

Exact solutions to nonlinear delay differential equations of hyperbolic type

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Abstract. We consider nonlinear delay differential equations of hyperbolic type, including equations with varying transfer coefficients and varying delays. The equations contain one or two arbitrary functions of a single argument. We present new classes of exact generalized and functional separable solutions. All the solutions involve free parameters and can be suitable for solving certain model problems as well as testing numerical and approximate analytical methods for similar and more complex nonlinear differential-difference equations.

Keywords: delay reaction-diffusion equations, hyperbolic delay equations, differential-difference equations, exact solutions

1. Introduction

The classical parabolic heat-conduction and diffusion equation has a physically paradoxical property, namely, an infinite disturbance propagation rate, which is not observed in nature. This problem does not appear in the heat-conduction and diffusion equation of hyperbolic type [1, 2]

$$\tau T_{tt} + T_t = a\Delta T,$$

which have a finite disturbance propagation rate at $\tau > 0$. Here T is temperature, t is time, a is thermal diffusivity, Δ is the Laplace operator, and τ is a relaxation (delay) time. The thermal and diffusion relaxation times can vary in extremely wide limits from milliseconds (or less) to several tens of seconds [3–5] and should be taken into account while solving numerous heat- and mass-transfer problems.

The second important feature of evolutionary processes is that the rate of change of the unknowns in chemical, biological, biomedical, ecological and other systems generally depends not only on the current state of the system but also on its entire previous evolution [3, 6]. These systems are called *hereditary systems*. In particular, when the state is determined by a specific moment in the past, rather than the entire evolution, the system is referred to as a *delayed feedback system*.

Delayed feedback systems are frequently modeled by delay reaction-diffusion equations with the kinetic function F (the rate of a chemical or biochemical reaction) that depends on both $u = u(x, t)$ and its delayed counterpart $w = u(x, t - \tau)$. The special case when $F(u, w) = f(w)$ implies a simple physical interpretation: a heat- and mass-transfer process in local non-equilibrium media possesses



inertial properties, i.e., the system responds to an impact not immediately at the time t , as opposed to the classical local-equilibrium case, but the delay time τ later.

The paper deals with the nonlinear delay reaction-diffusion equations of hyperbolic type

$$\varepsilon u_{tt} + \sigma u_t = au_{xx} + F(u, w), \quad w = u(x, t - \tau), \quad (1)$$

where $a > 0$, $\varepsilon \geq 0$, $\sigma \geq 0$ ($\varepsilon + \sigma \neq 0$). We will present several exact solutions to the equations of the form (1), which have been obtained by applying the methods of generalized and functional separation of variables [7, 8] as well as the functional constraints method [9, 10]. All the solutions involve free parameters and can be suitable for solving certain model problems as well as testing numerical and approximate analytical methods for similar and more complex nonlinear differential-difference equations.

2. The concept of 'exact solution'

The term 'exact solution' (see [9]) with regard to nonlinear delay partial differential equations is used in cases when the solution is expressible in terms of:

- (i) elementary functions or definite/indefinite integrals;
- (ii) solutions of ordinary differential equations with or without delay (or systems of these equations);
- (iii) solutions of linear partial differential equations.

3. The functional constraints method. Some examples

Now we are going to briefly describe the functional constraints method from the article [9]. For definiteness, we consider the nonlinear delay reaction-diffusion equations of hyperbolic type (1) with the kinetic function that depends on one argument:

$$\varepsilon u_{tt} + \sigma u_t = au_{xx} + F(z), \quad z = z(u, w), \quad w = u(x, t - \tau), \quad (2)$$

where $z = z(u, w)$ is a function to be determined.

We search for generalized separable solutions of the form

$$u = \sum_{n=1}^N \varphi_n(x) \psi_n(t). \quad (3)$$

The simplest cases are a multiplicative, $u = \varphi(x)\psi(t)$, and an additive, $u = \varphi(x) + \psi(t)$, separable solution.

The form of the argument $z = z(u, w)$ of the arbitrary function $F(z)$ is determined by the method of functional constraints. It is based on searching for exact solutions of the form (3) that satisfy one of two functional constraints:

$$z(u, w) = p(x), \quad w = u(x, t - \tau); \quad (4)$$

$$z(u, w) = q(t), \quad w = u(x, t - \tau). \quad (5)$$

The constraints represent difference equations in t with x playing the role of a free parameter.

Below we give some examples of constructing exact solutions to the equations of the form (2) by use of the functional constraints method.

