

Digraphs Structures Corresponding to the Analogue Realisation of Fractional Continuous-Time System

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Abstract. This paper presents a method of the determination of a minimal realisation of the fractional continuous-time linear system. For the proposed method, a digraph-based algorithm was constructed. In this paper, we have shown how we can perform the transfer matrix using electrical circuits consisting of resistances, capacitance and source voltages. We have also shown how after using the constant phase element method we can realize such a system. The proposed method was discussed and illustrated with some theoretical and practical numerical examples.

1. Introduction

In the last two decades, integral and differential calculus of an fractional order has become a subject of great interest in different areas of physics, biology, economics and other sciences. Fractional calculus is a generalization of traditional integer order integration and differentiation actions onto non-integer order. The idea of such a generalization was mentioned in 1695 by Leibniz and L'Hospital. The first definition of the fractional derivative was introduced by Liouville and Riemann at the end of the 19th century [1]. However, only just in the late 60' of the 20th century, this idea drew attention of engineers. Fractional calculus was found to be a very useful tool for modelling the behaviour of many materials and systems. Mathematical fundamentals of fractional calculus are given in the monographs [1–6]. Some other applications of fractional-order systems can be found in [7–12].

A lot of problems arise in one-dimensional systems, and they remain not completely solved; for example: realisation problem, minimal realisation problem, abilities problem, etc. As stated in the paper [13] Professor Benvenuti and Professor Farina *First of all, it is not clear what kind of mathematical "instruments" should be used to effectively tackle the minimality problem.* In many research studies in the area of the realisation problem or minimal realisation problem, we can find a canonical form of the system, i.e. constant matrix form, which satisfies the system described by the transfer function. Absolutely, in general we have a lot of solutions. This means that we can find many sets of matrices which fit into the system transfer function. In [14, 15] first attempts to determine a set of solutions for finding a set of possible realisations of the characteristic polynomial was proposed. If we add fractionality to the minimal positive realisation problem, then the realisation problem becomes much more complicated. In bibliography [3, 16–20] we can find some methods of determination of a positive realisation in the canonical form. In [21–24]



the first proposition of a solution of the positive fractional realisation problem based on digraph theory was presented. The proposed method determines all possible digraph structures which correspond to the characteristic polynomial.

The main purpose of this paper is to present a method based on digraphs theory for computation of a minimal realisation of a given proper transfer function of all-pole continuous-time fractional linear systems. Conditions for the existence of a minimal realisation of a given proper transfer function will be established. For the determined realisation using constant phase element method an electrical analogue model will be created. Finally, a comparison of analogue models with theoretical models using a state diagram method and state-space method will be performed. It should be noted that this work is the next step in research on the realisation of the electrical circuits by using a digraph theory started in the publication [25] and [26].

This work has been organised as follows. Section 2 presents: some notations and basic definitions of a digraph theory, constant phase element method and basic properties of the fractional continuous-time system. Then, in Section 3 an algorithm for determination of a minimal realisation of the fractional continuous-time system was presented and electrical analogue network model using a constant phase element is given. Finally, in Section 4 theoretical and practical models are compared and concluding remarks, open problems and bibliography positions are given.

2. Background

2.1. Digraphs

A directed graph (or just digraph) \mathcal{D} consists of a non-empty finite set $\mathbb{V}(\mathcal{D})$ of elements called vertices and a finite set $\mathbb{A}(\mathcal{D})$ of ordered pairs of distinct vertices called arcs [27]. We call $\mathbb{V}(\mathcal{D})$ the vertex set and $\mathbb{A}(\mathcal{D})$ the arc set of digraph \mathcal{D} . We will often write $\mathcal{D} = (\mathbb{V}, \mathbb{A})$ which means that \mathbb{V} and \mathbb{A} are the vertex set and arc set of \mathcal{D} , respectively. The order of \mathcal{D} is the number of vertices in \mathcal{D} . The size of \mathcal{D} is the number of arcs in \mathcal{D} . For an arc (v_1, v_2) the first vertex v_1 is its tail and the second vertex v_2 is its head.

