

Normal Vector Projection Method used for Convex Optimization of Chan-Vese Model for Image Segmentation

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Abstract. The variational level set method is one of the main methods of image segmentation. Due to signed distance functions as level sets have to keep the nature of the functions through numerical remedy or additional technology in an evolutionary process, it is not very efficient. In this paper, a normal vector projection method for image segmentation using Chan-Vese model is proposed. An equivalent formulation of Chan-Vese model is used by taking advantage of property of binary level set functions and combining with the concept of convex relaxation. Threshold method and projection formula are applied in the implementation. It can avoid the above problems and obtain a global optimal solution. Experimental results on both synthetic and real images validate the effects of the proposed normal vector projection method, and show advantages over traditional algorithms in terms of computational efficiency.

1. Introduction

The variational method and partial differential equation method have already become main methods in fields of image processing and computer vision[1-5]. With advantages such as universality of multi-model integration, expression flexibility of complex topology and stability of numerical computation method, variational level set method has been one of the basic methods of image segmentation. Chan-Vese model[6] is a combination of Mumford-Shah model[7] and variational level set method[8]. It is a variational method of image segmentation which could implement multi-phase image segmentation[9], motion segmentation, texture image segmentation[10] and implicit surface segmentation[11].

Traditional method which uses signed distance function as level sets has certain disadvantages:(1) Image segmentation method, which based on the variational level set method, generally uses gradient descent method to solve the energy functional, so that the minimum obtained from energy functional is probably a local minimum[12], which leads to a high requirement of function initialization. Nevertheless, results acquired sometimes are not really ideal, and even wrong; (2) Complicated level



set evolution equation needs to be computed in order to get solution of energy functional. This will slow down computation speed. To this end, Lie [13] and Bresson [14] replaced signed distance functions with binary tag function, transformed Chan-Vese model of two phase image segmentation into global optimal model by constraint relaxing, and finally a threshold method can be applied to the result to acquire global optimal solution of the original problem. This method avoids impact on segmentation results due to initialization of different level sets. Four methods were used to solve this model respectively: Gradient Descent Method(GDM) [15], Dual Method (DM) [16], Split-Bregman Method (SBM) [17] and Alternating Direction Method of Multipliers(ADMM) [18]. On this basis, a new fast segmentation method called Normal Vector Projection Method(NVPM) is proposed. The effectiveness and efficiency of the proposed method are validated by comparison of experimental results. Also, the proposed method can be generalized to other variational model of image processing.

The rest of this paper is organized as follows. In Section 2, the binary level set formulation of the functional of Chan-Vese model used in our paper is reviewed along with its traditional solution method. Our proposed method is discussed in Section 3 and its iterative discrete formulas for implementation will be presented in detail. In Section 4, some numerical experiments are given to illustrate the effectiveness of our method by comparing with other methods. Finally a conclusion is given in section 5.

2. Convex optimization of Chan-Vese model

Chan-Vese model, which is based on variational level set method, is a two-phase image segmentation Mumford-Shah model approximated by piece-wise constant. It divides the image into two types of heterogeneous area. Chan-Vese model can be transformed into the following energy functional minimization problem through combining the method that use binary tag functions as level sets which is proposed by Lie and Bresson.

$$\underset{c, \phi \in \{0,1\}}{\text{Min}} \left\{ E(c, \phi) = \alpha_1 \int_{\Omega} (c_1 - f)^2 \phi dx + \alpha_2 \int_{\Omega} (c_2 - f)^2 (1 - \phi) dx + \gamma \int_{\Omega} |\nabla \phi| dx \right\} \quad (1)$$

$f(x): \Omega \rightarrow R$ is the image that is yet to be segmented which is defined on Ω . $\alpha_1, \alpha_2, \gamma$ are penalty parameters, $c_1, c_2 \in c$ are piece-wise constants which are used for segmenting the image into different areas, $\phi(x)$ is level set function. We assume that there is a closed region $\Omega_1 \subset \Omega$, then the definition of binary tag function is as follows.

