Alfven wave absorption in dissipative plasma

M B Gavrikov[1] **and A A Taiurskii**[1]
[1]Keldysh Institute of Applied Mathematics of Russian Academy of Sciences

E-mail: nadya_p@cognitive.ru, tayurskiy2001@mail.ru

Abstract. We consider nonlinear absorption of Alfven waves due to dissipative effects in plasma and relaxation of temperatures of electrons and ions. This study is based on an exact solution of the equations of two-fluid electromagnetic hydrodynamics (EMHD) of plasma. It is shown that in order to study the decay of Alfven waves, it suffices to examine the behavior of their amplitudes whose evolution is described by a system of ordinary differential equations (ODEs) obtained in this paper. On finite time intervals, the system of equations on the amplitudes is studied numerically, while asymptotic integration (the Hartman-Grobman theorem) is used to examine its large-time behavior.

1. Introduction

We examine time decay of Alfven EMHD waves due to dissipative factors (magnetic and hydrodynamic viscosities of electrons and ions, as well as relaxation of their temperatures) under the assumption that the Alfven wave has been initially excited in plasma occupying the entire space. This study is based on the the equations of electromagnetic hydrodynamics (EMHD) of plasma [1,2] that take into account the electron-ion structure of plasma and are written out in Section 2. In Section 3, it is shown that the nonlinear absorption of an Alfven wave due to dissipation is described by a system of ordinary differential equations (ODEs) for the amplitudes of the Alfven wave parameters. Solutions of the ODEs for the amplitudes on finite time intervals are studied numerically in Section 5, while large time solutions are obtained in Section 4 by asymptotic integration with the help of the Hartman-Grobman theorem [3]. This investigation allows us to find some important relationships characterising the conversion of magnetic and kinetic energies of an Alfven wave into thermal energy of electrons and ions. These relationships are of principal value for explaining abnormal heating of plasma.

2. EMHD Equations of Plasma

In view of the electron-ion structure of plasma, in particular, taking full account of electron inertia, we can write the equations of hydrodynamics in the form

$$\begin{gathered}
\frac{\partial\rho}{\partial t}+\mathrm{div}\rho\mathbf{U}=0,\quad \frac{\partial\rho\mathbf{U}}{\partial t}+\mathrm{Div}\Pi=\mathrm{DivP} \\
\frac{\partial T_\pm}{\partial t}+\mathbf{U}\cdot\nabla T_\pm+T_\pm(\gamma-1)\mathrm{div}\mathbf{U}\pm\lambda_\mp\rho^{\gamma-2}\mathbf{j}\cdot\nabla\left(\frac{T_\pm}{\rho^{\gamma-1}}\right)= \\
=\frac{\lambda_\Sigma e_\pm(\gamma-1)}{k\rho}\{\mathrm{div}(\chi_\pm\nabla T_\pm)+\mathrm{tr}(\Pi_\pm \mathrm{D}_\pm)+\frac{m_\mp}{m_\Sigma}\frac{j^2}{\sigma}\pm b(T_- - T_+)\} \\
\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t}+\mathrm{rot}\mathbf{E}=0,\quad \mathrm{div}\mathbf{H}=0,\quad \mathbf{j}=\frac{c}{4\pi}\mathrm{rot}\mathbf{H}
\end{gathered}\tag{1}$$

$$\mathbf{E}+\frac{c^2\lambda_+\lambda_-}{4\pi\rho}\mathrm{rotrot}\mathbf{E}=\frac{\mathbf{j}}{\sigma}-\frac{1}{c}[\mathbf{U},\mathbf{H}]+\frac{1}{\rho}\mathrm{Div}\mathrm{W}$$