We consider the equation

$$\varepsilon u_{tt} + \sigma u_t = au_{xx} + bu + f(u - w), \quad w = u(x, t - \tau), \quad (6)$$

where b is an arbitrary constant, f is an arbitrary function.

The functional constraint of the first kind (4) has the form

$$u - w = p(x), \quad w = u(x, t - \tau). \quad (7)$$

The difference equation (7) can be satisfied with the generalized separable solution

$$u = t\varphi(x) + \psi(x), \quad (8)$$

which leads to $p(x) = \tau\varphi(x)$. Substituting (8) into (6) and separating the variables, we get ordinary differential equations for $\varphi(x)$ and $\psi(x)$:

$$\begin{aligned} a\varphi''(x) + b\varphi(x) &= 0, \\ a\psi''_{xx}(x) + b\psi(x) + f(\tau\varphi(x)) - \sigma\varphi(x) &= 0. \end{aligned} \quad (9)$$

The general solution of equation (9) has the form

$$\varphi(x) = \begin{cases} C_1 \cos(\lambda x) + C_2 \sin(\lambda x), & \lambda = \sqrt{b/a} \quad b > 0; \\ C_1 \exp(-\lambda x) + C_2 \exp(\lambda x), & \lambda = \sqrt{-b/a} \quad b < 0, \end{cases}$$

where C_1 and C_2 are arbitrary constants.

The functional constraint of the second kind (5) has the form

$$u - w = q(t), \quad w = u(x, t - \tau). \quad (10)$$

The difference equation (10) can be satisfied with the additive separable solution

$$u = \varphi(x) + \psi(t), \quad (11)$$

which leads to $q(t) = \psi(t) - \psi(t - \tau)$. Substituting (11) into (6), we get an equation for $\varphi(x)$ that coincides with equation (9) and an equation for $\psi(t)$:

$$\varepsilon\psi''(t) + \sigma\psi'(t) = b\psi(t) + f(\psi(t) - \psi(t - \tau)).$$

4. Exact solutions

4.1. Equations of the form (1)

In Table 1 we present some exact solutions obtained by use of the method described above. Details about these solutions are given below. Other solutions to the equations of the form (1) and more complex equations can be found in [11, 12]. Functions f, g, h are arbitrary.

1–2. Here C_1, C_2, λ are arbitrary constants and the function $\psi(t)$ is described by the ordinary delay differential equation

$$\varepsilon\psi''(t) + \sigma\psi'(t) = -a\lambda^2\psi(t) + \psi(t)f(\psi(t - \tau)/\psi(t)).$$

3. Here $\alpha, \beta, \lambda, \gamma$ are arbitrary constants and the function $\theta(z)$ satisfies the ordinary delay differential equation

$$\begin{aligned} (a\lambda^2 - \varepsilon\gamma^2)\theta''(z) + (2a\alpha\lambda - 2\varepsilon\beta\gamma - \sigma\gamma)\theta'(z) + \\ + (a\alpha^2 - \varepsilon\beta^2 - \sigma\beta)\theta(z) + \theta(z)f(e^{-\beta\tau}\theta(z - \delta)/\theta(z)) = 0, \quad \delta = \gamma\tau. \end{aligned}$$

4–5. Here b, β, γ are arbitrary constants, $\lambda = \sqrt{|b|/a}$ and the function $\theta(z)$ is determined by the ordinary delay differential equation

$$(\varepsilon\gamma^2 - a\beta^2)\theta''(z) + \sigma\gamma\theta'(z) = b\theta(z) + f(\theta(z) - \theta(z - \delta)), \quad \delta = \gamma\tau.$$

Table 1. Exact solutions of the equations $\varepsilon u_{tt} + \sigma u_t = au_{xx} + F(u, w)$

