# Feedback control of monotonic shocks

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**Abstract.**    The feedback control algorithm is developed to suppress oscillations caused by high resolution schemes. The stable propagation of the shock of the nonlinear advection equation is studied using the Lax-Wendroff and Warming- Beam second- order schemes. It is shown that the addition of one and the same control artificial term in both schemes results in efficient suppression of oscillations. The switch on/off the control is studied to demonstrate the role of the control in the stable monotonic shock propagation.

## Introduction

An important problem in utilization of higher-order shock capturing schemes is their ability to maintain the total Variation Diminishing (TVD) condition, see. e.g., [1]. In other words, the shock profile should not contain oscillations caused by the features of the scheme. Second-order schemes, e.g., the Lax-Wendroff and Warming-Beam schemes, are not TVD schemes, and an improvement of the numerical algorithm is needed.

As noted in [1] natural way to suppress the oscillations is to add artificial viscosity term in the scheme keeping approximate consistence with the original equation. It turns out that artificial viscosity may help to suppress oscillations caused by the dispersion of the scheme, however, it also produces unnecessary smearing of the front of the shock. Reducing of smearing requires further development of artificial viscosity by so-called adaptive viscosity [2, 3] which is introduced only in the areas where it is necessary to suppress dispersive oscillations. Despite some progress in this direction problem of adding optimal amount of viscosity is still difficult [1] while improvements are rather complicated [2].

Another modification is utilization of the so-called limiters whose role is to act as a nonlinear switching between numerical methods applied for equation under study. A variety of limiters may be found in the literature, see, e.g., [1, 4, 5, 6, 7] and references therein. There is no universal limiter, and their use is computationally expensive [2]. A TVD flux-limiter may be developed for nonlinear hyperbolic equations [7], it improves the similarity with moving monotonic discontinuity in comparison with, e.g., the Minmod limiter, however, there still remains small differences between required and numerically obtained shocks.

Method of equivalent equation (or differential approximation) [8, 9] may be used to analyze both artificial viscosity [9] and limiters [6]. Also asymptotic and exact solutions of the equivalent

equation may help for finding suitable artificial additional terms [10, 11, 12]. However, equivalent equations are usually very complicated for simulations of nonlinear equations and coupled equations that makes their analysis impossible.

Recently [13, 14], it was shown that the feedback control may provide stable propagation of the waves with monotonic shape for the sine- Gordon equation. It was found that oscillations caused by imperfect initial conditions may be suppressed by inclusion of additional control terms in the equation. An additional control term may be added to the discrete scheme of equation following the methods of control [15, 16], however, the addition of the term should be justified. It may be done using the speed-gradient control approach [15].