Here, k is the Botzmann constant; $\lambda_\pm = m_\pm/e_\pm$, $\lambda_\Sigma=\lambda_++\lambda_-$, $m_\Sigma = m_+ + m_-$, $\rho=\rho_++\rho_-$, $\mathbf{U}=(\rho_+\mathbf{v}_++\rho_-\mathbf{v}_-)/\rho$, where $\rho_\pm$, $\mathbf{v}_\pm$, $m_\pm$, $e_\pm$ are, respectively, the densities, hydrodynamic velocities, masses, and absolute values of charges of electrons and ions, which are assumed to be ideal polytropic gasses with the common adiabatic exponent γ. Thus, we have a closed system of equations for the unknown functions ρ, $\mathbf{U}$, $T_\pm$, $\mathbf{H}$, $\mathbf{E}$. The momentum flux density tensor $\Pi=\Pi^{(h)}+\Pi^{(p)}+\Pi^{(c)}$, the viscous stress tensor $\mathrm{P}=\Pi_*^c+\Pi^U$, and the Hall stress tensor W have the form

$$\Pi^{(h)}=\rho\mathbf{U}\mathbf{U}+p_\Sigma \mathrm{I}_3,\ \Pi^{(p)}=\frac{H^2}{8\pi}\mathrm{I}_3-\frac{\mathbf{HH}}{4\pi},\ \Pi^{(c)}=\lambda_+\lambda_-\frac{\mathbf{jj}}{\rho},\ p_\Sigma=p_++p_-$$

$$\Pi^U=2\mu_\Sigma \mathrm{D}^U+(\nu_\Sigma-2\mu_\Sigma/3)\mathrm{trD}^U\mathrm{I}_3,\ \Pi^c=2\mu^*\mathrm{D}^c+(\nu^*-2\mu^*/3)\mathrm{trD}^c\mathrm{I}_3$$

$$\mathrm{W}=(\lambda_--\lambda_+)(\Pi^{(p)}+\Pi^{(c)})+(\lambda_-p_+-\lambda_+p_-)\mathrm{I}_3+\lambda_+\lambda_-(\mathbf{jU}+\mathbf{Uj})-\Pi_*^U-\Pi^c$$

$$\Pi_*^U=2\mu_*\mathrm{D}^U+(\nu_*-2\mu_*/3)\mathrm{trD}^U\mathrm{I}_3,\ \Pi_*^c=2\mu_*\mathrm{D}^c+(\nu_*-2\mu_*/3)\mathrm{trD}^c\mathrm{I}_3$$

where $\mathrm{D}^U=\mathrm{def}\mathbf{U}$, $\mathrm{D}^c=\mathrm{def}(\mathbf{j}/\rho)$ are strain tensors. The viscous stress tensors of electrons and ions are assumed to have the form $\Pi_\pm=2\mu_\pm\mathrm{D}_\pm+(\nu_\pm-2\mu_\pm/3)\mathrm{trD}_\pm\mathrm{I}_3$, where $\mathrm{D}_\pm=\mathrm{def}\mathbf{v}_\pm$; $\mu_\pm$, $\nu_\pm$ are the first and the second hydrodynamic viscosities of electrons and ions, respectively; $\mu_\Sigma=\mu_++\mu_-$, $\mu_*=\lambda_-\mu_+-\lambda_+\mu_-$, $\mu^*=\lambda_-^2\mu_++\lambda_+^2\mu_-$, and ν_Σ, ν_*, ν^* are expressed similarly. Finally, b and $\chi_\pm$ are the coefficients of thermal relaxation and heat conductivity of electrons and ions, respectively. Below, it is assumed that $\sigma=RT_-^{3/2}$, $\mu_\pm=T_\pm^{5/2}R_\pm$, $b=R_0\rho^2T_-^{-3/2}$, $\nu_\pm=0$, where the constants R, $R_\pm$, R_0 have the form [4,5]

$$R_+=\frac{4\pi^{1/2}e^4Z^4L}{0.96\cdot 3m_i^{1/2}k^{5/2}},\quad R_-=\frac{4(2\pi)^{1/2}e^4ZL}{0.733\cdot 3m_e^{1/2}k^{5/2}}$$

$$R_0=\frac{5m_e^{1/2}e^4Z^3L}{m_i^3k^{1/2}},\quad \sigma=\frac{3k^{3/2}}{4(2\pi m_e)^{1/2}e^2ZL\cdot 0.5129}$$

Here, $e_-=e$ is the electron charge, $Z=e_+/e_-$ is the ion charge multiplicity, $L=15$ is the Coulomb logarithm, $m_i=m_+$, $m_e=m_-$. We omit the expressions of $\chi_\pm$, since the results obtained below do not depend on $\chi_\pm$.