$$\phi(x) = \begin{cases} 1, & x \in \Omega_1 \\ 0, & x \notin \Omega_1 \end{cases} \quad (2)$$

Eq.(1) is the energy function contains more than one variable, we solve it through alternating optimization method generally, i.e. solve the function under fixed ϕ with respect to c .

$$c_1 = \int_{\Omega} f \phi dx / \int_{\Omega} \phi dx, \quad c_2 = \int_{\Omega} f (1 - \phi) dx / \int_{\Omega} (1 - \phi) dx \quad (3)$$

Then, fixing c and solving the function with respect to ϕ . By Eq.(1), the formula of energy functional minimization about ϕ can be written as

$$\text{Min}_{c, \phi \in \{0,1\}} \left\{ E(\phi) = \int_{\Omega} Q_{12}(c_1, c_2) \phi dx + \gamma \int_{\Omega} |\nabla \phi| dx \right\} \quad (4)$$

where $Q_{12}(c_1, c_2) = \alpha_1(c_1 - f)^2 - \alpha_2(c_2 - f)^2$. Eq.(4) can be transformed into following problem via convex relaxation and threshold on binary tag function ϕ

$$\text{Min}_{\phi \in [0,1]} \left\{ E(\phi) = \int_{\Omega} Q_{12}(c_1, c_2) \phi dx + \gamma \int_{\Omega} |\nabla \phi| dx \right\} \quad (5)$$

The according gradient descent function about Eq.(5) is

$$\begin{cases} \partial \phi / \partial t = (\gamma \nabla \cdot (\nabla \phi / |\nabla \phi|) - Q_{12}(c_1, c_2)) & \text{in } \Omega \\ \partial \phi / \partial \bar{n} = 0 & \text{on } \partial \Omega \end{cases} \quad (6)$$

After getting ϕ , then project it into the interval between 0 and 1.

$$\tilde{\phi}^{k+1} = \text{Max}(\text{Min}(\phi^{k+1}, 1), 0) \quad (7)$$

Finally, ϕ is obtained through threshold technology.

$$\phi^{k+1} = \begin{cases} 1 & \tilde{\phi}^{k+1} > \alpha = 0.5 \\ 0 & \text{otherwise} \end{cases} \quad (8)$$

3. Normal vector projection method of Chan-Vese convex model and its optimization method

Through introduction of four methods above, the method[19][20] which uses low-order term instead of high-order term and optimize alternately, not only improves computation efficiency, but also reduces complexity of energy functional. On the basis of this thought, the normal vector projection method is proposed, which uses approximated variable to replace normal vector of curvature term in gradient descent function. This will simplify solving process, and then conduct the projection of normal vector to constrain the introduced approximated variable directly. This method is more concise on the structure of algorithm, moreover, it can guarantee the fast convergence of the energy functional. This chapter not only put forward NVPM of Chan-Vese model, but also the optimization algorithm of NVPM.

3.1 Vector projection method

It is easy to get Euler-Lagrange equation according to Eq.(5) which is depicted as follow.

$$Q_{12}(c_1, c_2) - \gamma \nabla \cdot (\nabla \phi / |\nabla \phi|) = 0 \quad (9)$$

Here the four-order curvature term which is difficult to calculate and the primary source of low iteration speed. According to method that has been introduced which utilizes low-order term to replace high-order term and alternately optimize each variable so that calculation efficiency is improved. Introducing instrumental variable $\vec{p} = [p_1, p_2]^T$ to approximately represent normal vector of ϕ , so

$\bar{p} \approx \nabla \phi / |\nabla \phi|$ can be obtained, and the above equation is converted as

$$Q_{12}(c_1, c_2) - \gamma \nabla \cdot \bar{p} = 0 \quad (10)$$

Now fix \bar{p} to find ϕ^{k+1} , the formula above should be calculated by using gradient descent method.

