Analysis of the incomplete Galerkin method for modelling of smoothly-irregular transition between planar waveguides

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Abstract. The paper deals with a numerical solution of the problem of waveguide propagation of polarized light in smoothly-irregular transition between closed regular waveguides using the incomplete Galerkin method. This method consists in replacement of variables in the problem of reduction of the Helmholtz equation to the system of differential equations by the Kantorovich method and in formulation of the boundary conditions for the resulting system. The formulation of the boundary problem for the ODE system is realized in computer algebra system Maple. The stated boundary problem is solved using Maples libraries of numerical methods.

1. Introduction

The paper deals with a numerical solution to the problem of waveguide propagation of polarized light in smoothly-irregular transition between closed regular waveguides using the incomplete Galerkin method [1, 2]. This method consists in replacement of variables in the problem of reduction of the Helmholtz equation to the system of differential equations by the Kantorovich method [3] and in formulation of the boundary conditions for the resulting system. The Kantorovich expansion is made in the system of functions that satisfy the reduced boundary conditions [3]. Coefficient functions of the Kantorovich expansion are the desired values of the boundary problem. The formulation of the boundary problem for the ODE system is realized by using computer algebra system Maple. The stated problem is solved using Maples libraries of numerical methods.

2. Statement of the problem

Consider smoothly-irregular along z closed waveguide transition between closed planar regular waveguides. The lower and upper boundaries of the transition are given by the equations $x = 0$ and $x = h(z)$. The condition of smooth irregularity of the waveguide transition is represented by continuity of $h(z)$ and $h'(z)$ for all z.

Function $h(z)$ describes the variable height of irregular waveguide transition, and in order to meet the conditions of smooth irregularity it is necessary that $h(0) = h_1$, $h(d) = h_2$ and $h'(0) = h'(d) = 0$. Let us consider the problem of propagation of polarized electromagnetic field in the described structure made of material with refractive index n_f.

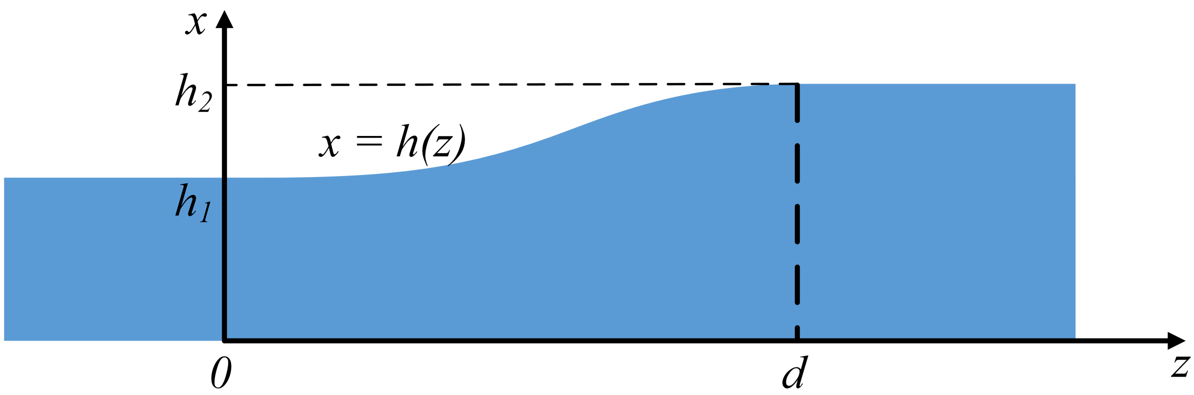

Figure 1. Waveguide transition between two closed planar waveguides of constant cross-section.

Electromagnetic field propagating in the waveguide satisfies the Maxwell's equations, constitutive relations and boundary conditions [4, 5]. For the described planar waveguide structure (Fig. 1) the Maxwells equations can be reduced to two independent subsystems with respect to the desired component of the electromagnetic field – subsystems for TE- and TM-modes. Subsystem for TE-modes can be formulated as the Helmholtz differential equation with respect to the component E_y and two differential equations relating the components H_x and H_z with E_y. Let us denote E_y by $u = u(x, z)$ and write down the Helmholtz differential equation[1]