The law of conservation of total energy holds on the solution of system (1):

$$\frac{\partial}{\partial t}\left[\rho\left(\frac{U^2}{2}+\varepsilon+\frac{\lambda_+\lambda_-j^2}{2\rho^2}\right)+\frac{H^2}{8\pi}\right]+\mathrm{div}\left[\rho\mathbf{U}\left(\frac{U^2}{2}+\varepsilon+\frac{p_\Sigma}{\rho}+\frac{\lambda_+\lambda_-j^2}{2\rho^2}\right)+\frac{c}{4\pi}[\mathbf{E},\mathbf{H}]+A\mathbf{j}\right]= \tag{2}$$

$$=\mathrm{div}\{\chi_+\nabla \mathrm{T}_++\chi_-\nabla \mathrm{T}_-+\Pi_+\mathbf{v}_++\Pi_-\mathbf{v}_-\}$$

where $A=\lambda_+\lambda_-\langle\mathbf{U},\mathbf{j}\rangle/\rho+\lambda_+\lambda_-(\lambda_--\lambda_+)j^2/(2\rho^2)+\gamma(\lambda_-p_+-\lambda_+p_-)/(\rho(\gamma-1))$ and $\varepsilon=(\lambda_+\varepsilon_++\lambda_-\varepsilon_-)/\lambda_\Sigma$ is plasma internal energy density.

3. Decay of Alfven Waves in EMHD

Consider plane flows ($\partial/\partial y=\partial/\partial z=0$) of uniform plasma ($\rho=\mathrm{const}$) described by system (1). For such flows, we have $H_x=\mathrm{const}$, $\mathrm{U}_x=0$. In the absence of dissipation ($\nu_\pm=\mu_\pm=0$, $b=0$, $\chi_\pm=0$, $\sigma=+\infty$), system (1) in the plane case admits solutions on the line $x\in R$ with the initial conditions

$$t=0:\quad U_\perp=u_0e^{i\kappa x},\ H_\perp=h_0e^{i\kappa x},\ E_\perp=e_0e^{i\kappa x},\ T_\pm=\mathrm{T}_\pm^0 \tag{3}$$

where $\kappa > 0$ is a constant (wave number) and complex notation $U_\perp = U_y + iU_z$, $H_\perp = H_y + iH_z$, etc., is used for the transverse components of the corresponding vector fields. This solution has the form

$$U_\perp = u(t)e^{i\kappa x}, H_\perp = h(t)e^{i\kappa x}, E_\perp = e(t)e^{i\kappa x}, T_\pm = \mathrm{T}_\pm^0 \tag{4}$$

where $u(t)$, $h(t)$, $e(t)$ are found by substituting the expressions (4) into system (1):

$$u(t) = C_1 e^{i\omega_+ t} + C_2 e^{i\omega_- t},\ h(t) = \frac{(4\pi\rho)^{1/2}}{\kappa \mathrm{v_A}}\{C_1\omega_+ e^{i\omega_+ t} + C_2\omega_- e^{i\omega_- t}\}$$

$$e(t) = \frac{i}{1+r^2}\left\{\left(\frac{H_x}{c} + \Lambda\sqrt{\lambda_+\lambda_-}\,\omega_+\right)C_1 e^{i\omega_+ t} + \left(\frac{H_x}{c} + \Lambda\sqrt{\lambda_+\lambda_-}\,\omega_-\right)C_2 e^{i\omega_- t}\right\},\ r = \frac{\kappa c}{\omega_p},\ \omega_p = \left(\frac{4\pi\rho}{\lambda_+\lambda_-}\right)^{1/2} \tag{5}$$