In this paper, a distributed feedback control algorithm will be developed to achieve suitable monotonic moving shock wave solution to the nonlinear advection equation solved by the second-order numerical methods. The paper is organized as follows. In Sec.1 the known Lax-Wendroff and Warming -Beam schemes for the advection equation are presented. Their utilization leads to oscillatory shock wave solution. Next Sec.2 is devoted to the development of the control algorithm, and artificial additions to the schemes are found. In the next Section, numerical simulations are performed for the control that may be switched on/off at any time. Conclusions summarize the paper.
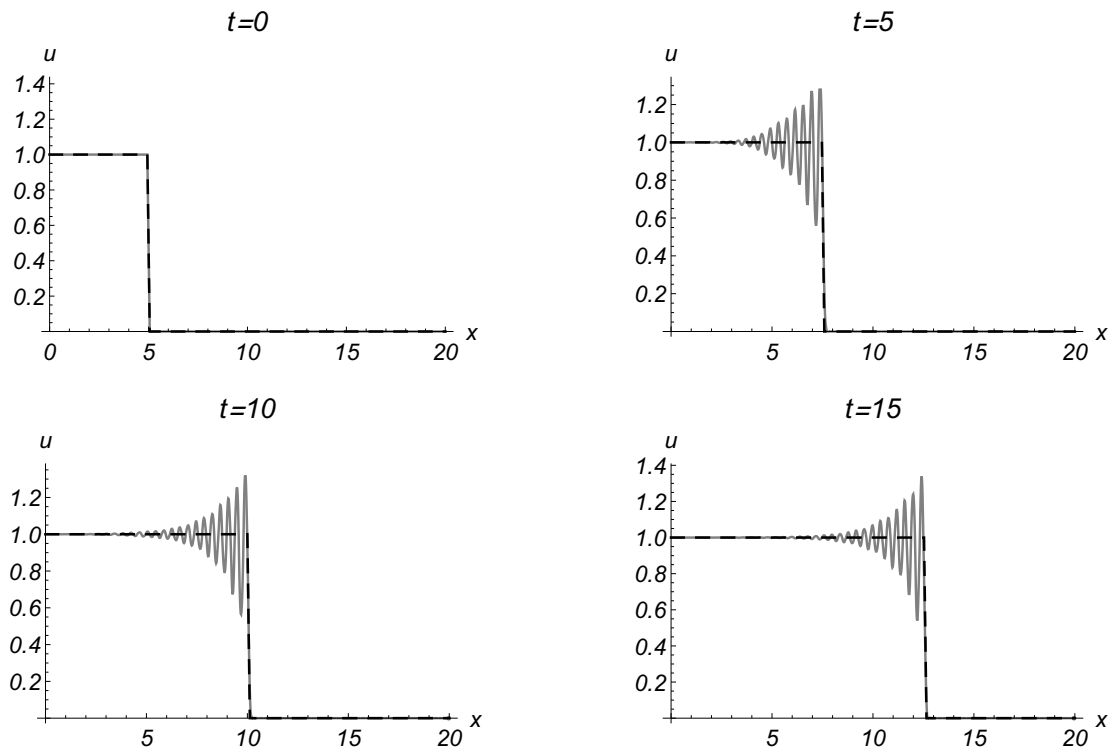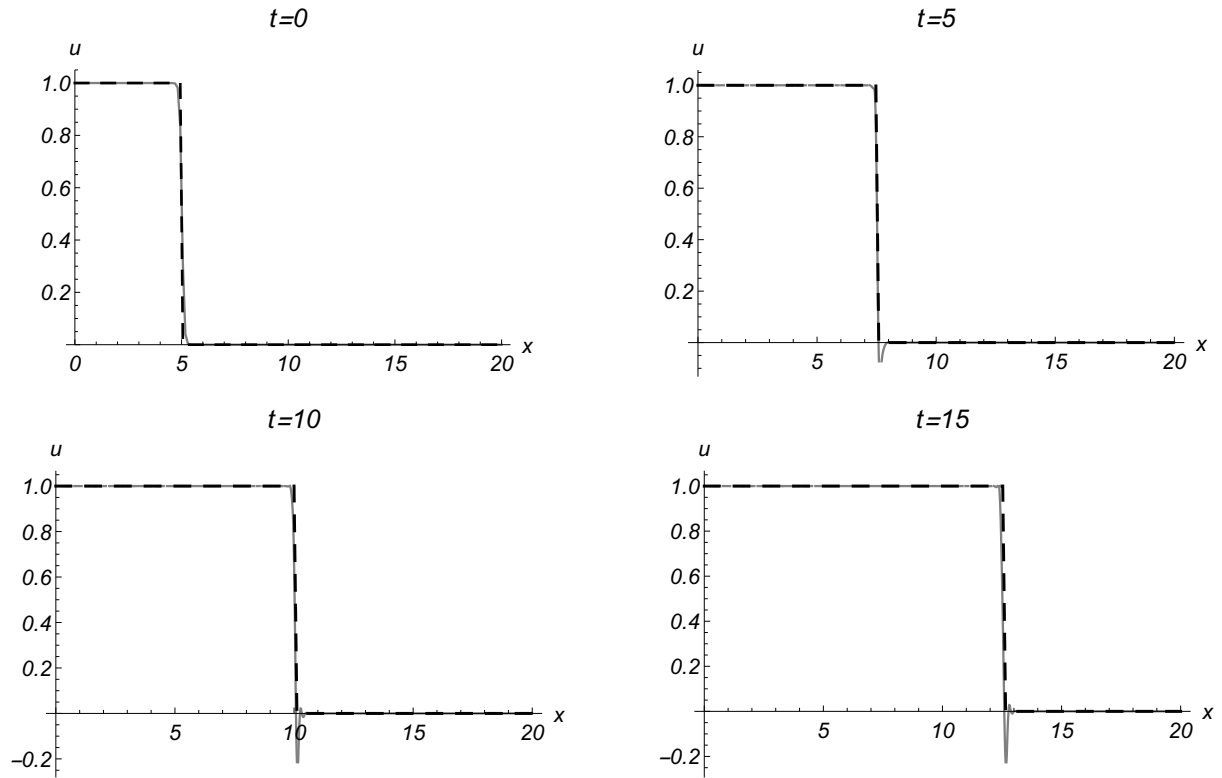