$$\partial \phi / \partial t = \gamma \nabla \cdot \bar{p} - Q_{12}(c_1, c_2) \quad (11)$$

The time step is τ_p , now we obtain the iteration function on ϕ .

$$\phi^{k+1} = \phi^k + \tau_p (\gamma \nabla \cdot \bar{p}^k - Q_{12}(c_1, c_2)) \quad (12)$$

After obtained ϕ^{k+1} , we need to make a projection of ϕ^{k+1} according to Eq.(7). Then fix ϕ^{k+1} . We can obtain \bar{p} with normal vector projection method directly based on $\bar{p} \approx \nabla \phi / |\nabla \phi|$.

$$\bar{p}^{k+1} = \nabla \phi^{k+1} / \max(|\nabla \phi^{k+1}|, 1) \quad (13)$$

Finally, threshold ϕ^{k+1} according to Eq.(8).

In summary, we can list the NVPM in a pseudo code as follows.

Algorithm 1: NVPM for CV model

Step 1: Initialize unknown values as ϕ^0, \bar{p}^0

Step 2: For $k \geq 1$, solve the following problems alternatively

2.1 Sub-problem 1 for ϕ^{k+1} : $\phi^{k+1} = \text{Arg min} \{ \varepsilon_1(\phi) = E(\phi, \bar{p}^k) \}$,

$$\phi^{k+1} = \text{Max}(\text{Min}(\phi^{k+1}, 1), 0).$$

2.2 Sub-problem 2 for \bar{p}^{k+1} : $\bar{p}^{k+1} = \text{Arg min} \{ \varepsilon_2(\bar{p}) = E(\phi^{k+1}, \bar{p}) \}$.

2.3 Threshold ϕ^{k+1} : $\phi^{k+1} = \begin{cases} 1 & \tilde{\phi}^{k+1} > \alpha = 0.5 \\ 0 & \text{otherwise} \end{cases}$.

Step 3: The overall loop will be terminated if the stopping criteria (described in section 3) are satisfied.

3.2 Optimization algorithm of normal vector projection method

The method proposed by Nesterov[21] could accelerate and optimize the process of gradient descent through over relaxation. A kind of optimization algorithm of the normal vector projection method is given to further improve computation efficiency. An over relaxation parameter α and an intermediate variable $\tilde{\phi}$ are introduced before we solve the gradient descent process of Eq.(24).

$$\tilde{\phi}^{k+1} = \phi^k + \tau_p (\gamma \nabla \cdot \bar{p}^k - Q_{12}(c_1, c_2)) \quad (14)$$

After getting $\tilde{\phi}^{k+1}$, we need to update the over relaxation parameter and then obtain α^{k+1} .

$$\alpha^{k+1} = \left(1 + \sqrt{4\alpha^k + 1}\right) / 2 \quad (15)$$

Then, update ϕ through intermediate parameter $\tilde{\phi}^{k+1}$ and over relaxation α^k and α^{k+1} .

$$\phi^{k+1} = \tilde{\phi}^{k+1} + (\alpha^k - 1)(\tilde{\phi}^{k+1} - \tilde{\phi}^k) / \alpha^{k+1} \quad (16)$$

We remain need to make a projection of ϕ^{k+1} according to Eq.(7). The rest of the steps are similar as the non-optimized NVPM. Via normal vector projection presented by Eq.(26) \vec{p}^{k+1} can be got, and according to Eq.(8) we obtain ϕ^{k+1} by threshold.

The pseudo code is shown as follows.

Algorithm 2: NVPM+ for CV model

Step 1: Initialize unknown values as ϕ^0, \vec{p}^0

Step 2: For $k \geq 1$, solve the following problems alternatively

2.1 Sub-problem 1 for ϕ^{k+1} : $\tilde{\phi}^{k+1} = \text{Arg min} \{ \varepsilon_1(\phi) = E(\phi, \vec{p}^k) \}$.

Update over relaxation parameter α : $\alpha^{k+1} = \left(1 + \sqrt{4\alpha^k + 1}\right) / 2$.