$$\omega_\pm = \frac{\kappa \mathrm{v}_A}{2}\left\{\frac{r\Lambda}{1+r^2} \pm \left[\frac{r^2\Lambda^2}{(1+r^2)^2} + \frac{4}{1+r^2}\right]^{1/2}\right\},\ \Lambda = \sqrt{\frac{\lambda_+}{\lambda_-}} - \sqrt{\frac{\lambda_-}{\lambda_+}},\ \mathrm{v_A} = H_x\big/\sqrt{4\pi\rho}$$

$$C_1 = \left(\frac{\kappa \mathrm{v_A}}{\sqrt{4\pi\rho}}h_0 - \omega_- u_0\right)(\omega_+ - \omega_-)^{-1},\ C_2 = \left(\frac{\kappa \mathrm{v_A}}{\sqrt{4\pi\rho}}h_0 - \omega_+ u_0\right)(\omega_- - \omega_+)^{-1}$$

The solution (4), (5), called an *Alfven wave,* is a superposition of transverse waves moving in the direction of the magnetic field or in the opposite direction with different phase velocities depending on the wave length $2\pi/\kappa$. In the MHD-limit, $r \ll 1$, the solution (4), (5) turns into the classical Alfven wave.

For a plane flow, consider the solution of system (1) on the line with the initial conditions (3) and dissipation taken into account. This solution describes the decay of the Alfven wave (4), (5) due to dissipative effects and has the form

$$U_\perp = u(t)e^{i\kappa x}, H_\perp = h(t)e^{i\kappa x}, E_\perp = e(t)e^{i\kappa x}, T_\pm = T_\pm(t) \tag{6}$$

where the amplitudes $u(t)$, $h(t)$, $T_\pm(t)$ satisfy the system of ODEs obtained by substituting the functions (6) into system (1):

$$\mathrm{d}u/\mathrm{d}t = a_{11}u + a_{12}h,\ \mathrm{d}h/\mathrm{d}t = a_{21}u + a_{22}h$$

$$\frac{\mathrm{d}T_\pm}{\mathrm{d}t} = Z_\pm a_*\left\{\mu_\pm\kappa^2\left|u - \frac{c\kappa\lambda_\mp}{4\pi\rho}h\right|^2 + \frac{m_\mp}{m_\Sigma}\frac{c^2\kappa^2}{16\pi^2\sigma}|h|^2 \pm b(T_- - T_+)\right\} \tag{7}$$

where $Z_+ = Z$, $Z_- = 1$, $a_{11} = -\kappa^2\mu_\Sigma\rho^{-1}$, $a_{12} = \kappa c(\mathrm{i}H_x c^{-1} + \kappa^2\mu_*\rho^{-1})(4\pi\rho)^{-1}$, $a_{21} = \lambda_+\lambda_-\omega_p^2 a_{12}(1+r^2)^{-1}$, $a_{22} = \kappa c\left[-c\kappa(4\pi\sigma)^{-1} + \mathrm{i}\kappa\Lambda \mathrm{v_A}\omega_p^{-1} - c\kappa^3\mu^*(4\pi\rho^2)^{-1}\right](1+r^2)^{-1}$, $a_* = \lambda_\Sigma e(\gamma-1)/(k\rho)$. Here, $e(t) = (a_{21}u + a_{22}h)(\kappa c)^{-1}$. From (6), (7), it follows that: (i) the decay of plane waves is purely temporal, in the sense that only the amplitudes $u(t)$, $h(t)$, $e(t)$, $T_\pm(t)$ are varying in time, while the spatial sine distribution of the plasma parameters remains unchanged; (ii) the decay of Alfven waves does not depend on the thermal conductivity of electrons and ions. From (2), it follows that the conservation law

$$\frac{\rho|u(t)|^2}{2} + (1+r^2)\frac{|h(t)|^2}{8\pi} + \frac{T_+(t)}{Za_*} + \frac{T_-(t)}{a_*} = C_0 = \mathrm{const} \tag{8}$$

holds on the solution (6), where C_0 is determined by the initial condition (3).

