Analytical solution of the second Stokes problem with variable amplitude on behavior of gas over an oscillation surface

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Abstract. The analytical solution of the second Stokes problem is found. The case of the variable amplitude of fluctuation of a surface is considered. The linear kinetic equation with boundary conditions is derived. Eigen solutions of the equation are found. Properties of dispersion function are investigated. The general solution of the kinetic equation with boundary conditions in terms of the eigen solutions decomposition is composed.

In the last time the second Stokes problem has a great interest of scientists. It is connected to development of technologies and in particular nanotechnologies. There are many articles devoted to the second Stokes problem [1-5], where the numerical solution is given [3-5]. The problem has the analytical solution is shown in the present work. In this article we continue works by V A Akimova, A V Latyshev and A A Yushkanov, in which the second Stokes problem has solution in case of the constant amplitude of fluctuation of a surface [1]. In this paper we consider the variable amplitude of fluctuation of a surface.

Let one-atomic rarefied gas fills half-space $x>0$. Surface makes harmonical oscillations lengthwise an axis Oy under the law $u_s(t)=u_0e^{-i\omega t}$, where $\omega=\omega_1+i\omega_2$. Here ω_1 is the oscillation frequency of plates and $A(t)=u_0e^{\omega_2 t}$ is the variable amplitude of fluctuation of a surface.

We will consider a model kinetic BGK equation (Bhatnagar, Gross, Krook)

$$\frac{\partial f}{\partial t}+\mathrm{v}_x\frac{\partial f}{\partial x}=\frac{f_{eq}-f}{\tau},$$

where $\nu = 1/\tau$ is the collision frequency of gaseous molecules, τ is the time between two consecutive collisions of molecules, f_{eq} is the equilibrium distribution function,

$$f_{eq}=n\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}}\exp\left[-\frac{m(\mathbf{v}-\mathbf{u})^2}{2kT}\right].$$

Here $u_y(t,x)$ is the mass velocity of gas, k is the Boltzmann constant, m is the mass of molecule of gas, n is the concentration of gas, T is the temperature of gas.

Considering, that the velocity of molecule of gas much less than the thermal velocity of molecule $|u_y(t_1,x_1)|<<\upsilon_T$, where $\upsilon_T=1/\sqrt{\beta}$ is the thermal velocity of molecule of gas, then problem can be linearized. Concentration of gas and temperature are considered as constants. We will search the distribution function in the form

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$$f = f_M(\mathrm{v})\big(1+\varphi(t,x,\mathbf{v})\big).$$

$f_M(\mathrm{v})$ is the absolute Maxwellian

$$f_M(\mathrm{v}) = n\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} \exp\left[-\frac{m\mathrm{v}^2}{2kT}\right],$$

We introduce dimensionless parametres: dimensionless velocity of molecules $\mathbf{C} = \sqrt{\beta}\mathbf{v} = \dfrac{\mathbf{v}}{\upsilon_{\mathrm{T}}}$, $\beta = \dfrac{m}{2kT}$, dimensionless mass velocity $U_y(t,x) = \sqrt{\beta}u_y(t,x)$, dimensionless coordinate $x_1 = xv\sqrt{\beta}$, dimensionless time $t_1 = tv$.

Making necessary calculations, we obtain the kinetic equation

$$\frac{\partial\varphi}{\partial t_1} + C_x\frac{\partial\varphi}{\partial x_1} + \varphi(t,x_1,C_x) = \frac{2C_y}{\pi^{3/2}}\int \exp(-\mathbf{C}'^2)C'_y\,\varphi(t_1,x_1,\mathbf{C}')d^3\mathbf{C}', \tag{1}$$

with the boundary conditions

$$\varphi(t_1,0,C_x) = 2qC_yU_s(t_1) + (1-q)\varphi(t_1,0,-C_x,C_y,C_z),\quad C_x>0, \tag{2}$$

and $\varphi(t_1,x_1\to\infty,\mathbf{C}) = 0$.