There are two well-known methods of representation of digraph: list and incidence matrix. In this paper we are using incidence matrix to represent all digraphs. Method of constructing digraphs by this method is presented for example in [27], [28] or [29]. There exists \mathcal{A} -arc from vertex v_j to vertex v_i if and only if the (i, j) -th entry of the matrix \mathbf{A} is non-zero. There exists \mathcal{B} -arc from source s_m to vertex v_i if and only if the (i, m) -th entry of the matrix \mathbf{B} is non-zero. Let be given the positive system single input described by the following matrices

$$(\mathbf{A}, \mathbf{B}) = \left(\begin{array}{c|ccc} v_i \backslash v_j & v_1 & v_2 & v_3 \\ \hline v_1 & \textcolor{red}{1} & 0 & \textcolor{red}{1} \\ v_2 & \textcolor{red}{1} & 0 & \textcolor{red}{1} \\ v_3 & 0 & \textcolor{red}{1} & 0 \end{array} , \begin{array}{c|c} v_i \backslash s_m & s_1 \\ \hline v_1 & \textcolor{green}{1} \\ v_2 & 0 \\ v_3 & 0 \end{array} \right) \quad (1)$$

we can draw one-dimensional digraph \mathcal{D} consisting of vertices v_1, v_2, v_3 and source s_1 . One-



Figure 1: One-dimensional digraph

dimensional digraph corresponding to system (1) is presented in Figure 1.

We present below some basic notions from graph theory which are used in further considerations. A walk in a digraph \mathcal{D} is a finite sequence of arcs in which every two vertices

v_i and v_j are adjacent or identical. A walk in which all of the arcs are distinct is called a path. The path that goes through all vertices is called a finite path. If the initial and the terminal vertices of the path are the same, then the path is called a cycle. More information about use digraph theory in dynamical system is given in [30], [31].

2.2. Constant Phase Element Method

The constant phase element method relies on construction of the model consisting of resistors and capacitors. In relation to the method described in paper [32] in method presented in [33] a newer network model with correcting elements R_p and C_p (Figure 2) was constructed.

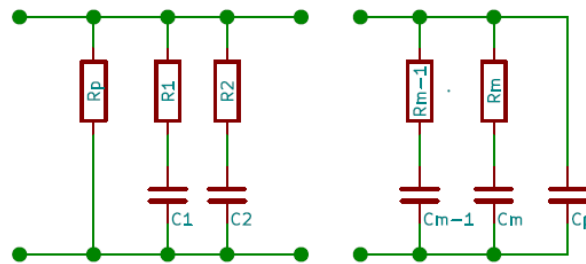


Figure 2: Network model with correcting elements

The impedance of the ideal constant phase element is defined as $Z(s) = Ds^\alpha$. For $s = j\omega$ we have $Z(j\omega) = D(j\omega)^\alpha = D\omega^\alpha(\cos \varphi + j \sin \varphi)$. The model parameters are selected according to the following procedure:

- Set the following parameters: resistance R_1 , conductance C_1 , number of lines m , argument value φ and oscillates around it with amplitude $\Delta\varphi$;
- Determine the following parameters:

$$\alpha = \frac{\varphi}{90}, \quad ab = \frac{0.24}{1 + \Delta\varphi}, \quad a = 10^{\alpha \log ab}, \quad b = \frac{ab}{a}, \quad R_p = R_1 \frac{1-a}{a}, \quad C_p = C_1 \frac{b^m}{1-b},$$

$$R_k = R_1 a^{k-1}, \quad C_k = C_1 b^{k-1}, \quad k = 1, 2, \dots, m. \quad (2)$$

2.3. Fractional Order System

Let us consider the continuous-time fractional linear system described by state-space equations:

$$\begin{aligned} {}_0\mathfrak{D}_t^\alpha x(t) &= \mathbf{A}x(t) + \mathbf{B}u(t), \quad 0 < \alpha \leq 1, \\ y(t) &= \mathbf{C}x(t) + \mathbf{D}u(t) \end{aligned} \quad (3)$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$ are the state, input and output vectors respectively and $\mathbf{A} \in \mathbb{R}^{n \times n}$, $\mathbf{B} \in \mathbb{R}^{n \times m}$, $\mathbf{C} \in \mathbb{R}^{p \times n}$ and $\mathbf{D} \in \mathbb{R}^{p \times m}$.

The following Caputo definition of the fractional derivative will be used:

$${}_a^C \mathfrak{D}_t^\alpha = \frac{d^\alpha}{dt^\alpha} = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau, \quad (4)$$

where $\alpha \in \mathbb{R}$ is the order of a fractional derivative, $f^{(n)}(\tau) = \frac{d^n f(\tau)}{d\tau^n}$ and $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$ is the gamma function.

Theorem 1 *The Laplace transform of the derivative-integral (4) has the form*

$$\mathcal{L} \left[\begin{smallmatrix} C \\ 0 \end{smallmatrix} \mathcal{D}_t^\alpha \right] = s^\alpha F(s) - \sum_{k=1}^n s^{\alpha-k} f^{(k-1)}(0^+).$$

The proof of the Theorem 1 is given in [3].