**Figure 1.** Arising of oscillations at the wave front for the LW scheme. Shown by dashed line is propagation of shock wave (2) with velocity $c = 1/2$.

## 1. Statement of the problem
Consider the nonlinear advection equation,

$$u_t + u\, u_x\ =\ 0. \tag{1}$$

**Figure 2.** Arising of oscillations at the wave front for the WB scheme. Shown by dashed line is propagation of shock wave (2) with velocity $c = 1/2$.

whose solution behaves as

$$u \to 1 \text{ at } x \to -\infty,$$

$$u \to 0 \text{ at } x \to \infty.$$

The initial condition has the form of discontinuity,

$$u_0 = 1 \text{ at } x - x_0 \leq 0, \ u_0 = 0 \text{ at } x - x_0 > 0, \tag{2}$$

$x_0$ is a constant accounting for the position of discontinuity. Propagation of this discontinuity is carried out with velocity equal to the average of the values before and after discontinuity [17] and equal to $c = 1/2$ in our case.

Correct numerical description of the shock wave propagation requires use of the higher-order schemes, e.g., the Lax- Wendroff (LW) or Warming- Beam (WB) schemes. However, these schemes possess their internal dispersion that results in an appearance of oscillations on the wave front. In particular, the LW scheme for Eq. (1) is

$$\frac{u_i^{n+1} - u_i^n}{\triangle t} + \frac{1}{2\triangle x}\left(f_{i+1}^n - f_{i-1}^n\right) - \frac{\triangle t}{2\triangle x^2}\left(u_{i+1/2}\left(f_{i+1}^n - f_i^n\right) - u_{i-1/2}\left(f_i^n - f_{i-1}^n\right)\right) = 0, \tag{3}$$

