

Parametrically defined differential equations

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Abstract. The paper deals with nonlinear ordinary differential equations defined parametrically by two relations. It proposes techniques to reduce such equations, of the first or second order, to standard systems of ordinary differential equations. It obtains the general solution to some classes of nonlinear parametrically defined ODEs dependent on arbitrary functions. It outlines procedures for the numerical solution of the Cauchy problem for parametrically defined differential equations.

Keywords: parametrically defined differential equations, nonlinear differential equations, numerical methods, Cauchy problem, exact solutions

1. Introduction

1.1. Preliminary remarks

Von Mises and Crocco type transformations are used to reduce the order of boundary layer equations and some other nonlinear PDEs [1–6]. Such transformations suggest choosing suitable first- or second-order partial derivatives to be new independent variables. Sometimes, the reduced equations admit exact solutions in parametric form [7]. As a result, finding exact solutions to the original PDEs reduces to integrating parametric ODEs. Moreover, parametrically defined nonlinear differential equations also arise directly in the theory of ODEs [7].

The study [7] describes a few classes of nonlinear parametric ordinary differential equations of the first and second order for which the authors managed to find general solutions. Note that parametric ODEs are non-classical and are not treated in the literature dealing with standard ODEs (e.g., see [8–12]). The general solutions obtained in [7] were further used to construct exact solutions to unsteady axisymmetric boundary-layer equations.

1.2. Ordinary differential equations defined parametrically

In general, a parametrically defined n th-order ordinary differential equation is given by two equations of the form [7]

$$y_x^{(m)} = F(x, y, y'_x, \dots, y_x^{(m-1)}, t), \quad y_x^{(n)} = G(x, y, y'_x, \dots, y_x^{(n-1)}, t), \quad (1)$$



where $y = y(x)$ is the unknown function with $y_x^{(n)}$ denoting the n th derivative $d^n y/dx^n$, $t = t(x)$ is a functional parameters, and $n \geq m$. In what follows, we assume that the parameter t cannot generally be eliminated from Eqs. (1).

2. First-order parametrically defined ODEs

2.1. Reduction to a standard system of ODEs. An example of an exact solution

Consider parametric first-order ODEs defined by

$$F(x, y, t) = 0, \quad y'_x = G(x, y, t). \quad (2)$$

In the theory of differential-algebraic equations (DAEs), equations of the form (2) are referred to a system of semi-explicit DAEs or ODEs with constraints [13, 14]. Normally, such equations are reduced to a standard system of ODEs for $y = y(x)$ and $t = t(x)$ by differentiating the first equation in (2) with respect to x [13]. However, there is a more convenient approach that suggests using an alternative system of ODEs, for $y = y(t)$ and $x = x(t)$, which is derived below.

Eqs. (2) can be rewritten in terms of differentials as

$$F_x dx + F_y dy + F_t dt = 0, \quad dy = G(x, y, t) dx, \quad (3)$$

where F_x , F_y , and F_t are respective partial derivatives of $F = F(x, y, t)$. By eliminating dy and then dx from (3), we arrive at

$$(F_x + GF_y)x'_t + F_t = 0, \quad (4)$$

$$(F_x + GF_y)y'_t + GF_t = 0. \quad (5)$$

Eqs. (4) and (5) represent a standard system of first-order equations for $x = x(t)$ and $y = y(t)$. If we manage to solve this system, we automatically obtain a solution to the original equations (2) in parametric form. In some cases, it suffices to use either Eq. (4) or (5) and the first equation in (2).

Example 1. Let us look at the nonlinear first-order ODE defined parametrically

$$x = f(t)y, \quad y'_x = g(t), \quad (6)$$

where $f = f(t)$ and $g = g(t)$ are arbitrary functions ($fg - 1 \neq 0$).

Eq. (6) is a special case of Eq. (2) with $F = x - fy$ and $G = g$. Then Eq. (5) becomes $(1 - fg)y'_t - gf'_t y = 0$. Integrating this equation in view of the first relation in (6) yields the general solution to Eq. (6) in parametric form

$$x = Cf \exp\left(\int \frac{gf'_t dt}{1 - fg}\right), \quad y = C \exp\left(\int \frac{gf'_t dt}{1 - fg}\right), \quad (7)$$

where C is an arbitrary constant.

Remark 1. If equations (4) and (5) are used, isolated solutions satisfying $F_x + GF_y = 0$ may be lost; this issue calls for further analysis.

2.2. Cauchy problem. Numerical solution procedure

Consider the Cauchy problem for the parametric first-order ODE (2) with the initial condition

$$y(x_0) = y_0. \quad (8)$$

The initial value of the parameter, $t = t_0$, can be found from the algebraic (generally transcendental) equation

$$F(x_0, y_0, t_0) = 0, \quad (9)$$

where x_0 and y_0 are the values appearing in the initial condition (8).