Update ϕ : $\phi^{k+1} = \tilde{\phi}^{k+1} + (\alpha^k - 1)(\tilde{\phi}^{k+1} - \tilde{\phi}^k) / \alpha^{k+1}$,

$$\phi^{k+1} = \text{Max}(\text{Min}(\phi^{k+1}, 1), 0).$$

2.2 Sub-problem 2 for \vec{q}^{k+1} : $\vec{p}^{k+1} = \text{Arg min} \{ \varepsilon_2(\vec{p}) = E(\phi^{k+1}, \vec{p}) \}$.

2.3 Threshold ϕ^{k+1} : $\phi^{k+1} = \begin{cases} 1 & \tilde{\phi}^{k+1} > \alpha = 0.5 \\ 0 & \text{otherwise} \end{cases}$.

Step 3: The overall loop will be terminated if the stopping criteria (described in section 3) are satisfied.

4. Numerical experiments and analysis

In this section, the numerical results of our proposed methods are applied on some real cases and they will be compared with different methods (Gradient Descent Method (GDM), Dual Method (DM), Split-Bregman Method (SBM), Alternating Direction Method of Multipliers (ADMM)) to demonstrate the effectiveness and efficiency of our methods. All the experiments are operated on the same platform (Matlab7.8) on a PC(Intel(R), Core(TM), CPU i7 2. 60GHz) . The same initial contours and initiations of variables for all the methods in each experiment are used in order to have a relatively neutral criterion for comparison.

As described in [22] the iterations need to be terminated when the following criteria are satisfied:

i We need to monitor the constraints errors in iterations:

$$(R_1^k) = \left(\|\vec{R}_1^k\|_{L^1} / \|\vec{R}_1^0\|_{L^1} \right) \quad (17)$$

with

$$\vec{R}_1^k = \vec{p}^k - \nabla \phi^k / |\nabla \phi^k|$$

ii During iterating, the relative errors of the solution ϕ^k should be noticed. They should reduce to a sufficiently small level:

$$\|\phi^k - \phi^{k-1}\|_{L^1} / \|\phi^{k-1}\|_{L^1} \tag{18}$$

iii The relative energy error can be chosen as stopping criterion

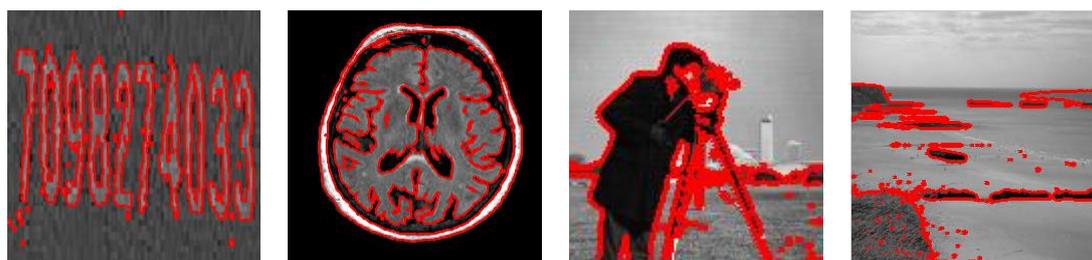
$$|E^{k+1} - E^k| / |E^k| \leq \varepsilon \tag{19}$$

where, E^k is the energy value. The computation stops automatically when $|E^{k+1} - E^k| / |E^k|$ is less than a predefined tolerance, which means the energy approaches its steady state. Until now, the proposed NVPM and NVPM+ are completed and the effectiveness will be given with extensive experiments in the following.