After using the Laplace transform to (3), Theorem 1 and taking into account

$$\begin{aligned} X(s) &= \mathcal{L}[x(t)] = \int_0^\infty x(t)e^{-st}dt, \\ \mathcal{L}[\mathcal{D}^\alpha x(t)] &= s^\alpha X(s) - s^{\alpha-1}x_0 \end{aligned} \quad (5)$$

we obtain:

$$\begin{aligned} X(s) &= [\mathbf{I}_n s^\alpha - \mathbf{A}]^{-1} [s^{\alpha-1}x_0 + \mathbf{B}U(s)], \\ Y(s) &= \mathbf{C}X(s) + \mathbf{D}U(s), \quad U(s) = \mathcal{L}[u(t)]. \end{aligned} \quad (6)$$

After using (6) we can determine the transfer matrix of the system in the following form:

$$\mathbf{T}(s) = \mathbf{C} [\mathbf{I}_n s^\alpha - \mathbf{A}]^{-1} \mathbf{B} + \mathbf{D} \in \mathbb{R}^{p \times m}(s). \quad (7)$$

Matrices \mathbf{A} , \mathbf{B} , \mathbf{C} and \mathbf{D} are called a realisation of the transfer matrix $\mathbf{T}(s)$ if they satisfy the equality (7). The realisation is called minimal if the dimension of the state matrix \mathbf{A} is minimal among all possible realisations of $\mathbf{T}(s)$.

Our task is the following: *For a given transfer function (7) determine a minimal realisation of the system (3) using one-dimensional \mathcal{D} digraphs theory. The dimension of the system must be the minimal among possible. For a designed realisation, build the analogue model and compare it with the theoretical model.*

3. Main results

The solution to the minimal realisation problem will be presented using special cases of the single-input single-output (SISO) system described by the transfer function of an all-pole system (which is the transfer function with only poles) in the following form:

$$T(s) = \frac{b_n s^{\alpha n}}{s^{\alpha n} - \sum_{i=0}^{n-1} a_i s^{\alpha i}} = \frac{b_n s^{\alpha n}}{s^{\alpha n} - a_{n-1} s^{\alpha(n-1)} - a_{n-2} s^{\alpha(n-2)} - \dots - a_1 s^\alpha - a_0}. \quad (8)$$

3.1. Theoretical Realisations

Using the transfer function (8) we have:

$$\mathbf{D} = \lim_{s^\alpha \rightarrow \infty} T(s) = [b_n] \quad (9)$$

since $\lim_{s^\alpha \rightarrow \infty} [\mathbf{I}s^\alpha - \mathbf{A}] = 0$. The strictly proper transfer function is given by the equation:

$$T_{sp}(s) = T(s) - \mathbf{D} = \frac{\tilde{b}_{n-1} s^{\alpha(n-1)} + \tilde{b}_{n-2} s^{\alpha(n-2)} + \dots + \tilde{b}_1 s^\alpha + \tilde{b}_0}{s^{\alpha n} - a_{n-1} s^{\alpha(n-1)} - a_{n-2} s^{\alpha(n-2)} - \dots - a_1 s^\alpha - a_0} \quad (10)$$

where $\tilde{b}_{n-i} = b_n a_{n-i}$ for $i = 1, \dots, n$.

The task of determining minimal realisations using digraphs theory, can be divided into two parts:

- (a) **Part 1: Determine state matrix \mathbf{A} .** To determine state matrix, we must multiplay the denominator of the strictly proper transfer function by $s^{-\alpha \cdot n}$ and make the decomposition of the characteristic polynomial

$$d(s) = 1 - a_{n-1}s^{-\alpha} - a_{n-2}s^{-2\alpha} - \dots - a_1s^{\alpha(1-n)} - a_0s^{-\alpha n} \quad (11)$$

into a set of monomials. Using Proposition 1 presented in paper [34] we create all possible digraph representations. Then using Theorem 1 presented in paper [14] and [15] we can create all digraph realisations of the characteristic polynomials. Each digraph corresponding to a characteristic polynomial must satisfy two conditions. The first condition relates to the existence of the common part in the digraph (blue vertex), the second condition relates to non-existence of additional cycles in the digraph. One of the digraph structure corresponding to characteristic polynomial (11) presented in Figure 3 and state matrix corresponding to digraph has the form:

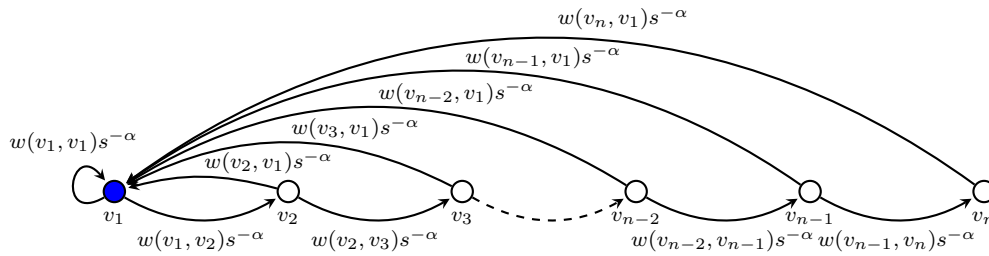


Figure 3: One of the possible realisation of the characteristic polynomial $d(s)$

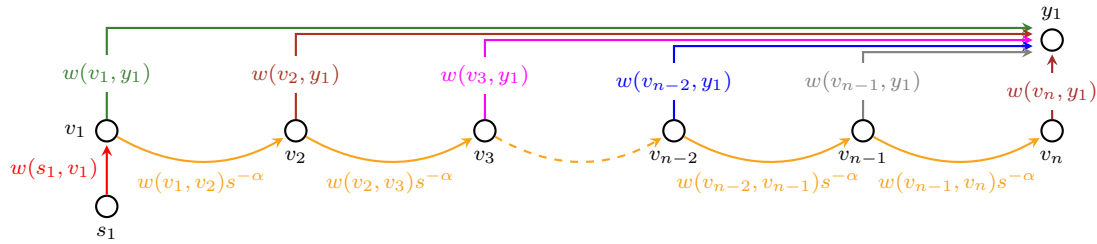
$$\mathbf{A} = \begin{matrix} v_i \backslash v_j \\ \begin{matrix} v_1 & v_2 & v_3 & \dots & v_{n-2} & v_{n-1} & v_n \end{matrix} \\ \begin{matrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_{n-2} \\ v_{n-1} \\ v_n \end{matrix} \end{matrix} \begin{bmatrix} w(v_1, v_1) & w(v_2, v_1) & w(v_3, v_1) & \dots & w(v_{n-2}, v_1) & w(v_{n-1}, v_1) & w(v_n, v_1) \\ w(v_1, v_2) & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & w(v_2, v_3) & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & w(v_{n-2}, v_{n-1}) & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & w(v_{n-1}, v_n) & 0 \end{bmatrix} \quad (12)$$

- (b) **Part 2: Determine matrices \mathbf{B} and \mathbf{C} .** To determine matrices, we must multiply the nominator of the transfer function (10) by $s^{-\alpha(n-1)}$. Then we obtain the following polynomial

$$n(s) = \tilde{b}_{n-1} + \tilde{b}_{n-2}s^{-\alpha} + \dots + \tilde{b}_1s^{-\alpha \cdot n} + \tilde{b}_0s^{-\alpha(n-1)} \quad (13)$$

and expand the digraph created in the first step. We have two possible structures:

- **Structure 1:** We connect source vertex s corresponding to matrix \mathbf{B} with vertex v_1, \dots, v_n and output vertex y corresponding to matrix \mathbf{C} with vertex v_i belonging to a set of common parts of a digraph. This digraphs structure was considered in the paper [34].
- **Structure 2:** We connect source vertex s corresponding to matrix \mathbf{B} with vertex v_i belonging to a set of common parts of a digraph, and output vertex y corresponding to matrix \mathbf{C} with vertex v_1, \dots, v_n (Figure 4). This digraphs structure was considered in the paper [34].

Figure 4: One-dimensional digraph corresponding to polynomial $n(s)$

In the future consideration, we will investigate only the Structure 2. From digraph presented in Figure 4, we can write the set of the equations which we compare with the same power of the polynomial (13). We obtain the following equality:

$$\begin{cases} s^{-\alpha} & w(s_1, v_1) \cdot w(v_1, y_1) & = \tilde{b}_{n-1} \\ s^{-2\alpha} & w(s_1, v_1) \cdot w(v_1, v_2) \cdot w(v_2, y_1) & = \tilde{b}_{n-2} \\ & w(s_1, v_1) \cdot w(v_1, v_2) \cdot w(v_2, v_3) \cdot w(v_3, y_1) & = \tilde{b}_{n-3} \\ \vdots & \vdots & \vdots \\ s^{-\alpha(n-3)} & w(s_1, v_1) \cdot w(v_1, v_2) \cdot w(v_2, v_3) \cdot \dots \cdot w(v_{n-2}, y_1) & = \tilde{b}_2 \\ s^{-\alpha(n-2)} & w(s_1, v_1) \cdot w(v_1, v_2) \cdot w(v_2, v_3) \cdot \dots \cdot w(v_{n-2}, v_{n-1}) \cdot w(v_{n-1}, y_1) & = \tilde{b}_1 \\ s^{-\alpha(n-1)} & w(s_1, v_1) \cdot w(v_1, v_2) \cdot w(v_2, v_3) \cdot \dots \cdot w(v_{n-2}, v_{n-1}) \cdot w(v_{n-1}, v_n) \cdot w(v_n, y_1) & = \tilde{b}_0 \end{cases} \quad (14)$$