where $f_i^n = (u_i^n)^2/2$, $u_{i\pm1/2}^2 = (u_{i\pm1}^n + u_i^n)/2$, $\triangle t$ and $\triangle tx$ are the temporal and spatial steps respectively. Simulations of Eq. (3) were realized using the Wolfram Mathematica programm. One can see in Fig. 1 that oscillations appear at the upper side of the wave front as time goes
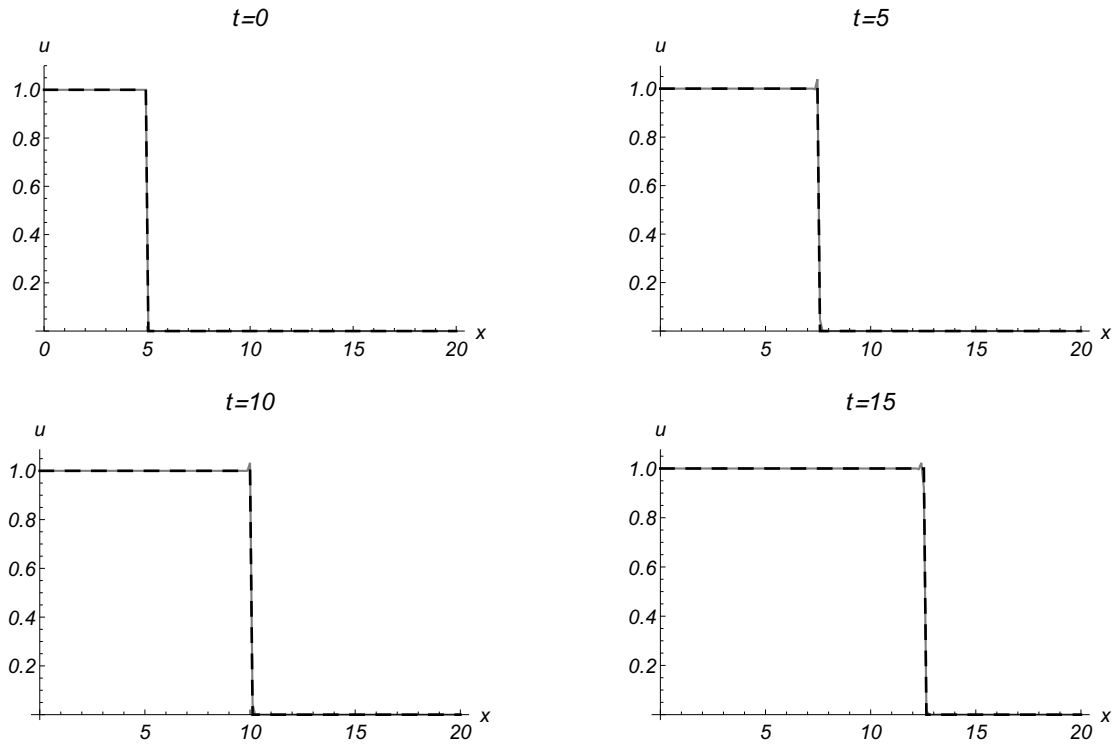
**Figure 3.** Control action (8) for both the LW and WB schemes. Shown by dashed line is propagation of shock wave (2) with velocity $c = 1/2$.

on while the slope and the velocity of the wave are similar to those of the desired wave shown by dashed line.

For the WB scheme one obtains

$$\frac{u_i^{n+1} - u_i^n}{\triangle t} + \frac{1}{2\triangle x}\left( 3(f_i^n - f_{i-1}^n) - (f_{i-1}^n - f_{i-2}^n)\right) +$$

$$\frac{\triangle t}{2\triangle x^2}\left( u_{i-3/2}(f_{i-1}^n - f_{i-2}^n) - u_{i-1/2}(f_i^n - f_{i-1}^n)\right) = 0. \tag{4}$$

Simulations of Eq. (4) reveal oscillations at the lower side of the front of the shock as shown in Fig. 2. Again the slope and the velocity of the wave are similar to those of the desired wave shown by dashed line.

There is a need in a modification of the schemes that keeps the steepness of the profile of the shock and its velocity but suppresses parasitic oscillations caused by the scheme (shock capturing).