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + k^2\right) u = 0 \tag{1}$$

with coefficient $k^2 = (2\pi n_f/\lambda)^2$ (λ is wavelength), and the boundary conditions of the first kind[2]:

$$u|_{x=0} = u|_{x=h(z)} = 0 \tag{2}$$

The function $u = u(x, z)$ meets the conditions of excitation and emission at infinity[2]:

$$u|_{z\le 0} = \sum_{n=1}^{\infty} R_n e^{-i\gamma_n z} \sin\left(\frac{\pi n x}{h_1}\right) + \ A e^{i\gamma_{n_0} z} \sin\left(\frac{\pi n_0 x}{h_1}\right) \tag{3}$$

$$u|_{z\ge d} = \sum_{n=1}^{\infty} T_n e^{i\Gamma_n z} \sin\left(\frac{\pi n x}{h_2}\right) \tag{4}$$

where $\gamma_n = k^2 - \left(\frac{\pi n}{h_1}\right)^2$, $\Gamma_n = k^2 - \left(\frac{\pi n}{h_2}\right)^2$, A is the given amplitude of the incident mode of number n_0, R_n and T_n are unknown amplitude coefficients.

According to the algorithm proposed in [1, 2], we introduce new variables which transform an irregular area in the strip:

$$\begin{aligned} \xi(x, z) &= x/h(z) \\ \eta(x, z) &= z. \end{aligned} \tag{5}$$

In the new coordinates the Helmholtz equation takes the form:

$$\left(\frac{1}{h^2(\eta)} + \xi^2 b^2(\eta)\right) \times \frac{\partial^2 u}{\partial \xi^2} - 2\xi b(\eta) \times \frac{\partial^2 u}{\partial \xi\, \partial \eta} + \frac{\partial^2 u}{\partial \eta^2} + \xi c(\eta) \times \frac{\partial u}{\partial \xi} + k^2 u = 0 \tag{6}$$

where $b(\eta) = h'(\eta)/h(\eta)$, $c(\eta) = b^2(\eta) - b'(\eta)$. The boundary conditions (2) and the conditions of excitation and emission at infinity (3), (4) in the new coordinates take the following form:

$$u|_{\xi=0} = u|_{\xi=1} = 0 \tag{7}$$

$$u|_{\eta\le 0} = \sum_{n=1}^{\infty} R_n e^{-i\gamma_n \eta} \sin(\pi n\xi) + \; A e^{i\gamma_{n_0}\eta} \sin(\pi n_0 \xi) \tag{8}$$

$$u|_{\eta\ge d} = \sum_{n=1}^{\infty} T_n e^{i\Gamma_n \eta} \sin(\pi n\xi) \tag{9}$$

Following [1, 2], we apply the Kantorovich method [3] for the formulation of a system of differential equations and boundary conditions for this system. We use the eigenfunctions of the waveguide problem in the zeroth approximation by $h'(\eta)$ as the basis functions of the Kantorovich method, and we justify such a choice for the problem in the zeroth approximation.

3. Zeroth approximation

Consider the case, when $h'(\eta)$ is a small quantity: $h'(\eta) \le \delta \ll 1$. Equation (6) in the zeroth approximation by the parameter δ takes the form:

$$-h^2(\eta)\left(\frac{\partial^2 u}{\partial \eta^2} + k^2 u\right) = \frac{\partial^2 u}{\partial \xi^2} \tag{10}$$

We seek a solution of (10) that can be represented as a product $u(\xi,\eta) = V(\eta)\varphi(\xi)$. We substitute the estimated form of solution to the equation (10) and divide the equation by $V\varphi$. The resulting equation must be satisfied for all values of the variables, for $u(\xi,\eta) = V(\eta)\varphi(\xi)$ to be a solution of (10), which means:

$$-h^2(\eta)\left(\frac{V''}{V}(\eta) + k^2\right) = \frac{\varphi''}{\varphi}(\xi) = -\lambda \tag{11}$$

where λ is an unknown constant. Relation (11) and conditions (7) formulate the Sturm-Liouville problem:

$$\begin{aligned} &\varphi'' + \lambda\varphi = 0, \\ &\varphi(0) = \varphi(1) = 0. \end{aligned} \tag{12}$$

Eigenvalues of the problem (12), given by $\lambda_n = (\pi n)^2$ where $n = 1, 2, 3$ ..., correspond to eigenfunctions (defined with an accuracy up to a factor), which take the form:

$$\varphi_n(\xi) = \sin\left(\sqrt{\lambda_n}\xi\right), \quad n = 1, 2, 3 \; ... \tag{13}$$

Functions (13) depend on z parametrically, and the system of functions (13) is orthogonal for all z from $[0,\ d]$ and coincides with the system of eigenfunctions of the left regular waveguide at $z = 0$ and with the system of eigenfunctions of the right regular waveguide at $z = d$.

We use the system of functions (13) as the basis in the Kantorovich method [3], because the functions (13) satisfy the exact boundary conditions (7).