After solving the set of the equation (14), we can write **B** and **C** matrix in the following form:

$$\underbrace{\begin{matrix} v_i \setminus s \\ v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_{n-1} \\ v_n \end{matrix} \begin{bmatrix} s_1 \\ w(s_1, v_1) \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{B} \in \mathbb{R}^{n \times 1}}, \underbrace{\begin{matrix} y \setminus v_j \\ y_1 \end{matrix} \begin{bmatrix} v_1 & v_2 & v_3 & \dots & v_{n-1} & v_n \\ w(v_1, y_1) & w(v_2, y_1) & w(v_3, y_1) & \dots & w(v_{n-1}, y_1) & w(v_n, y_1) \end{bmatrix}}_{\mathbf{C} \in \mathbb{R}^{1 \times n}} \quad (15)$$

where:

$$\begin{aligned} w(v_1, y_1) &= \frac{\tilde{b}_{n-1}}{w(s_1, v_1)}, w(v_2, y_1) = \frac{\tilde{b}_{n-2}}{w(s_1, v_1)w(v_1, v_2)}, w(v_3, y_1) = \frac{\tilde{b}_{n-3}}{w(s_1, v_1)w(v_1, v_2)w(v_2, v_3)}, \\ w(v_{n-1}, y_1) &= \frac{\tilde{b}_1}{w(s_1, v_1)w(v_1, v_2)w(v_2, v_3) \dots w(v_{n-2}, v_{n-1})}, \\ w(v_n, y_1) &= \frac{\tilde{b}_0}{w(s_1, v_1)w(v_1, v_2)w(v_2, v_3) \dots w(v_{n-2}, v_{n-1})w(v_{n-1}, v_n)}. \end{aligned}$$

Example 1 Find a positive realisation of the all-pole rational positive continuous-time linear system with a proper transfer function:

$$\mathbf{T}(s) = \frac{4s^{2.1}}{s^{2.1} + 0.5s^{1.4} + 0.5s^{0.7} + 0.7}. \quad (16)$$

The matrix **D** for a given proper transfer function (16) can be computed using equation (9). In the considered example, we have:

$$\mathbf{D} = \lim_{s^\alpha \rightarrow \infty} T(s) = 4. \quad (17)$$

After using (16) and (17), we can determine strictly proper transfer function in the form:

$$T_{sp}(s) = T(s) - \mathbf{D} = \frac{-2s^{1.4} - 2s^{0.7} - 2.8}{s^{2.1} + 0.5s^{1.4} + 0.5s^{0.7} + 0.7}. \quad (18)$$

One of the possible digraph structure corresponding to transfer function (18) is presented in Figure 5. From the digraph, we can write the state matrix in the following form:

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & w(v_3, v_1) \\ w(v_1, v_2) & 0 & w(v_3, v_2) \\ 0 & w(v_2, v_3) & w(v_3, v_3) \end{bmatrix} = \begin{bmatrix} 0 & 0 & -0.7 \\ 1 & 0 & -0.5 \\ 0 & 1 & -0.5 \end{bmatrix} \quad (19)$$

and we can write a set of equations in the form:

$$\begin{cases} s^{-0.7} \begin{cases} w(s_1, v_3) \cdot w(v_3, y_1) & = -2 \\ w(s_1, v_3) \cdot w(v_3, v_2) \cdot w(v_2, y_1) + w(s_1, v_3) \cdot w(v_3, v_1) \cdot w(v_1, y_1) & = -2 \\ w(s_1, v_3) \cdot w(v_3, v_1) \cdot w(v_1, v_2) \cdot w(v_2, y_1) & = -2.8 \end{cases} \end{cases} \quad (20)$$

After solving the set of the equation (20), we can write the input and output matrix in the following form:

$$\mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ w(s_1, v_3) \end{bmatrix}, \mathbf{C} = \begin{bmatrix} 0 & \frac{4}{w(s_1, v_3)} & \frac{-2}{w(s_1, v_3)} \end{bmatrix}. \quad (21)$$

The desired realisation of the (16) is given by (17), (19), (21).