N	The kinetic function $F(u, w)$	The form of the solution $u(x, t)$
1	$uf(w/u)$	$(C_1 \cos(\lambda x) + C_2 \sin(\lambda x))\psi(t)$
2	$uf(w/u)$	$(C_1 e^{-\lambda x} + C_2 e^{\lambda x})\psi(t)$
3	$uf(w/u)$	$e^{\alpha x + \beta t} \theta(z), \quad z = \lambda x + \gamma t$
4	$bu + f(u - w), \quad b > 0$	$C_1 \cos(\lambda x) + C_2 \sin(\lambda x) + \theta(z), \quad z = \beta x + \gamma t$
5	$bu + f(u - w), \quad b < 0$	$C_1 e^{-\lambda x} + C_2 e^{\lambda x} + \theta(z), \quad z = \beta x + \gamma t$
6	$f(u - w)$	$C_1 x^2 + C_2 x + \theta(z), \quad z = \beta x + \gamma t$
7	$uf(u - kw) +$ $+wg(u - kw) + h(u - kw)$	$e^{ct} \sum_{n=1}^N (\varphi_n(x) \cos(\beta_n t) + \psi_n(x) \sin(\beta_n t)) + e^{ct} \theta(x) + \xi(x)$
8	$uf(u + kw) +$ $+wg(u + kw) + h(u + kw)$	$e^{ct} \sum_{n=1}^N (\varphi_n(x) \cos(\beta_n t) + \psi_n(x) \sin(\beta_n t)) + \xi(x)$

6. Here C_1, C_2, β, γ are arbitrary constants and the function $\theta(z)$ satisfies the ordinary delay differential equation

$$(\varepsilon \gamma^2 - a \beta^2) \theta''(z) + \sigma \gamma \theta'(z) = 2C_1 a + f(\theta(z) - \theta(z - \delta)), \quad \delta = \gamma \tau.$$

7. Here $k > 0$ is an arbitrary constant, N is a positive integer, $c = (\ln k)/\tau$, $\beta_n = 2\pi n/\tau$, the functions $\varphi_n(x)$ and $\psi_n(x)$ are described by the system

$$\begin{aligned} a\varphi_n'' + \left[f(\eta) + \frac{1}{k}g(\eta) + \varepsilon(\beta_n^2 - c^2) - \sigma c \right] \varphi_n - (2\varepsilon c + \sigma)\beta_n \psi_n &= 0, \\ a\psi_n'' + \left[f(\eta) + \frac{1}{k}g(\eta) + \varepsilon(\beta_n^2 - c^2) - \sigma c \right] \psi_n + (2\varepsilon c + \sigma)\beta_n \varphi_n &= 0, \quad \eta = (1 - k)\xi, \end{aligned}$$

the function $\theta(x)$ is determined by the ordinary differential equation

$$a\theta'' + \left[f(\eta) + \frac{1}{k}g(\eta) - \varepsilon c^2 - \sigma c \right] \theta = 0,$$

and the function $\xi(x)$ satisfies the independent equation

$$a\xi'' + [f(\eta) + g(\eta)]\xi + h(\eta) = 0, \quad \eta = (1 - k)\xi.$$

8. Here $k > 0$ is an arbitrary constant, N is a positive integer, $c = (\ln k)/\tau$, $\beta_n = (2n - 1)\pi/\tau$, the functions $\varphi_n(x)$ and $\psi_n(x)$ are described by the system

$$\begin{aligned} a\varphi_n'' + \left[f(\eta) - \frac{1}{k}g(\eta) + \varepsilon(\beta_n^2 - c^2) - \sigma c \right] \varphi_n - (2\varepsilon c + \sigma)\beta_n \psi_n &= 0, \\ a\psi_n'' + \left[f(\eta) - \frac{1}{k}g(\eta) + \varepsilon(\beta_n^2 - c^2) - \sigma c \right] \psi_n + (2\varepsilon c + \sigma)\beta_n \varphi_n &= 0, \quad \eta = (1 + k)\xi, \end{aligned}$$

and the function $\xi(x)$ is determined by the independent equation

$$a\xi'' + [f(\eta) + g(\eta)]\xi + h(\eta) = 0, \quad \eta = (1 + k)\xi.$$

Remark 1. Equations of the form (1) with varying delay times $\tau = \tau(t)$ admit the solutions from lines 1, 2. A few other solutions for these equations are presented in [11].

Table 2. Exact solutions of the equations $\varepsilon u_{tt} + \sigma u_t = a(G(u)u_x)_x + F(u, w)$