4. Asymptotic Integration of Amplitude Equations for $t \to +\infty$

Let us write system (7) in dimensionless form, choosing the following characteristic scales of the density, the magnetic field strength, velocity, etc.: $\rho_0 = \rho$, $H_0 = H_x$, $U_0 = \mathrm{v_A}$, $L_0 = c\omega_p^{-1}$, $t_0 = L_0 U_0^{-1}$, $T_0 = \mathrm{v_A^2}\lambda_\Sigma e(2k)^{-1}$. Thus, we obtain the system

$$\frac{\mathrm{d}u}{\mathrm{d}t} = r\left(\mathrm{i} + \frac{r^2}{\zeta}\alpha_1\right)h + \frac{r^2}{\zeta}\beta_1 u, \quad \frac{\mathrm{d}h}{\mathrm{d}t} = \frac{r}{1+r^2}\left\{\left(\mathrm{i} + \frac{r^2}{\zeta}\alpha_1\right)u + r\left(-\frac{\xi}{T_-^{3/2}} + \mathrm{i}\Lambda + \frac{r^2}{\zeta}\beta\right)h\right\}$$

$$\frac{\mathrm{d}T_\pm}{\mathrm{d}t} = 2Z_\pm(\gamma-1)\left\{\frac{m_\mp}{m_\Sigma}r^2\zeta\frac{|h|^2}{T_-^{3/2}} \pm \zeta\eta\frac{(T_- - T_+)}{T_-^{3/2}} + \alpha^\pm\frac{r^2}{\zeta}\left[|u|^2 + r^2\frac{\lambda_\mp}{\lambda_\pm}|h|^2 \mp r\left(\frac{\lambda_\mp}{\lambda_\pm}\right)^{1/2}(\bar{u}h + u\bar{h})\right]\right\} \tag{9}$$

where $\alpha_1 = \alpha^+(\lambda_-/\lambda_+)^{1/2} - \alpha^-(\lambda_+/\lambda_-)^{1/2}$, $\beta_1 = -(\alpha^+ + \alpha^-)$, $\beta = -\alpha^+(\lambda_-/\lambda_+) - \alpha^-(\lambda_+/\lambda_-)$, $\alpha^\pm = T_\pm^{5/2}R_\pm^{-1}$, and $R_\pm$, η are universal constants; r, ζ are similarity numbers,

$$\zeta = 0.386LZ^3\frac{ce^3}{m_+^2}\frac{(4\pi\rho)^{5/2}}{H_x^4}\left(\frac{\lambda_+}{\lambda_\Sigma}\right)^{3/2}, \quad r = \frac{\kappa c}{\omega_p} = \frac{c\sqrt{\lambda_+\lambda_-}}{\sqrt{4\pi\rho}}$$

$$R_+ = 2.87\left(\frac{m_-}{m_+}\right)^{1/2}\frac{\lambda_+}{\lambda_\Sigma},\ R_- = 5.313\frac{\lambda_+}{\lambda_\Sigma},\ \eta = 1.46\frac{m_-}{m_+}\frac{\lambda_\Sigma}{\lambda_+} \tag{10}$$

Thus, if $\ell = 2\pi/\kappa$ is the length of an Alfven wave, then the problem of wave decay has two determining parameters: $\rho^{5/2}/H_x^4$ and $\ell\rho^{1/2}$. Moreover, the energy integral (8) can be rewritten in the dimensionless form

$$|u|^2 + (1+r^2)|h|^2 + \frac{T_+}{Z(\gamma-1)} + \frac{T_-}{(\gamma-1)} = C_0 \tag{11}$$

Separating the real and the imaginary parts in system (9), we pass to the real unknown functions u_1, u_2, h_1, h_2, with $u_1 + \mathrm{i}u_2 = u$, $h_1 + \mathrm{i}h_2 = h$, and exclude T_+ from the unknown quantities with the help of the energy integral (11). Thus we obtain a modification of system (9) that consists of five ODEs for five real unknown functions (T, u_1, u_2, h_1, h_2). It is not difficult to verify the following statements:

1) The modified system (9) has a unique singular point $(T^0, 0,0,0,0)$, where $T^0 = Z(\gamma-1)C_0(1+Z)^{-1}$, with $C_0 = |u_0|^2 + (1+r^2)|h_0|^2 + T_+^0(Z(\gamma-1))^{-1} + T_-^0(\gamma-1)^{-1}$ being the value of the energy integral calculated on the basis of the initial values.