## 2. Feedback speed- gradient control method

To suppress scheme oscillations caused by the LW and WB schemes, a control algorithm is developed. Let us add an artificial control function, $w(x,t)$ in Eq. (1),

$$u_t + u \, u_x + w(x,t) = 0, \tag{5}$$

The *distributed* error of the shape of the wave is

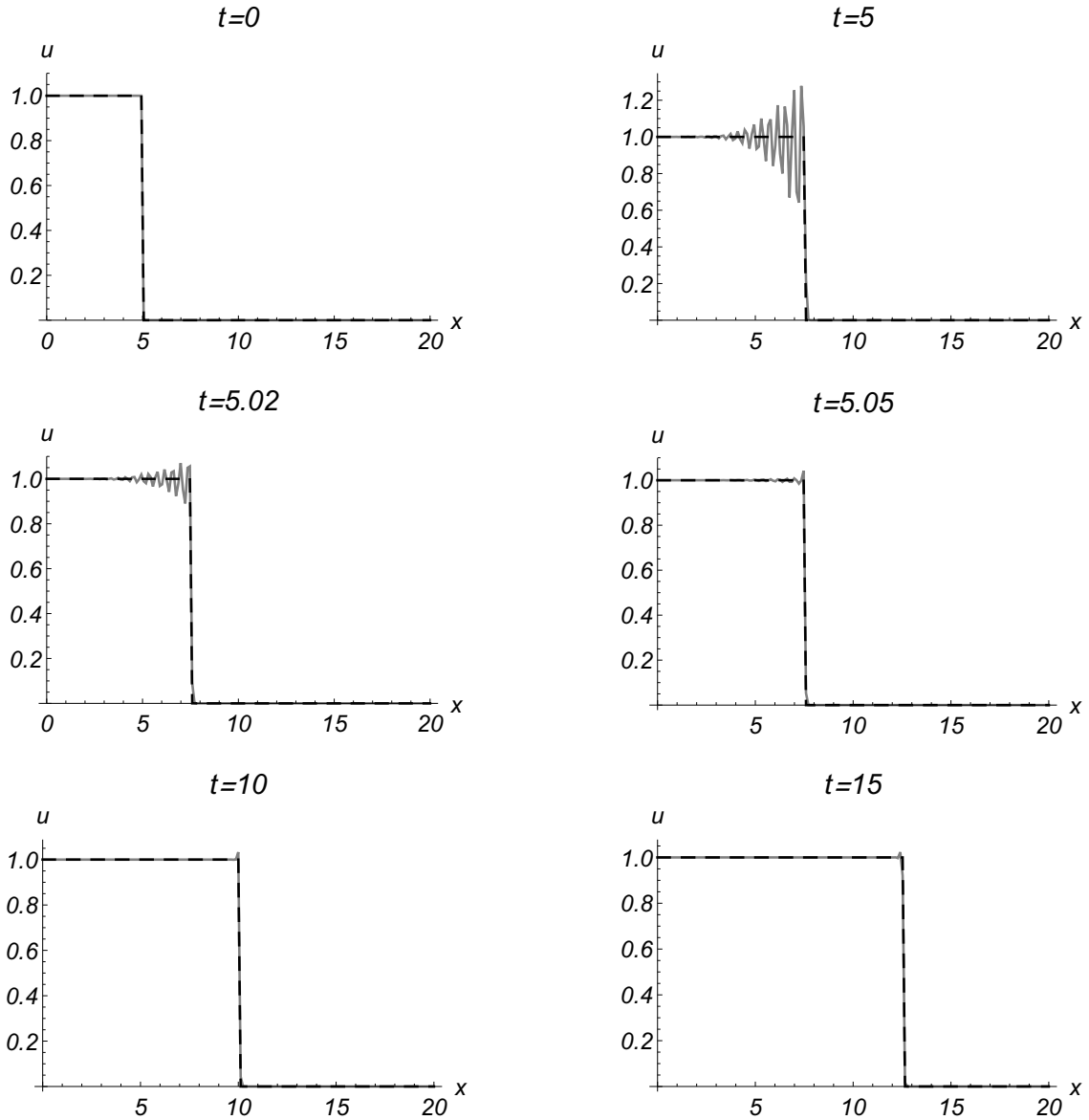$$e(x,t) = u(x,t) - u_0(x,t). \tag{6}$$

**Figure 4.** Control action (9) for the LW scheme switched on at $t_b = 5$. Shown by dashed line is propagation of shock wave (2) with velocity $c = 1/2$.