4. Reduction to the boundary problem for ODE system

We seek an approximate solution to the equation (6) in the form of a partial sum of the series in the system of functions (13) satisfying the boundary conditions (7):

$$u^N(\xi,\,\eta) = \sum_{n=1}^{N} V_n(\eta)\sin(\pi n\xi) \tag{14}$$

where N is the number of terms in the partial sum of the series.

We substitute the solution (14) into the equation (6) and the boundary conditions (8), (9) and apply projection scheme of the Galerkin method [6, 7]. As a result we get the boundary problem for the system of $N \times N$ ordinary differential equations of the second order:

$$\mathbf{A}(\eta)\,\vec{v}'' + \mathbf{P}(\eta)\,\vec{v}' + \mathbf{Q}(\eta)\,\vec{v} = \vec{0} \tag{15}$$

$$\begin{array}{l} \left.(\vec{v}' + i\mathbf{D}_\gamma\vec{v})\right|_{\eta=0} = 2i\gamma_{n_0}\vec{\delta}_{n,n_0}A/h_1 \\ \left.(\vec{v}' - i\mathbf{D}_\Gamma\vec{v})\right|_{\eta=d} = 0 \end{array} \tag{16}$$

where $\vec{v} = (V_1(\eta), V_2(\eta), \ldots, V_N(\eta))^T$ is the vector of the desired amplitude (coefficient) functions of the Kantorovich expansion, $\mathbf{A}(\eta)$, $\mathbf{P}(\eta)$, $\mathbf{Q}(\eta)$ are matrix functions, elements of which are calculated using Maple, $\mathbf{D}_\gamma = diag\,\{\gamma_n\}_{n=1}^N$, $\mathbf{D}_\Gamma = diag\,\{\Gamma_n\}_{n=1}^N$, $\vec{\delta}_{n,n_0}$ is a vector with zero components, with the exception of one, standing on n_0-th position.

The described algorithm of stating the boundary problem (15), (16) by the incomplete Galerkin method is realized in computer algebra system Maple. The boundary problem (15), (16) is being solved numerically in Maple, the results of calculations are presented below.

5. Numerical calculation

We solve numerically the problem of waveguide propagation of monochromatic light of wavelength $0,55\,\mu$ in the waveguide transition made of material with refractive index $n_f = 1,51$. Function $h(z)$ is third-degree polynomial and its value changes smoothly from $h_1 = 0,70\,\mu$ to $h_2 = 0,92\,\mu$ along the interval of length $d = 2,00\,\mu$. In the waveguide of the thickness $h_1 = 0,70\,\mu$ there exist 3 non-evanescent modes, in the waveguide of the thickness $h_2 = 0,92\,\mu$ - 5 non-evanescent modes.

We consider two cases: in the first case the first non-evanescent mode ($n_0 = 1$) is incident on the described structure, in the second case the first evanescent mode ($n_0 = 4$) is incident on the structure. The total number of modes in the expansion $N = 16$. The boundary problem is solved by finite difference scheme with partition into 2048 points using Maple function `dsolve`.

The numerical calculation (see Fig. 2) shows the evolution of the amplitude functions of all allowable modes when the mode with the number $n_0 = 1$ is incident on the irregular transition. We can observe that the second and the third non-evanescent (or guided) modes are excited. Amplitude of incident mode ($n_0 = 1$) decreases because of the energy redistribution between guided modes of irregular transition.

The second example (see Fig. 3) shows the evolution of the amplitude functions in the case when the first evanescent mode ($n_0 = 4$) is incident on the irregular transition. Amplitude of the incident evanescent mode decreases with the thickness of the waveguide decreasing till the critical value. When the thickness of the waveguide becomes less than the critical thickness, the evanescent mode becomes the guided mode with decreased amplitude. And as in the first example, other guided modes are excited in the waveguide as well.

The results of numerical experiments in both examples are in good agreement with the theoretical expectation of the behavior of waveguide modes.