2) The eigenvalues of the Jacobi matrix J coincide with $\lambda_0 = -2(\gamma-1)\eta\zeta(1+Z)(T^0)^{-3/2}$ and the roots of two quadratic equations

$$\lambda^2(1+r^2) - \lambda r^2[\beta_0 + \beta_1\zeta^{-1}(1+r^2) \pm \mathrm{i}\Lambda] + r^4\zeta^{-1}[\beta_1\beta_0 - r^2\zeta^{-1}\alpha_1^2 \pm \mathrm{i}(\Lambda\beta_1 - 2\alpha_1)] + r^2 = 0. \tag{12}$$

3) If $\lambda_1 \neq \lambda_2$ are the roots of (12) with the upper sign, then $\bar{\lambda}_1 \neq \bar{\lambda}_2$ are the roots of (12) with the lower sign.

4) Each equation in (12) has neither multiple, nor conjugate, nor real roots.

5) All roots of equations (12) have negative real parts.

6) All eigenvalues $\{\lambda_0, \lambda_1, \lambda_2, \bar{\lambda}_1, \bar{\lambda}_2\}$ of the matrix J are single and there is a basis of the space C^5 that consists of eigenvectors of J.

It follows that the only singular point of the modified system (9) is an attractive stable multidimensional focus and, by the Hartman-Grobman theorem, the topology of the integral curves in a neighborhood of the singular point of this system coincides with that of its linearization at this singular point. Thus, for $t \to +\infty$, the decay of the Alfven wave is correctly described by the linearization of the modified system (9) at the singular point $(T^0, 0,0,0,0)$. The solutions of the linearized system

$$(\dot{T}, \dot{u}_1, \dot{u}_2, \dot{h}_1, \dot{h}_2)^* = \mathrm{J}(T, u_1, u_2, h_1, h_2)^* \tag{13}$$

(here the dot and the asterisk indicate differentiation in t and transposition, respectively) can be easily obtained in explicit form. Let $\lambda_1 \neq \lambda_2$ be the roots of the characteristic equation (12) with the upper sign and let $x_j + iy_j \neq 0$ be the eigenvector of J corresponding to λ_j, $j = 1,2$. If $\lambda_j = a_j + ib_j$, $j = 1,2$, $x_0 = (1,0,0,0,0)$, then $\{x_0, x_1, y_1, x_2, y_2\}$ is a basis of R^5 in which the Jacobi matrix J has the form

$$\mathrm{J} = \mathrm{diag}\left\{1, \begin{pmatrix} a_1 & b_1 \\ -b_1 & a_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ -b_2 & a_2 \end{pmatrix}\right\}$$

Therefore, if $(z_0, z_1, z_2, z_3, z_4)$ are the coordinates of a vector in R^5 in the basis $\{x_0, x_1, y_1, x_2, y_2\}$, then system (13), in that basis, splits into three independent subsystems,

$$\dot{z}_0 = \lambda_0 z_0, \quad \begin{cases} \dot{z}_1 = a_1 z_1 + b_1 z_2 \\ \dot{z}_2 = -b_1 z_1 + a_1 z_2 \end{cases}, \quad \begin{cases} \dot{z}_3 = a_2 z_3 + b_2 z_4 \\ \dot{z}_4 = -b_2 z_3 + a_2 z_4 \end{cases}$$

whose solutions can be easily written out, which gives us the solution of system (13). This solution represents a two-frequency spiral (with frequencies b_1, b_2) in five-dimensional space. The spiral winds around the origin with the decrements of distance from the origin being equal to $|a_j|$, $j = 1,2$, $|\lambda_0|$. In particular, we have

$$(u_1, u_2, h_1, h_2)^* = \sum_{j=1}^{2} D_j e^{a_j t} (\cos(\varphi_j - b_j t) x_j + \sin(\varphi_j - b_j t) y_j)$$