where $u_0$ is the desired wave profile, e.g., moving discontinuity like Eq. (2). Then the objective functional $Q$ is

$$Q(u) = \frac{1}{2}e(x,t)^2. \qquad (7)$$

Let us introduce an auxiliary control goal: to diminish the functional (7). However, it does not depend *explicitly* on the control function $w$. To involve the dependence, consider the first derivative of $Q$ with the use of Eq.(5):

$$Q_t(u) = e(x,t)e_t(x,t) \;=\; -e(x,t)(uu_x + w)$$

Then $\partial Q_t(u)/\partial w$ is evaluated to characterize decrease in $Q_t(u)$, and the distributed control function $w$ is assumed to be
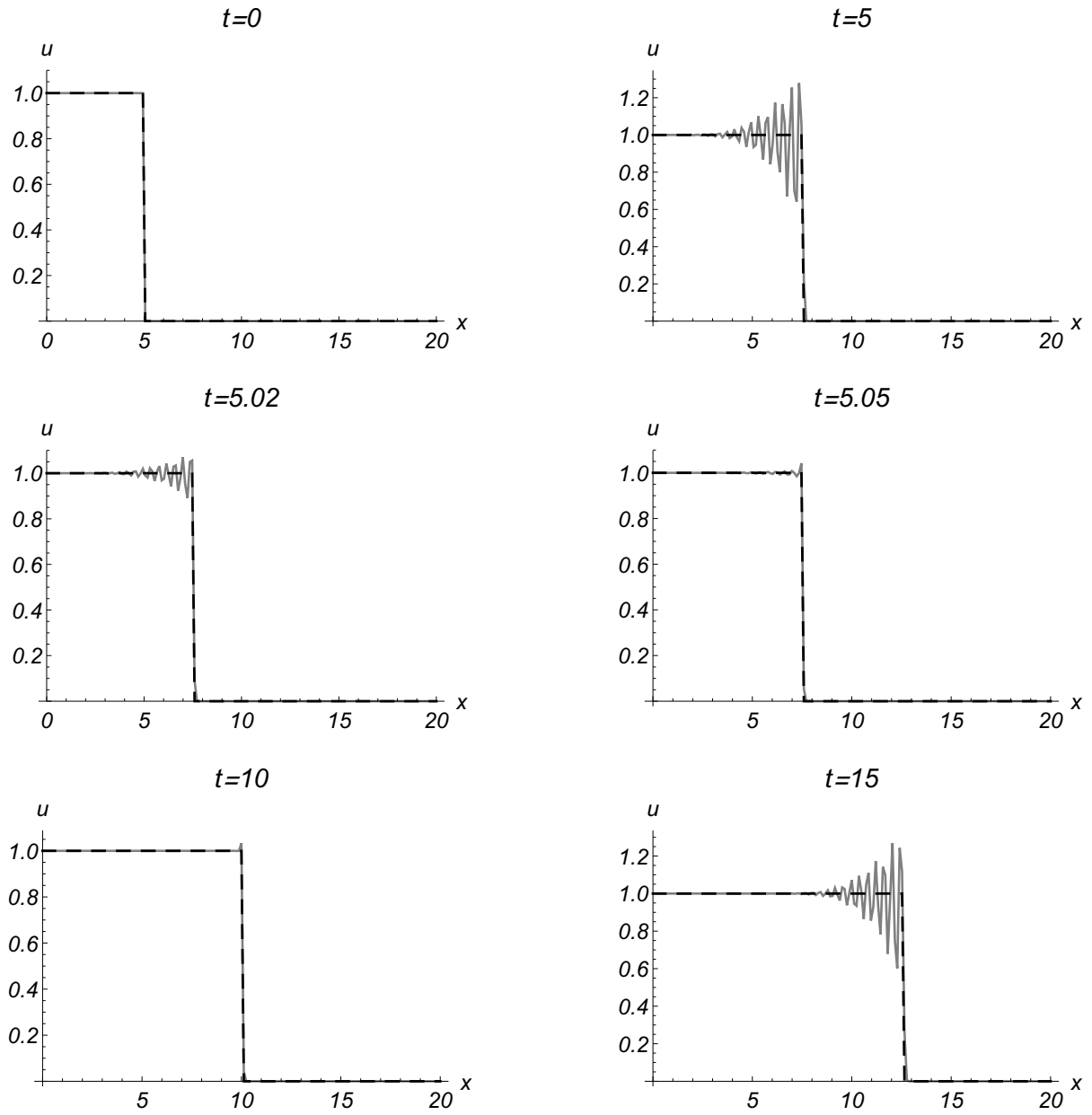
**Figure 5.** Control action (10) for the LW scheme switched on at $t_b = 5$ and switched off at $t_f = 10$ . Shown by dashed line is propagation of shock wave (2) with velocity $c = 1/2$.