Considering, that plate oscillations are considered along an axis y, we will search function $\varphi(t,x_1,\mathbf{C})$ in the form

$$\varphi(t,x_1,\mathbf{C}) = C_y e^{-i\Omega t_1}h(x_1,C_x).$$

Thus, the equation (1), can be rewritten as

$$C_x\frac{\partial h}{\partial x_1} + (1-i\Omega)h(x_1,C_x) = \frac{1}{\sqrt{\pi}}\int\limits_{-\infty}^{\infty}\exp(-C'^2_x)h(x_1,C'_x)dC'_x, \tag{3}$$

and boundary conditions (2) can be written as

$$h(0,C_x) = 2qU_0 + (1-q)h(0,-C_x),\quad C_x>0,$$

and $h(x_1\to\infty,C_x) = 0$.

We will seek the solution to equation (3) in the form

$$h_\eta(x_1,\mu) = \exp\left(-\frac{x_1z_0}{\eta}\right)\Phi(\eta,\mu),\qquad \frac{1}{z_0}\int\limits_{-\infty}^{\infty}\exp\left(-\mu'^2\right)\Phi(\eta,\mu')\,d\mu' = 1. \tag{4}$$

Here η is the spectral parameter or the separation parameter, it is complex one. Using these equalities (4), we obtain from equation (3) the characteristic equation

$$(\eta-\mu)\Phi(\eta,\mu) = \frac{\eta}{\sqrt{\pi}},\quad \eta\in C.$$

Then, we find the eigen functions of the characteristic equation

$$\Phi(\eta,\mu) = \frac{1}{\sqrt{\pi}}\eta P\frac{1}{\eta-\mu} + \exp(\eta^2)\lambda(\eta)\delta(\eta-\mu),$$

where $-\infty<\eta,\mu<+\infty$, *δ(x)* is the Dirac delta function, $\lambda(z)$ is the dispersion function, which present in the form

$$\lambda(z) = 1 - i\Omega + \frac{z}{\sqrt{\pi}}\int\limits_{-\infty}^{\infty}\frac{\exp(-\tau^2)d\tau}{\tau - z}.$$

or $\lambda(z) = -i\Omega + \lambda_0(z)$, where $$\lambda_0(z) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-\tau^2} \tau\, d\tau}{\tau - z}.$$

Using the Sokhotsky formulas [6], we will find boundary value of dispersion function

$$\lambda^{\pm}(\mu) = \pm i\sqrt{\pi}\mu\, e^{-\mu^2} - i\Omega + \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-\tau^2} \tau\, d\tau}{\tau - \mu}.$$

Let's separate at function $\lambda^{\pm}(\mu)$ the real and imaginary parts (figure 1 and 2)

$$\operatorname{Re}\lambda^{\pm}(\mu) = \Omega_2 + \lambda_0(\mu), \quad \operatorname{Im}\lambda^{\pm}(\mu) = -\Omega_1 \pm \sqrt{\pi}\mu\, e^{-\mu^2},$$

$$s(\mu) = \sqrt{\pi}\mu e^{-\mu^2}, \quad \lambda_0(\mu) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-\tau^2} \tau\, d\tau}{\tau - \mu}.$$

It's clear that the function $\lambda_0(\mu)$ has two complex zero $\pm\mu_0$, $\mu_0 = 0.924$ on the real axis, which differing only signs.

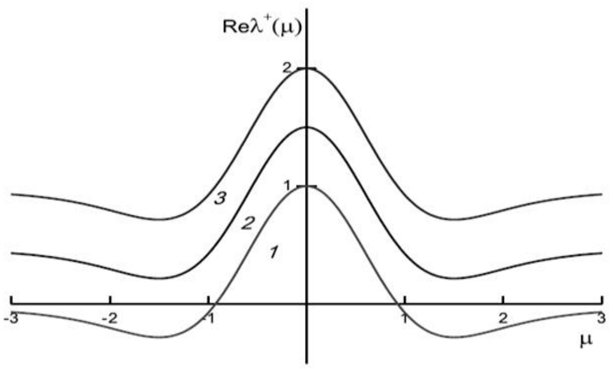

Figure 1. Real part of dispersion function $\lambda^{+}(\mu) = \Omega_2 + \lambda_0(\mu)$ on the real axis. Curves 1, 2, 3 correspond to values of parameter $\Omega_2 = 0, 0.5, 1$.