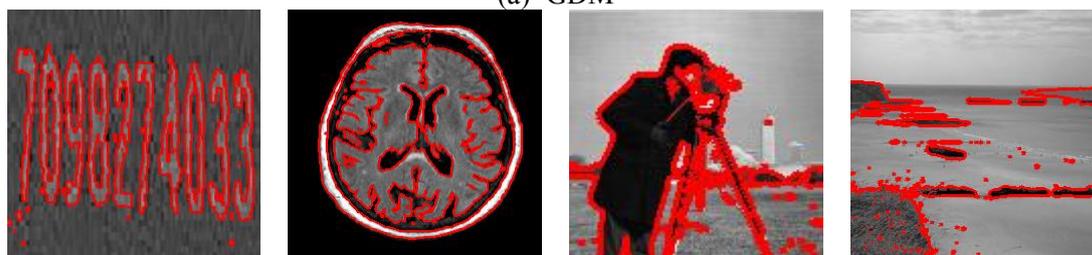


Figure 1. Images for segmentation

Four images to be segment are presented in Fig.1. There are blurred image, MRI image, character image and nature scene image.



(a) GDM



(b) DM

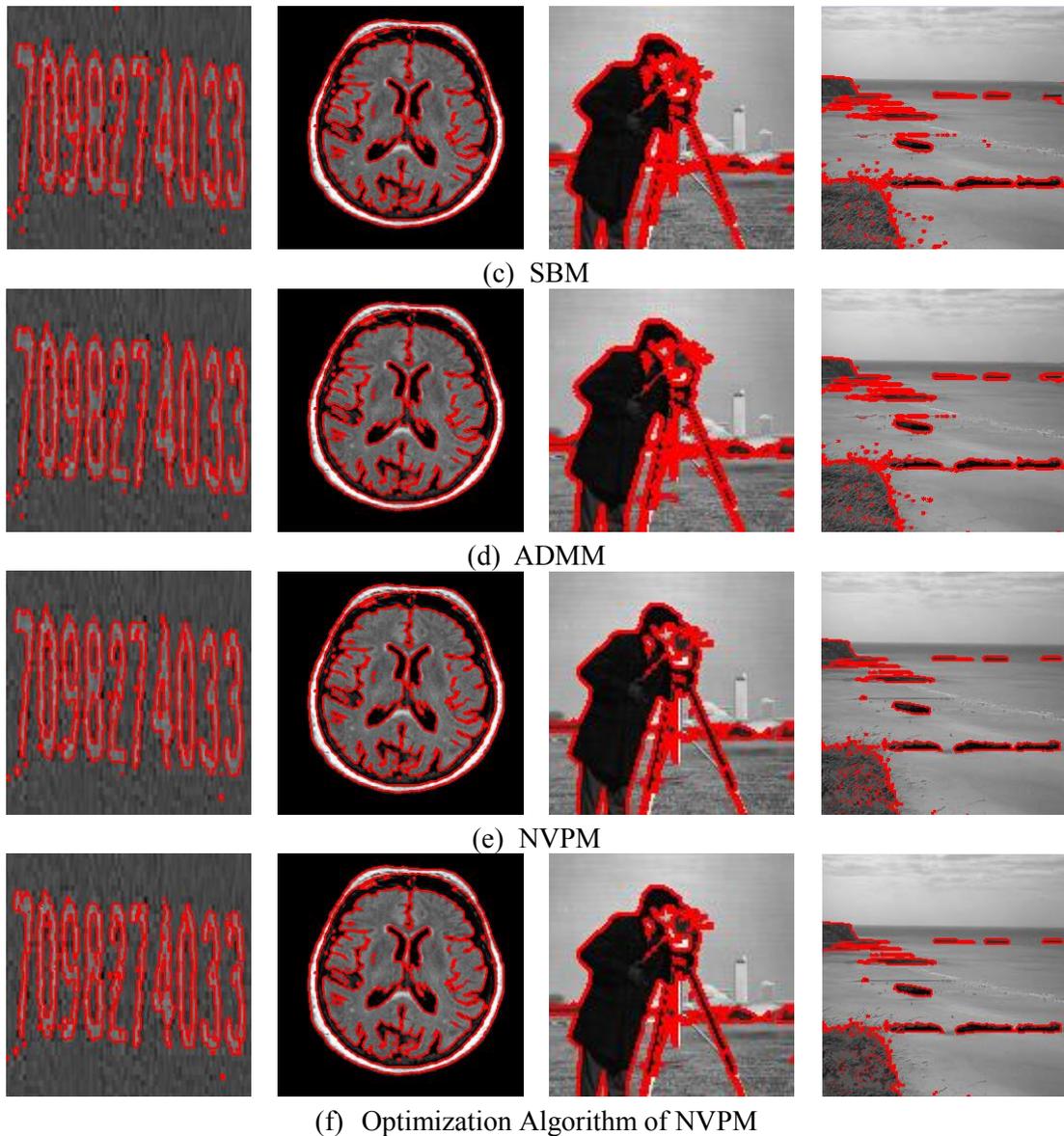
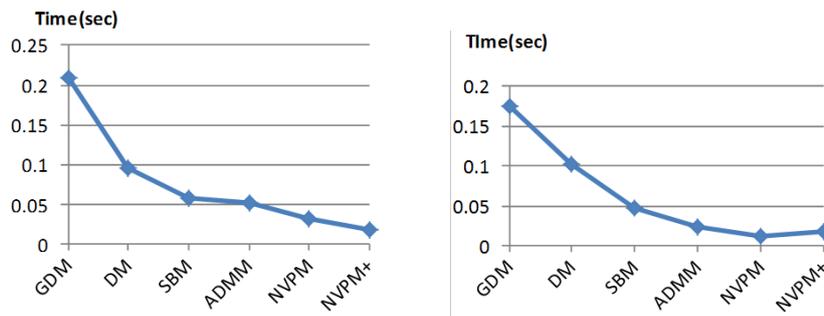