Using the technique described in Section 2.1, one first reduces the parametric ODE (2) to the standard system of first-order ODEs (4) and (5) for $x = x(t)$ and $y = y(t)$. Then this system, subject to the initial conditions

$$x(t_0) = x_0, \quad y(t_0) = y_0, \quad (10)$$

can be solved numerical by the Runge–Kutta or another suitable method [10, 14–16].

Remark 2. The algebraic (or transcendental) equation (9) can generally have more than one root. In this case, the original problem (2), (8) will have the same number of solutions.

3. Second-order parametrically defined ODEs

3.1. Reduction to a standard system of first-order ODEs

Consider parametric second-order ODEs of the form

$$y'_x = F(x, y, t), \quad y''_{xx} = G(x, y, t). \quad (11)$$

On differentiating the first equation with respect to t , we get $(y'_x)'_t = F_x x'_t + F_y y'_t + F_t$. Taking into account that $y'_t = F x'_t$ and $(y'_x)'_t = x'_t y''_{xx}$, we further obtain

$$x'_t y''_{xx} = F_x x'_t + F_y y'_t + F_t. \quad (12)$$

On eliminating the second derivative y''_{xx} with the aid of the second equation in (11), we arrive at the first-order equation

$$(G - F_x - F F_y) x'_t = F_t. \quad (13)$$

In view of $y'_t = F x'_t$, Eq. (13) becomes

$$(G - F_x - F F_y) y'_t = F F_t. \quad (14)$$

Equations (13) and (14) represent a standard system of first-order equations for $x = x(t)$ and $y = y(t)$. If we manage to solve this system, we automatically get a solution to the original equation (11) in parametric form. In some cases, it suffices to use either Eq. (13) or (14) and the first equation in (11).

Remark 3. If equations (13) and (14) are used, isolated solutions satisfying $G - F_x - F F_y = 0$ may be lost; this issue calls for further analysis (see Example 3, Item 2 below).

Example 2. Consider the nonlinear second-order ODE defined parametrically

$$y'_x = f(t)y, \quad y''_{xx} = g(t)y, \quad (15)$$

where $f = f(t)$ and $g = g(t)$ are arbitrary functions ($g - f^2 \neq 0$).

Eq. (15) is a special case of Eq. (11) with $F = fy$ and $G = gt$. Then Eqs. (13) and (14) are independent first order linear equations

$$(g - f^2) x'_t = f'_t, \quad (g - f^2) y'_t = f f'_t y.$$

General solution in parametric form:

$$x = \int \frac{f'_t(t) dt}{g(t) - f^2(t)} + C_1, \quad y = C_2 \exp \left[\int \frac{f(t) f'_t(t) dt}{g(t) - f^2(t)} \right],$$

where C_1 and C_2 are arbitrary constants.

Example 3. Consider the equation

$$y'_x = f(t)x + g(t), \quad y''_{xx} = h(t). \quad (16)$$

where $f = f(t)$, $g = g(t)$, and $h = h(t)$ are arbitrary functions.

Eq. (16) is a special case of Eq. (11) with $F = fx + g$ and $G = h$. Then Eqs. (13) and (14) take the form

$$\begin{aligned} (h - f)x'_t &= f'_t x + g'_t, \\ (h - f)y'_t &= (fx + g)(f'_t x + g'_t). \end{aligned} \quad (17)$$

1. Suppose $h - f \neq 0$. Then the general solution to the first (linear) equation of system (17) has the form

$$x = C_1 E + E \int \frac{g'_t dt}{E(h - f)}, \quad E = \exp\left(\int \frac{f'_t dt}{h - f}\right),$$

where C_1 is an arbitrary constant. Substituting this expression of x into the second equation of the system, we arrive at a separable equation for $y = y(t)$ (its solution is omitted).

2. Suppose $h - f \equiv 0$. On differentiating the first equation in (16) with respect to x and comparing with the second equation, we see that $t'_x = 0$. Hence, $t = C_1$, where C_1 is an arbitrary constant. By replacing t in the first equation in (16) t with C_1 followed by integrating with respect to x , we obtain the general solution to the original equation

$$y = \frac{1}{2}f(C_1)x^2 + g(C_1)x + C_2,$$

where C_1 and C_2 are arbitrary constants.

3.2. Cauchy problem. Numerical solution procedure

Consider the Cauchy problem for the parametric second-order ODE (11) with the initial conditions

$$y(x_0) = y_0, \quad y'_x(x_0) = y_1. \quad (18)$$

The initial value of the parameter, $t = t_0$, is determined from the algebraic (generally transcendental) equation

$$y_1 = F(x_0, y_0, t_0), \quad (19)$$

where x_0 , y_0 , and y_1 are the values appearing in the initial conditions (18); Eq. (19) follows from the first equation in (11).

Using the technique outlined in Section 3.1, we first reduce the parametric differential equation (11) to the standard system of first-order ODEs (13), (14) for $x = x(t)$ and $y = y(t)$. Then this system, subject to the initial conditions (10), can be solved numerically using, for example, the Runge–Kutta or another suitable method [10, 14–16].