$$w(x,t) = -\gamma(u(x,t) - u_0(x,t)), \tag{8}$$

$\gamma > 0$ is the parameter of the algorithm. The algorithm does not contain derivatives of the function $u$ and may be easily incorporated as artificial addition in both Eqs. (3), (4) in one and the same form.

The control may be switched on at some time $t = t_b$. In this case the distributed control function $w$ is

$$w(x,t) = -\gamma(u(x,t) - u_0(x,t))H(t - t_b), \tag{9}$$

where $H$ is the unit-step function. Similarly, the control may be switched off at some time $t_f > t_b$, in this case we get

$$w(x,t) = -\gamma(u(x,t) - u_0(x,t))H(t - t_b)H(t_f - t). \qquad (10)$$

## 3. Shock capturing by control

Let us choose the target function $u_0$ in the form of shock wave (2) propagating with velocity $c = 1/2$. Then LW scheme (3) modified by addition of control function (8) at $\gamma = 0.5$ results in description of stable propagation of monotonic shock shown in Fig. 3. Similar results are obtained using simulations of the WB scheme (4) modified by addition of control function (8). Due to definition of the control function, it is small when $u$ is almost equal to $u_0$ and tends to zero as soon as these functions coincide. Therefore, the addition of the control function gives rise to a small deviation from the original discrete equations. However, even very small control function term does not mean switching off the control.

To see it, let us consider first the control which is realized according to Eq. (9). One can see in Fig. 4 for the LW scheme that oscillations are developing by $t = t_b$ when the control is switched on. Then oscillations are suppressed by the control very fast, see the second row in Fig. 4, and again stable propagation of the shock carries out, see last two sketches in Fig. 4. Similar behavior is observed for the WB scheme.

Further, the control may be both switched on and switched off using control function in the form of Eq. (10). One can see in Fig. 5 that oscillations recover after switching off the control. Therefore, despite the control term is very small after coincidence of the wave profiles after $t = 5$, it should be kept to support further stable propagation of the monotonic shock.

## 4. Conclusions

The developed feedback algorithm provides fast and efficient suppression of the scheme oscillations. The algorithm does not contain derivatives that makes it universal for different schemes. The structure of the control function results in very small addition to the scheme in the areas where the numerical and desired wave profiles coincide, then the equation with control slightly deviates from that of without control. The method does not require knowledge of the exact solution since the desired function is chosen according our choice.

The LW and WB schemes were used to demonstrate the efficiency of the method. Certainly it may be extended to more advanced finite-difference approaches using the procedure explained in Sec. 2. Also the algorithm may be extended by modification of the higher- order schemes used for the coupled gas dynamic equations and to two-dimensional problems although, in the last case, it may be difficult to get the reference solution for the control function. It will be the subject of future work.

## 5. Acknowledgment

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