where the constants D_0, D_1, D_2, φ_1, φ_2 are found by expanding the initial vector $(u_1^0, u_2^0, h_1^0, h_2^0)$ with respect to the basis $\{x_1, y_1, x_2, y_2\}$. The explicit expressions for x_j, y_j, $j = 1,2$, are the following:

$$x_j = \left(0, \zeta r \cdot \frac{(r^2\beta_1 - \zeta a_j) - r^2\alpha_1 b_j}{(r^2\beta_1 - \zeta a_j)^2 + \zeta^2 b_j^2}, -r \cdot \frac{b_j\zeta^2 + r^2\alpha_1(r^2\beta_1 - \zeta a_j)}{(r^2\beta_1 - \zeta a_j)^2 + \zeta^2 b_j^2}, 0, 1\right)$$

$$y_j = \left(0, r \cdot \frac{b_j\zeta^2 + r^2\alpha_1(r^2\beta_1 - \zeta a_j)}{(r^2\beta_1 - \zeta a_j)^2 + \zeta^2 b_j^2}, \zeta r \cdot \frac{(r^2\beta_1 - \zeta a_j) - r^2\alpha_1 b_j}{(r^2\beta_1 - \zeta a_j)^2 + \zeta^2 b_j^2}, -1, 0\right)$$

The constants a_j, b_j can be easily calculated by the formulas for the roots of quadratic equations and square roots of complex numbers. The resulting expressions are rather lengthy, but can be simplified in some special or limit cases. Thus, for $r >> 1$ (short waves), $h_0 \neq 0$, $\mu_\pm \neq 0$, we have the asymptotic formulas

$$a_{1,2} \sim \left(\frac{Z(\gamma-1)}{1+Z}\right)^{5/2} |h_0|^5 r^7 \frac{\lambda_\Sigma}{\lambda_\pm} \frac{1}{\zeta R_\pm}, \quad b_{1,2} \sim \frac{1}{2}\left\{\Lambda \pm \frac{\Lambda R^* - \Lambda R_\Sigma - 4R_*}{\lambda_\Sigma(\lambda_+^{-1}R_+^{-1} - \lambda_-^{-1}R_-^{-1})}\right\} \tag{14}$$

where $R^* = (\lambda_-/\lambda_+)R_+^{-1} + (\lambda_+/\lambda_-)R_-^{-1}$, $R_\Sigma = R_+^{-1} + R_-^{-1}$, $R^* = (\lambda_-/\lambda_+)^{1/2} R_+^{-1} + (\lambda_+/\lambda_-)^{1/2} R_-^{-1}$, with the upper and the lower signs in (14) being in agreement. For $r >> 1$ (long waves), we have

$$a_{1,2} \sim -\frac{r^2}{2\zeta A_0^{3/2}}[\zeta^2 + R_\Sigma A_0^4], \quad b_{1,2} \sim \pm r, \quad A_0 = \frac{Z(\gamma-1)}{Z+1}\left[|h_0|^2 + |u_0|^2 + \frac{T_+^0}{Z(\gamma-1)} + \frac{T_-^0}{\gamma-1}\right]$$

The coordinate T^0 of the singular point is equal to the equilibrium temperature established in plasma after the complete absorption of the Alfven wave and the relaxation of electron and ion temperatures:

$$T^0 = \frac{T_0^+ + ZT_0^-}{1+Z} + \frac{Z}{1+Z}\frac{\lambda_\Sigma e(\gamma-1)}{k}\left\{\frac{|u_0|^2}{2} + \frac{|h_0|^2}{8\pi\rho}\left[1 + \left(\frac{\kappa c}{\omega_p}\right)^2\right]\right\} \tag{15}$$

It follows from (15) that the equilibrium temperature does not depend on the plasma magnetization H_x, but depends on the wave length $2\pi/\kappa$. Theoretically, (15) indicates that fairly short Alfven waves, even of small amplitudes u_0, h_0, may heat up plasma to arbitrarily high temperatures.