Figure 2. Segmentation results with different methods

Fig.2 (a)-(f) shows the segmentation results of GDM, DM, SBM, ADMM, NVPM model and its optimization. From the experimental results, we can see that each method can achieve better segmentation results.



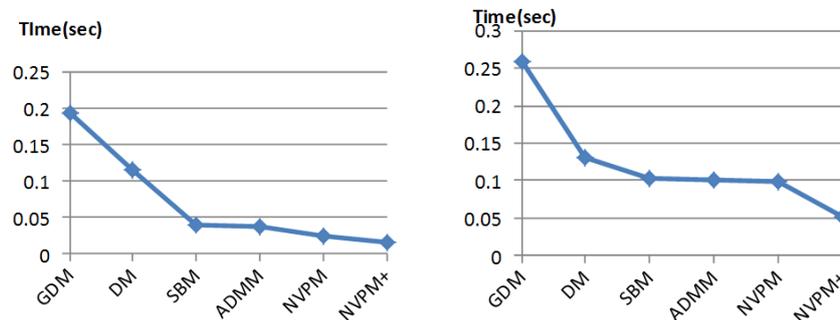


Figure 3. Comparisons of computational time with different methods

Comparisons of computational time using different methods for four kinds of images are given in Fig.3. According to the results of numerical experiments and comparing to other algorithm, no matter optimization or not, the computation rate of our method is always better than DM, GDM, SBM and ADMM. Among them, the NVPM optimization algorithm is the fastest, followed by the non-optimized NVPM, SBM and ADMM are equivalent and better than DM and GDM, the lowest calculation rate of GDM. The advantages of our method in terms of computational speed are verified through multiple sets of experiments.