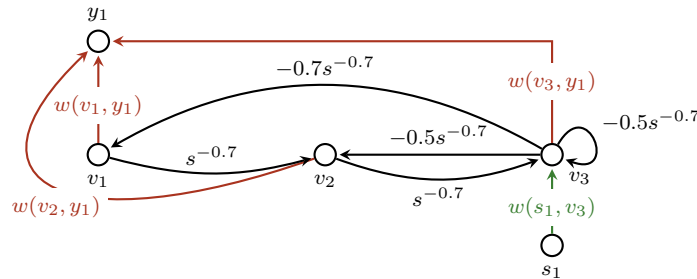


Figure 5: One of the possible realisation of the transfer function (18).

3.2. Practical Analogue Realisation

In this section, a practical realisation of the transfer using Constant Phase Element (CPE) – which was introduced in Section 2.2 – will be presented. The obtained realisation was compared with numerical simulations. The numerical simulations were performed with FOMCON toolbox [35, 36].

Using CPE method, we construct a network model consisting of resistance and capacitance. Consider two following models corresponding to $K(s) = D/s^{0.7}$ where $D \in \mathbb{R}$ is constant value, and it depends on resistance and capacitance. In considered simulations, the constant D has been determined experimentally.

- **Model 1:** Given $R_1 = 2k\Omega$, $C_1 = 1mF$, $m = 20$, $\Delta\varphi = 0.26$. For the given initial parameters we can determine model coefficients:

$$ab = \frac{0.24}{1 + \Delta\varphi} = 0.1905; \quad a = 10^{\alpha \log(ab)} = 0.3132; \quad b = \frac{ab}{a} = 0.6081;$$

$$R_p = \frac{R_1 \cdot (1 - a)}{a} = 4.3847k\Omega; \quad C_p = \frac{C_1 \cdot b^m}{1 - b} = 121.85nF.$$

Finally, the network model consists of the following elements:

– Resistors:

$$\begin{aligned} R_1 &= 2k\Omega; R_2 = 626.5\Omega; R_3 = 196.2\Omega; R_4 = 61.5\Omega; R_5 = 19.2\Omega; R_6 = 6\Omega; \\ R_7 &= 1.9\Omega; R_8 = 0.6\Omega; R_9 = 0.18\Omega; R_{10} = 0.06\Omega; R_{11} = 0.02\Omega; R_{12} = 5.7m\Omega \\ R_{13} &= 1.8m\Omega; R_{14} = 0.55m\Omega; R_{15} = 175.1\mu\Omega; R_{16} = 54\mu\Omega; R_{17} = 17\mu\Omega; \\ R_{18} &= 5.4\mu\Omega; R_{19} = 1.7\mu\Omega; R_{20} = 0.5\mu\Omega. \end{aligned}$$

– Capacitors:

$$\begin{aligned} C_1 &= 1mF; C_2 = 0.6mF; C_3 = 0.37mF; C_4 = 0.22mF; C_5 = 0.14mF; \\ C_6 &= 83\mu F; C_7 = 50\mu F; C_8 = 30\mu F; C_9 = 18.6\mu F; C_{10} = 11.4\mu F; \\ C_{11} &= 6.9\mu F; C_{12} = 4.2\mu F; C_{13} = 2.6\mu F; C_{14} = 1.6\mu F; C_{15} = 0.94\mu F; \\ C_{16} &= 574nF; C_{17} = 349nF; C_{18} = 212nF; C_{19} = 129nF; C_{20} = 78.5nF. \end{aligned}$$

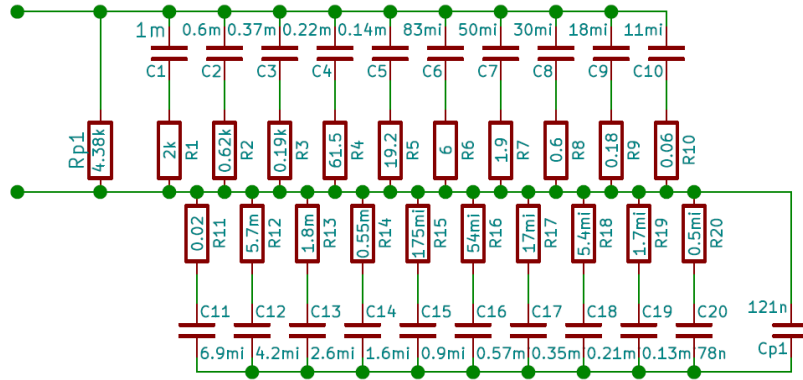


Figure 6: The first network model

In Figure 6 the network model containing determined elements has been presented. It should be noted that the complexity of the model is dependent on the number of lines. In our case, we have 22 lines: 2 lines including correcting elements (resistor R_p and capacitor C_p); 20 lines including resistors and capacitors.

