta} x^k. \quad (30)$$

The constitutive equation (28) after obvious transformations can be finally represented as:

$$\begin{aligned} -S_{\cdot j}^{\alpha} &= \frac{\partial \mathcal{L}}{\partial \epsilon_{\mu\alpha}} g_{jk} \partial_{\mu} x^k + \frac{\partial \mathcal{L}}{\partial \gamma_{\alpha}} g_{jk} (\partial_{\sigma} x^k d_{\alpha}^{\sigma} - d_{\alpha}^k) + \\ &+ \frac{\partial \mathcal{L}}{\partial \gamma_{\beta}} g_{jk} \partial_{\beta} x^k d_{\alpha}^{\alpha} + \frac{\partial \mathcal{L}}{\partial \mathcal{T}_{\mu\nu}} g_{jk} \partial_{\nu} d_{\alpha}^k. \end{aligned} \quad (31)$$

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8. Conclusions

The study of thermoelastic continuum by methods of field theory allows us to conclude. A non-linear mathematical model of thermoelastic micropolar continuum has been developed. The model has been presented in terms of 4-covariant field theoretical formalism. Thermal field has been incorporated in the theory by the two constitutive constants. The differential field equations corresponding to the action integral have been derived. The energy-momentum tensor for micropolar thermoelastic continuum has been obtained and the corresponding conservation laws have been formulated. Following the usual procedure, micropolar thermoelastic Lagrangians have been represented as functions of independent rotationally invariant arguments. The completeness of the rotationally invariant arguments system has been proved by means of the algebraic invariants theory. Relative strain measures have been discussed. The rotationally invariant form of the Helmholtz free energy has been given. Constitutive equations of micropolar thermoelasticity with three directors have been obtained and discussed.

9. References

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