Remark 4. The algebraic (transcendental) equation (19) can generally have more than one root. In this case, the original Cauchy problem (11), (18) will have the same number of solutions.

3.3. First boundary value problem. Numerical solution procedure

Let us look at the first boundary value problem for the parametric second-order ODE (11) in the range $x_1 \leq x \leq x_2$ with the boundary conditions

$$y(x_1) = y_1, \quad y(x_2) = y_2. \quad (20)$$

Below we present the main idea of a numerical procedure to solve this kind of problem.

Consider two auxiliary Cauchy problems for the equation (see the second equation in Eq. (11))

$$y''_{xx} = G(x, y, t) \quad (21)$$

subject to the initial conditions

$$y(x_1) = y_1, \quad y'_x(x_1) = F(x_1, y_1, t) \quad (\text{problem 1}); \quad (22)$$

$$y(x_2) = y_2, \quad y'_x(x_2) = F(x_2, y_2, t) \quad (\text{problem 2}). \quad (23)$$

By choosing a specific value of the parameter, $t = t_k$, we solve the auxiliary Cauchy problems numerically (e.g., by the Runge-Kutta method [10, 14–16]) to obtain $y^1 = y^1(x, t_k)$ and $y^2 = y^2(x, t_k)$, respectively (the superscripts indicate the problem number). To any t_k there corresponds a point (x_k, y_k) in the (x, y) plane at which the curves corresponding to the solutions $y^1 = y^1(x, t_k)$ and $y^2 = y^2(x, t_k)$ intersect. By choosing a different value, t_{k+1} , we find another point, (x_{k+1}, y_{k+1}) . The discrete set of points (x_k, y_k) with $k = 0, 1, 2, \dots$ defines an approximation to the solution $y = y(x)$ of the original boundary value problem (11), (20).

4. Higher-order parametrically defined ODEs. Cauchy problem. Numerical solution procedure

Below we outline a procedure for the numerical solution of the Cauchy problem for a parametric n th-order equation defined by two relations of the form (1), with $n > m$, subject to the initial conditions

$$y(x_0) = y_0, \quad y'_x(x_0) = y_1, \quad \dots, \quad y_x^{(n-1)}(x_0) = y_{n-1}. \quad (24)$$

By relying on Eqs. (1) directly, let us consider the following two auxiliary Cauchy problems:

$$y_x^{(m)} = F(x, y, y'_x, \dots, y_x^{(m-1)}, t),$$

$$y(x_0) = y_0, \quad y'_x(x_0) = y_1, \quad \dots, \quad y_x^{(m-1)}(x_0) = y_{m-1} \quad (\text{problem 1}); \quad (25)$$

$$y_x^{(n)} = G(x, y, y'_x, \dots, y_x^{(n-1)}, t),$$

$$y(x_0) = y_0, \quad y'_x(x_0) = y_1, \quad \dots, \quad y_x^{(n-1)}(x_0) = y_{n-1} \quad (\text{problem 2}). \quad (26)$$

Let $y_F = y_F(x, t)$ and $y_G = y_G(x, t)$ denote their respective solutions. We look at the solution difference

$$\Delta(x, t) = y_G(x, t) - y_F(x, t). \quad (27)$$

By choosing a specific value of the parameter, $t = t_k$, we find solutions $y_F(x, t_k)$ and $y_G(x, t_k)$ numerically using, for example, the Runge-Kutta method [10, 14–16]. Further, by varying x , we find an x_k such that $\Delta(x_k, t_k) = 0$. To this value of x_k there corresponds a value of the desired function: $y_k = y_F(x_k, t_k) = y_G(x_k, t_k)$. Hence, to each t_k there corresponds a point (x_k, y_k) in the (x, y) plane at which the curves $y_F = y_F(x, t_k)$ and $y_G = y_G(x, t_k)$ intersect.

By choosing a different value t_{k+1} , we find, in the same manner as above, the corresponding point (x_{k+1}, y_{k+1}) . The discrete set of points (x_k, y_k) , $k = 0, 1, 2, \dots$, defines an approximation to the solution $y = y(x)$ of the original problem (1), (24).

The initial value of the parameter, $t = t_0$, is determined from the algebraic (transcendental) equation

$$y_m = F(x_0, y_0, y_1, \dots, y_{m-1}, t_0), \quad (28)$$

where $x_0, y_0, y_1, \dots, y_m$ are the values appering in the initial conditions (24).

Remark 5. The algebraic (transcendental) equation (28) can generally have more than one root. In this case, the original Cauchy problem (1), (24) will have the same number of solutions.

5. Brief conclusions

For nonlinear parametrically defined ordinary differential equations of the form (1), we have proposed techniques to reduce such equations, of the first and second order, to standard systems of ordinary differential equations, in which x and y are the unknown functions and t is the independent variable. We have constructed the general solution to some classes of equations in question and suggested procedures for the numerical solution of the Cauchy problem for parametrically defined differential equations of the first, second, and higher order.

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