After this operations, we can determine the following transmittance

$$\begin{aligned} &1.55 \cdot 10^{-131} \cdot s^{20} + 4.614 \cdot 10^{-118} \cdot s^{19} + 2.198 \cdot 10^{-105} s^{18} + 1.935 \cdot 10^{-93} s^{17} + \\ &3.227 \cdot 10^{-82} s^{16} + 1.024 \cdot 10^{-71} s^{15} + 6.187 \cdot 10^{-62} s^{14} + 7.12 \cdot 10^{-53} s^{13} + \\ &1.561 \cdot 10^{-44} s^{12} + 6.518 \cdot 10^{-37} s^{11} + 5.184 \cdot 10^{-30} s^{10} + 7.854 \cdot 10^{-24} s^9 + \\ &2.266 \cdot 10^{-18} s^8 + 1.246 \cdot 10^{-13} s^7 + 1.304 \cdot 10^{-09} s^6 + 2.601 \cdot 10^{-06} s^5 + \\ &0.0009876 s^4 + 0.07136 s^3 + 0.9766 s^2 + 2.471 s + 1 \\ G_1 = &\frac{1.889 \cdot 10^{-138} s^{21} + 9.894 \cdot 10^{-125} s^{20} + 7.877 \cdot 10^{-112} s^{19} + 1.145 \cdot 10^{-99} s^{18} + \\ &3.141 \cdot 10^{-88} s^{17} + 1.639 \cdot 10^{-77} s^{16} + 1.629 \cdot 10^{-67} s^{15} + 3.084 \cdot 10^{-58} s^{14} + \\ &1.112 \cdot 10^{-49} s^{13} + 7.634 \cdot 10^{-42} s^{12} + 9.985 \cdot 10^{-35} s^{11} + 2.488 \cdot 10^{-28} s^{10} + \\ &1.181 \cdot 10^{-22} s^9 + 1.067 \cdot 10^{-17} s^8 + 1.837 \cdot 10^{-13} s^7 + 6.026 \cdot 10^{-10} s^6 + \\ &3.764 \cdot 10^{-07} s^5 + 4.474 \cdot 10^{-05} s^4 + 0.00101 s^3 + 0.004264 s^2 + \\ &0.003115 s + 0.0002281 \end{aligned} \quad (22)$$

corresponding to the considered model.

- **Model 2:** Given $R_1 = 10k\Omega$, $C_1 = 1mF$, $m = 10$, $\Delta\varphi = 0.26$. For the given initial parameters we can determine model coefficients:

$$ab = \frac{0.24}{1 + \Delta\varphi} = 0.1905; \quad a = 10^{\alpha \log(ab)} = 0.3132; \quad b = \frac{ab}{a} = 0.6081;$$

$$R_p = \frac{R_1 \cdot (1 - a)}{a} = 43.847k\Omega; \quad C_p = \frac{C_1 \cdot b^m}{1 - b} = 17.632\mu F.$$

Finally, the network model consists of the following elements:

– Resistors:

$$R_1 = 2k\Omega; R_2 = 6.26k\Omega; R_3 = 1.96k\Omega; R_4 = 614.75\Omega; R_5 = 192.57\Omega;$$

$$R_6 = 60.32\Omega; R_7 = 18.89\Omega; R_8 = 5.91\Omega; R_9 = 1.85\Omega; R_{10} = 0.58\Omega.$$

– Capacitors:

$$C_1 = 1mF; C_2 = 0.61mF; C_3 = 0.37mF; C_4 = 0.22mF; C_5 = 0.14mF;$$

$$C_6 = 83.13\mu F; C_7 = 50.55\mu F; C_8 = 30.74\mu F; C_9 = 18.69\mu F; C_{10} = 11.36\mu F.$$

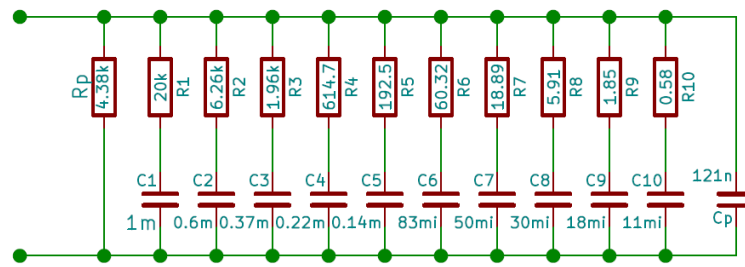


Figure 7: The second network model

In Figure 7 the network model containing determined elements has been presented. In this case, we have 12 lines: 2 lines including correcting elements (resistor R_p and capacitor C_p); 10 lines including resistors and capacitors.