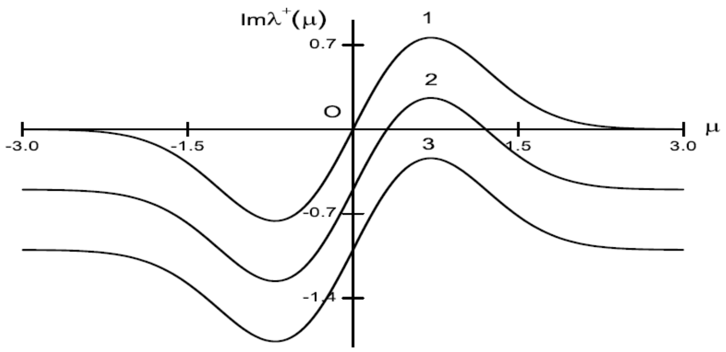

Figure 2. Imaginary part of dispersion function $\lambda^{+}(\mu) = -\Omega_1 + s(\mu)$ on the real axis. Curves 1, 2, 3 correspond to values of parameter $\Omega_1 = 0,\ 0.5,\ 1$.

Let’s expand dispersion function in asymptotic series on negative degrees variable z at the vicinity of infinitely remote z

$$\lambda(z) = -i\Omega - \frac{1}{2z^2} - \frac{3}{4z^4} - \frac{15}{8z^6} - \ldots, \quad z \to \infty. \tag{5}$$

From expansion (5) it is visible, that at small values Ω dispersion function has two complex zero differing only signs

$$\pm \eta_0^{(0)}(\Omega) = \frac{1+i}{2\sqrt{\Omega}}.$$

Considering a family of curves on the complex plane

$$\Gamma = \Gamma(\Omega): \; G(\mu) = \frac{\lambda^+(\mu)}{\lambda^-(\mu)}, \; 0 \le \mu \le +\infty,$$

we note that $G(0) = 1$, $\lim\limits_{\mu \to +\infty} G(\mu) = 1$. It means that curves $\Gamma(\Omega)$ are closed.

Making necessary calculations and considering principle of argument, we obtain that

$$N = \frac{1}{\pi}\left[\arg G(\tau)\right]_0^{+\infty} = 2\chi(G) \text{ или } N = 2\chi(G),$$

where $\chi = \chi(G)$ is the index of function $G(\mu)$, i.e. is the number of revolution, which are made by curve in the positive direction from the beginning of coordinates.

We will construct the frequency plane (Ω_1, Ω_2) (figure 3).

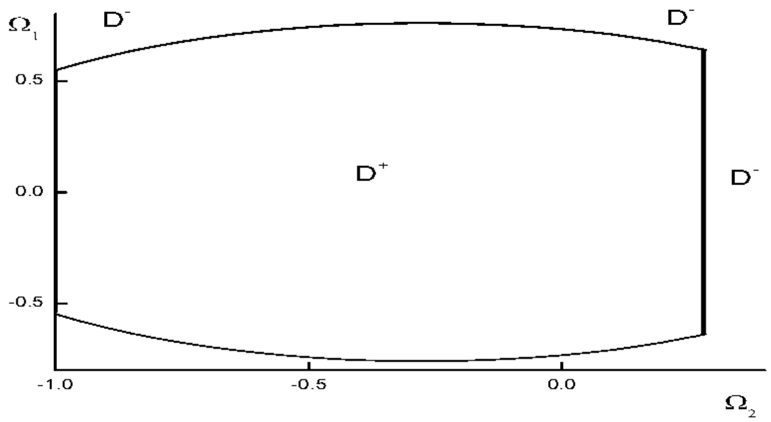

Figure 3. Regions D^+ and D^-. If $(\Omega_1, \Omega_2) \in D^+$ then $\chi = 1$, $(\Omega_1, \Omega_2) \in D^-$ then $\chi = 0$.