5. Conclusions

In this paper, normal vector projection method used for global convex active contour model is proposed. This method with a simple structure uses the properties of introduction of auxiliary variables to solve energy functional directly. The calculation procedure is simplified to a large extent, and the computational efficiency is accelerated at the same time. The comparison of results indicate that our approach owns good enough effects and it is a good way to efficiently minimize the difficult functional. In addition, the method can also be optimized by acceleration algorithm and good results have been obtained. It is supposed to yield shorter running time while the quality of results is identical. Our method can also be applied into surface restoration and segmentation, motion segmentation and medical image 3D reconstruction in the future work.

References

- [1] Aubert G 2001 *Mathematical Problems in Image Processing: Partial Differential Equations and the Calculus of Variations*. New York: Springer
- [2] Osher S 2003 *Geometric level set methods in imaging, vision, and graphics*. New York: Springer-Verlag
- [3] Paragios N 2006 *Handbook of mathematical models in computer vision*. New York: Springer
- [4] Chan T 2005 *Image Processing And Analysis: Variational, PDE, Wavelet, And Stochastic Methods*. Philadelphia: SIAM Society for Industrial and Applied Mathematics.
- [5] Tai X C 2007 *Image processing based on partial differential equations. Proceedings of the international conference on PDE-based image processing and related inverse problems* CMA Oslo Norway August 8–12 2005 Mathematics & Visualization
- [6] Chan T F 2001 *Active contours without edges*. IEEE Transactions on Image Processing A Publication of the IEEE Signal Processing Society, 10(2), 266-277

- [7] Mumford D 1989 *Optimal approximations by piecewise smooth functions and associated variational problems*. Communications on Pure and Applied Mathematics, 42(5):577-685
- [8] Zhao H 1996 *A variational level set approach to multiphase motion*. Journal of Computational Physics, 127(1):179-195
- [9] Vese L 2002 *A multiphase level set framework for image segmentation using the mumford and shah model*. International Journal of Computer Vision, 50(3):271-293
- [10] Liu J 2011 *A fast segmentation method based on constraint optimization and its applications: Intensity inhomogeneity and texture segmentation*. Pattern Recognition, 44(9):2093–2108.
- [11] Bertalmio M 2001 *Variational problems and partial differential equations on implicit surfaces*. Journal of Computational Physics 174(2):759-780
- [12] Chan T F 2001 *Active contours without edges*. IEEE Transactions on Image Processing 10(2): 266-277
- [13] Lie J 2006 *A binary level set model and some applications to Mumford-Shah image segmentation*. IEEE Transactions on Image Processing 15(5): 1171-1181
- [14] Bresson X 2007 *Fast global minimization of the active contour/snake model*. Journal of Mathematical Imaging and Vision 28(2):151-167
- [15] Rudin L 1992 *Nonlinear total variation based noise removal algorithms*. Physica D: Nonlinear Phenomena 60(1): 259-268.
- [16] Chambolle A 2004 *An algorithm for total variation minimization and applications*. Journal of Mathematical Imaging and Vision, 20(1-2): 89-97
- [17] Goldstein T. 2008 *The split Bregman method for L1 regularized problems*. SIAM Journal on Imaging Sciences, 2(2), 323-343
- [18] Goldstein T 2014 *Fast Alternating Direction Optimization Methods*. SIAM Journal on Imaging Sciences, 7(3):1588-1623.
- [19] Pan Z K 2009 *A Variational Level Set Method of Multiphase Segmentation for 3D Images*. Chinese Journal of Computers 32(12):2464-2474.
- [20] Wang Q 2010 *Split-Bregman Method and Dual Method for Multiphase Image Segmentation*. Journal of Computer-aided Design & Computer Graphics 22(9): 1561-1569
- [21] Nesterov Y 1983 *A method of solving a convex programming problem with convergence rate $\mathcal{O}(1/k^2)$* . Soviet Math(27):372-376
- [22] Tai X C 2014 *Fast numerical schemes related to curvature minimization: a brief and elementary review*. UCLA CAM Report: 14-40