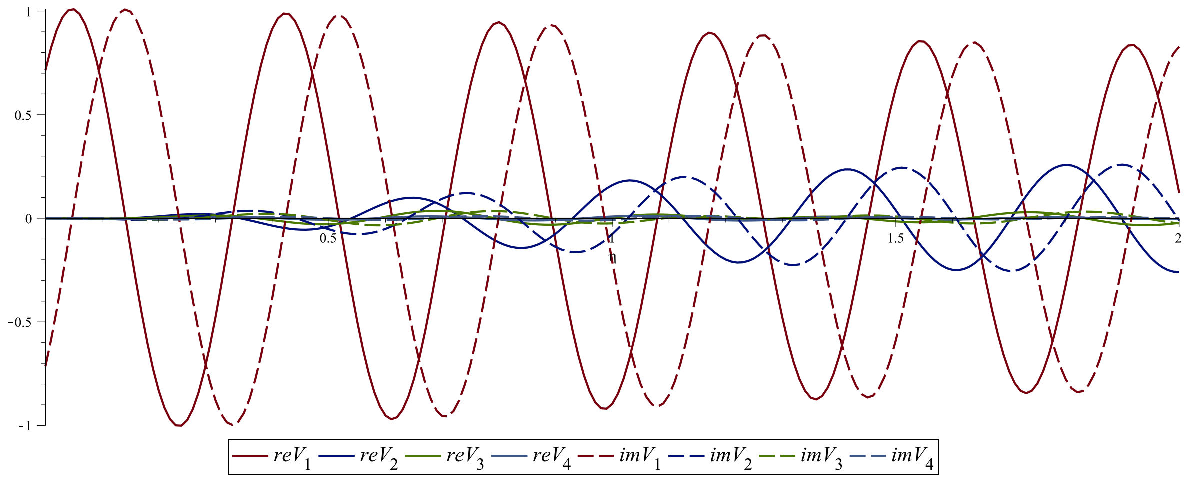

Figure 2. First four amplitude (coefficient) functions: real parts are shown by solid lines, imaginary parts – by dashed lines. Incident mode has the number $n_0 = 1$ and amplitude $A = (i+1)/2$.

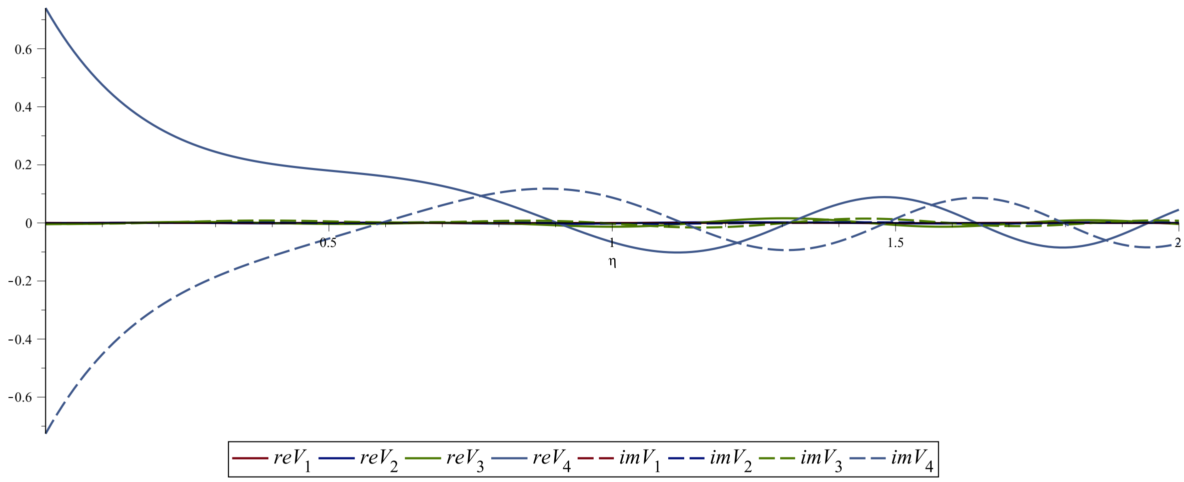

Figure 3. First four amplitude (coefficient) functions: real parts are shown by solid lines, imaginary parts - by dashed lines. Incident mode has the number $n_0 = 4$ and amplitude $A = (i+1)/2$.

6. Conclusion

The paper considered using of the incomplete Galerkin method to calculate the waveguide propagation of monochromatic polarized light in closed waveguide structures with irregularity along the propagation direction. Algorithms for symbolic transformations of the original problem for the Helmholtz equation to a boundary problem for ODE system and numerical solution of the reduced boundary problem are implemented using computer algebra system Maple. The numerical experiments show that using of eigenfunctions of waveguide problem in zeroth approximation as the basis in the Kantorovich method is adequate to the problem. The numerical results show qualitative agreement with the theoretically expected behavior of waveguide modes and evanescent modes.

Authors considered the waveguide propagation of monochromatic polarized light in an open irregular transition between two regular planar open waveguides in [8, 9]. Comparing the results of current research and the results from [8, 9] shows qualitative agreement and the ability to generalize the incomplete Galerkin method from the closed to the open irregular waveguide transitions.

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