On the frequency plane (Ω_1, Ω_2) there is such region D^+, that for the points (Ω_1, Ω_2), lying in this region, the index of the problem is equal to unit: $\chi(\Omega_1, \Omega_2) = 1$, $(\Omega_1, \Omega_2) \in D^+$, and the points (Ω_1, Ω_2), belonging region D^-, the index of the problem is equal to zero, i.e. $\chi(\Omega_1, \Omega_2) = 0$, $(\Omega_1, \Omega_2) \in D^-$. The border of this region ∂D^+ is called the line of critical frequency.

The region D^+ is built following in the way

$$D^+ = \left\{(\Omega_1, \Omega_2): |\Omega_1| < \Omega_1^*(\Omega_2), \; -1 < \Omega_2 < \max_\mu\{-\lambda_0(\mu)\}\right\}.$$

Here $|\Omega_1| = \Omega_1^*(\Omega_2) = \max\limits_\mu \sqrt{\left\{s^2(\mu) - [\Omega_2 + \lambda_0(\mu)]^2\right\}}$ is the line of critical frequency.

If $(\Omega_1,\Omega_2)\in D^+$, then the dispersion function has zero equal two, and ifи $(\Omega_1,\Omega_2)\in D^-$, then the dispersion function has no zero.

Let's separate at function $G(\mu)$ the real and imaginary parts and build the covers $\Gamma(\Omega)$ on the frequency plane, which are defined by the following equations (figure 4 и 5)

$$\Gamma(\Omega):\quad x=\operatorname{Re}G(\mu),\quad y=\operatorname{Im}G(\mu),\quad 0\le\mu\le+\infty .$$

In the case when $\Omega_1=\Omega_2=0$, then curve $\Gamma(0)$ once cover the beginning of coordinates. The function $\lambda_0(\mu)$ has single zero $\mu_0\approx 0.924$ on the real axis.

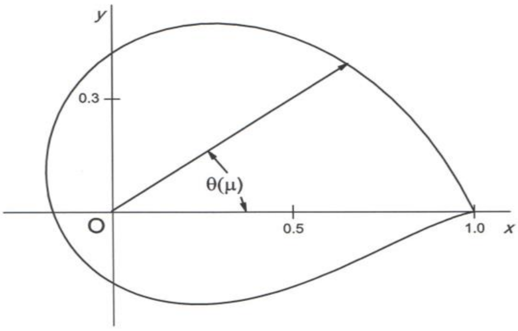

Figure 4. Curve $\Gamma(\Omega)$ cover the beginning of coordinates, when $(\Omega_1,\Omega_2)\in D^+$ The index of function $G(\mu)$ is equal to unit and dispersion function has zero equal two.

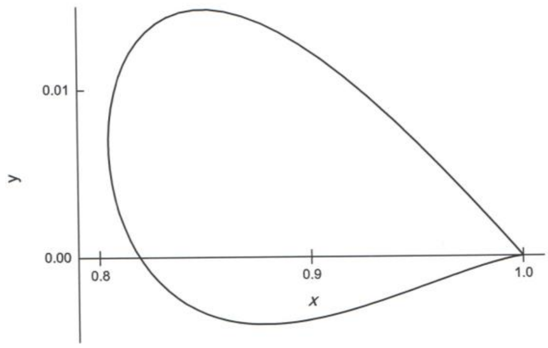

Figure 5. Curve $\Gamma(\Omega)$ doesn't cover the beginning of coordinates, when $(\Omega_1,\Omega_2)\in D^-$. The index of function $G(\mu)$ is equal to zero and dispersion function has no zero.