5. Results of Numerical Analysis

The absorption of an Alfven wave amounts to the conversion of its kinetic energy $\varepsilon_{\rm kin} = \rho|u(t)|^2/2$ and its total (with the kinetic energy of the relative motion of electrons taken into account) magnetic energy $\varepsilon_{\rm m} = (1+r^2)|h(t)|^2/(8\pi)$ into the thermal energy of electrons and ions $\varepsilon_- = T_- a_*^{-1}$, $\varepsilon_+ = T_+(Za_*)^{-1}$. This process is superimposed on the relaxation of the electron and ion temperatures determined by the coefficient b. Numerical solutions of the Cauchy problem for system (14) show that the absorption of an Alfven wave splits into two stages: (i) first, there is a rapid conversion of its magnetic energy and a considerable part of its kinetic energy into the thermal energy of (mostly) electrons; (ii) then, slow (for the most part) relaxation of temperatures occurs, which is approximated by the solution of system (9) with $u = 0$, $h = 0$; here, the remainder of the kinetic energy is converted into heat. The curves in Figure 1 represent typical values of thermal energies of electrons and ions, as well as the magnetic and the kinetic energies, versus time in the case of $r = 0.1$, $\zeta = 300$, $T_+^0 = 0.1$, $T_-^0 = 1$, $h_0 = 5$, $u_0 = 1.5$, $\mu_\pm = 0$. If the hydrodynamic viscosity of ions is taken into account, then the absorption process becomes much faster. For instance, the time of magnetic energy absorption becomes equal to $\sim(\omega_c^+\omega_c^-)^{-1/2}$, where $\omega_c^\pm$ are cyclotron frequencies of electrons and ions. If we additionally take into account electron viscosity, then the absorption process becomes even faster, occurring in a fraction of $(\omega_c^+\omega_c^-)^{-1/2}$, and the absorption time for the magnetic energy becomes $\sim 10^{-2}(\omega_c^+\omega_c^-)^{-1/2}$.

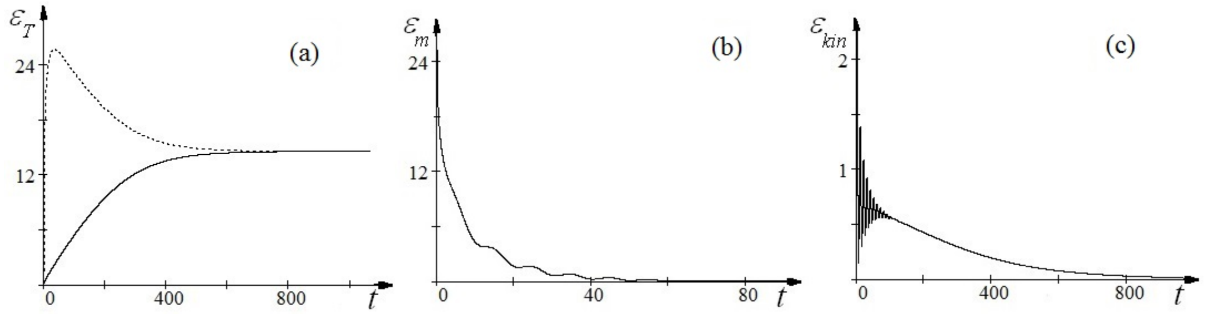

Figure 1. Time dependence of the thermal energy of electrons ("···") and ions ("—") in the Alfven wave (a), the magnetic energy of the Alfven wave (b), the kinetic energy of the Alfven wave (c)

Acknowledgments

The study was performed by the grant from the Russian Foundation for Basic Research (projects №15-01-03085, №16-31-00302)

Bibliography

[1] Braginskii S I 1965 *Reviews of Plasma Physics* vol 1 ed M A Leontovich (New York: Consultants Bureau) pp 205-311
[2] Gavrikov M B and Sorokin R V 2008 *Fluid Dynamics* **43** 6 pp 977-989
[3] Hartman P 2002 *Ordinary Differential Equations* (SIAM)
[4] Spitzer L 1962 *Physics of Fully Ionized Gases* (New York: 2nd ed Interscience)
[5] Chapman S and Cowling T G 1952 *The Mathematical Theory of Non-uniform Gases* (Cambridge University Press)