N	The transfer coefficient $G(u)$	The kinetic function $F(u, w)$	The form of the solution $u(x, t)$
1	u^n	$bu^{n+1} + uf(w/u),$ $b(n+1) > 0$	$(C_1 \cos(\lambda x) + C_2 \sin(\lambda x))^{1/(n+1)} \psi(t)$
2	u^n	$bu^{n+1} + uf(w/u),$ $b(n+1) < 0$	$(C_1 e^{-\lambda x} + C_2 e^{\lambda x})^{1/(n+1)} \psi(t)$
3	u^n	$uf(w/u) + u^{n+1}h(w/u)$	$e^{\lambda t} \varphi(x)$
4	u^{-1}	$b + uf(w/u)$	$C_1 \exp\left(-\frac{b}{2a}x^2 + C_2 x\right) \psi(t)$
5	$u^{-1/2}$	$f(u^{1/2} - w^{1/2}) +$ $+u^{1/2}h(u^{1/2} - w^{1/2})$	$(t\varphi(x) + \psi(x))^2$
6	$e^{\beta u}$	$f(u - w)$	$\frac{1}{\beta} \ln(C_1 x^2 + C_2 x + C_3) + \psi(t)$
7	$e^{\beta u}$	$be^{\beta u} + f(u - w), b\beta > 0$	$\frac{1}{\beta} \ln(C_1 + C_2 \cos(\lambda x) + C_3 \sin(\lambda x)) + \psi(t)$
8	$e^{\beta u}$	$be^{\beta u} + f(u - w), b\beta < 0$	$\frac{1}{\beta} \ln(C_1 + C_2 \exp(-\lambda x) + C_3 \exp(\lambda x)) + \psi(t)$

4.2. Equations with varying transfer coefficients

In Table 2 we present some exact solutions to nonlinear delay hyperbolic reaction-diffusion equations with varying transfer coefficients of the form

$$\varepsilon u_{tt} + \sigma u_t = a(G(u)u_x)_x + F(u, w). \quad (12)$$

The solutions have been obtained by use of the method briefly described above (for more explanation see [10]). Details about these solutions are given below. Other solutions to the equations of the form (12) can be found in [11, 12]. Functions f and h are arbitrary.

1–2. Here C_1, C_2, b, n are arbitrary constants, $\lambda = \sqrt{|b(n+1)|/a}$, and the function $\psi(t)$ is determined by the ordinary delay differential equation

$$\varepsilon \psi''(t) + \sigma \psi'(t) = \psi(t) f(\psi(t - \tau)/\psi(t)). \quad (13)$$

Equation (13) yields a particular solution $\psi(t) = Ae^{\beta t}$, where A is an arbitrary constant and β is a solution of the algebraic (transcendental) equation

$$\varepsilon \beta^2 + \sigma \beta - f(e^{-\beta \tau}) = 0.$$

3. Here n is an arbitrary constant, λ is a solution of the algebraic (transcendental) equation

$$\varepsilon \lambda^2 + \sigma \lambda = f(e^{-\lambda \tau}),$$

and the function $\varphi(x)$ is described by the ordinary differential equation

$$a(\varphi^n \varphi'_x)'_x + \varphi^{n+1} h(e^{-\lambda \tau}) = 0.$$

At $n \neq 1$, a substitution $\theta = \varphi^{n+1}$ leads to a linear ordinary differential equation of the second order; at $n = -1$, one should make a substitution $\theta = \ln \varphi$.

4. Here C_1, C_2, b are arbitrary constants and the function $\psi(t)$ is determined by equation (13).

5. Here the functions $\varphi(x)$ and $\psi(t)$ satisfy the ordinary differential equations

$$\begin{aligned} 2a\varphi''_{xx} + \varphi h(\tau\varphi) - 2\sigma\varphi^2 &= 0, \\ 2a\psi''_{xx} + \psi h(\tau\varphi) - 2\sigma\varphi\psi - 2\varepsilon\varphi^2 + f(\tau\varphi) &= 0. \end{aligned}$$

A particular solution of this system is

$$\varphi = A, \quad \psi = \frac{2\varepsilon A^2 - f(A\tau)}{4a}x^2 + C_1x + C_2,$$

where C_1, C_2 are arbitrary constants and the constant A is describes by the algebraic (transcendental) equation

$$h(A\tau) - 2\sigma A = 0.$$

6. Here C_1, C_2, C_3, β are arbitrary constants and the function $\psi(t)$ satisfies by the ordinary delay differential equation

$$\varepsilon\psi''(t) + \sigma\psi'(t) = \frac{2aC_1}{\beta}e^{\beta\psi(t)} + f(\psi(t) - \psi(t - \tau)).$$

7–8. Here b, β, C_1, C_2, C_3 are arbitrary constants, $\lambda = \sqrt{|b\beta|/a}$, and the function $\psi(t)$ is determined by the ordinary delay differential equation

$$\varepsilon\psi''(t) + \sigma\psi'(t) = bC_1\beta e^{\beta\psi(t)} + f(\psi(t) - \psi(t - \tau)).$$

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