At $(\Omega_1,\Omega_2)\in D^-$ the discrete spectrum is the empty set. At $(\Omega_1,\Omega_2)\in D^+$ the characteristic equation has two solutions, provided the eigen functions of the characteristic equation

$$\Phi(\pm\eta_0(\Omega),\mu)=\frac{1}{\sqrt{\pi}}\frac{\pm\eta_0(\Omega)}{\pm\eta_0(\Omega)-\mu}$$

and has two eigen solutions of the kinetic equation

$$h_{\pm\eta_0(\Omega)}(x_1,\mu) = \exp\left(-\frac{x_1 z_0}{\pm\eta_0(\Omega)}\right)\frac{1}{\sqrt{\pi}}\frac{\pm\eta_0(\Omega)}{\pm\eta_0(\Omega)-\mu}.$$

At the heart of the analytical solution of boundary problems the kinetic theory lays the solution of the homogeneous boundary value Riemann problem [6] with coefficient $G(\mu)=\lambda^{+}(\mu)/\lambda^{-}(\mu)$

$$\frac{X^{+}(\mu)}{X^{-}(\mu)} = G(\mu), \quad \mu>0 \qquad \text{or} \qquad \frac{X^{+}(\mu)}{X^{-}(\mu)} = \frac{\lambda^{+}(\mu)}{\lambda^{-}(\mu)}, \quad \mu>0. \tag{6}$$

Homogeneous boundary value Riemann problem (6) is called also the factorization problem of coefficient $G(\mu)$. Its solution [5], has the form

$$X(z) = \frac{1}{z}\exp V(z)$$

in the case, when the index of problem is equal to unit, where

$$V(z) = \frac{1}{2\pi i}\int\limits_0^\infty \frac{\ln|G(\mu)| + i\varsigma(u)}{u-z}du,$$

In the case, when the index of problem is equal to unit. The solution of Riemann problem (6) is given by a formula

$$X(z) = \exp V(z), \quad V(z) = \frac{1}{2\pi i}\int\limits_0^\infty \frac{\ln G(\tau)d\tau}{\tau - z}.$$

Here $\ln G(\tau)$ is the principal branch of logarithm, fixed at zero by condition $\ln G(0)=0$, angle is the principal value of argument. We will mark that $\ln G(\infty)=0$.

We can seek the general solution to problem with the boundary condition in the form

$$h(x_1,\mu) = \frac{a_0}{\eta_0-\mu}\exp\left(-\frac{x_1 z_0}{\eta_0}\right) + \int\limits_0^\infty \exp\left(-\frac{x_1 z_0}{\eta}\right)\Phi(\eta,\mu)a(\eta)d\eta, \tag{7}$$

$a(\eta)$ is the coefficient of the continuous spectrum, a_0 is the coefficient of the discrete spectrum.

Adding eigen functions of the characteristic equation in the formula (7), we can write it in a classical form

$$h(x_1,\mu) = \frac{\eta_0 a_0}{\sqrt{\pi}(\eta_0-\mu)}\exp\left(-\frac{x_1 z_0}{\eta_0}\right) + \frac{1}{\sqrt{\pi}}\int\limits_0^\infty \exp\left(-\frac{x_1 z_0}{\eta}\right)\frac{\eta a(\eta)d\eta}{\eta-\mu} +$$
$$+\exp\left(-\frac{x_1 z_0}{\mu}+\mu^2\right)\lambda(\mu)a(\mu)\theta_{+}(\mu),$$

where $\theta_{+}(\mu)$ is the Heaviside function: $\theta_{+}(\mu)=1,\ \mu>0,\ \theta_{+}(\mu)=0,\ \mu<0$.

This expression is proved with the help analytical methods [6], as shown in article [1, 7].

Conclusions. In this work the second Stokes problem in the case of the variable amplitude of fluctuation of a surface is formulated. Zero of the dispersion function are calculated and investigated. The region of critical frequency is built. Eigen functions of the characteristic equation and eigen solution of kinetic equation ate found. The general solution to problem with the boundary condition can be present the form of the sum of the discrete solution and the integral over the continuous spectrum of eigen solutions corresponding to the continuous spectrum is shown.

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