After this operations, we can determine the following transmittance

$$G_2 = \frac{4.01 \cdot 10^{-20}s^{10} + 7.504 \cdot 10^{-15}s^9 + 2.247 \cdot 10^{-10}s^8 + 1.244 \cdot 10^{-6}s^7 + 0.001304s^6 + 0.26s^5 + 9.876s^4 + 71.36s^3 + 97.66s^2 + 24.71s + 1}{7.07 \cdot 10^{-25}s^{11} + 2.328 \cdot 10^{-19}s^{10} + 1.165 \cdot 10^{-14}s^9 + 1.064 \cdot 10^{-10}s^8 + 1.836 \cdot 10^{-7}s^7 + 6.025 \cdot 10^{-5}s^6 + 0.003763s^5 + 0.04474s^4 + 0.101s^3 + 0.04264s^2 + 0.003115s + 2.281 \cdot 10^{-5}} \quad (23)$$

corresponding to considered model.

4. Comparison and Discussion

To compare the theoretical model with determined CPE network models in Simulink, test environment (Figure 8) has been built. It consists of three areas: the first represents a model as a state diagram system (Figure 9(a)); the second represents model as a state-space system and the third represents CPE network model (Figure 9(b)).

Let us consider the following two cases:

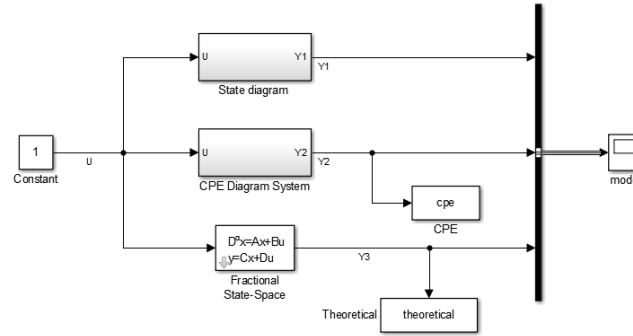


Figure 8: Test environment

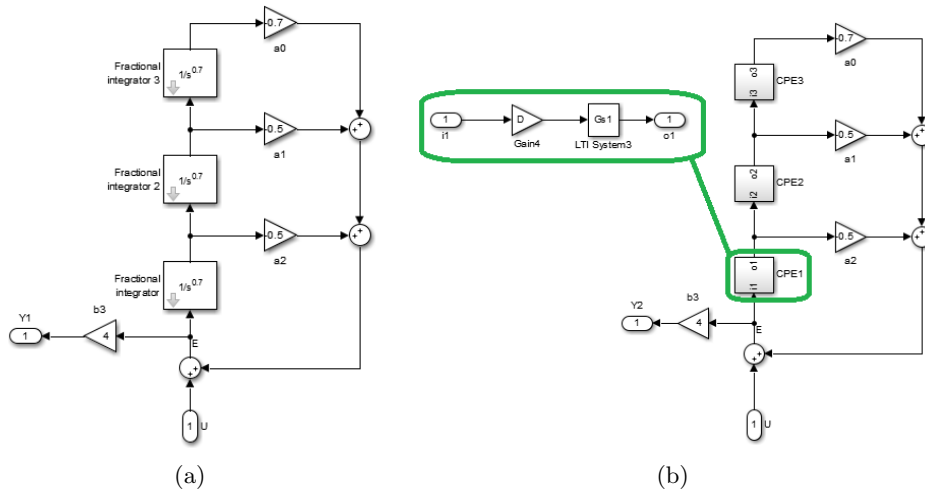


Figure 9: Models: (a) state diagram (b) CPE model.

- **Case 1:** Figure 10(a) presents a step response of the state-space model (yellow line), state diagram model (red line) and the first network model (green line). As we can see the fit of the model is not the best. We can notice quite clear difference between the theoretical and physical model. Particularly, it is clearly visible in the graph of mean square error presented in Figure 11 (blue line).
- **Case 2:** Figure 10(b) presents step response of the state-space model (yellow line), state diagram model (red line) and the second network model (green line). As we can see the fit of the model is very good. There are no significant differences between the theoretical and physical model. Particularly, it is clearly visible in the graph of mean square error presented in Figure 11 (red line).

From the above observation, we can write a very important conclusion. Before we start physically building an analogue model, we should carry out a series of simulations using Simulink, for example. In the paper [33] it has been proposed that for a higher value of the parameter m the model is more accurate. It appears, that the quality of the model and the adjustment to the theoretical model are influenced by many parameters. For example in Model 1 we used 20 lines and the fit is worse than in the case of Model 2 in which we used only 10 lines. Additionally, the quality of the model is affected by the way of selection of initial parameters: R_1 , C_1 , and $\Delta\varphi$.

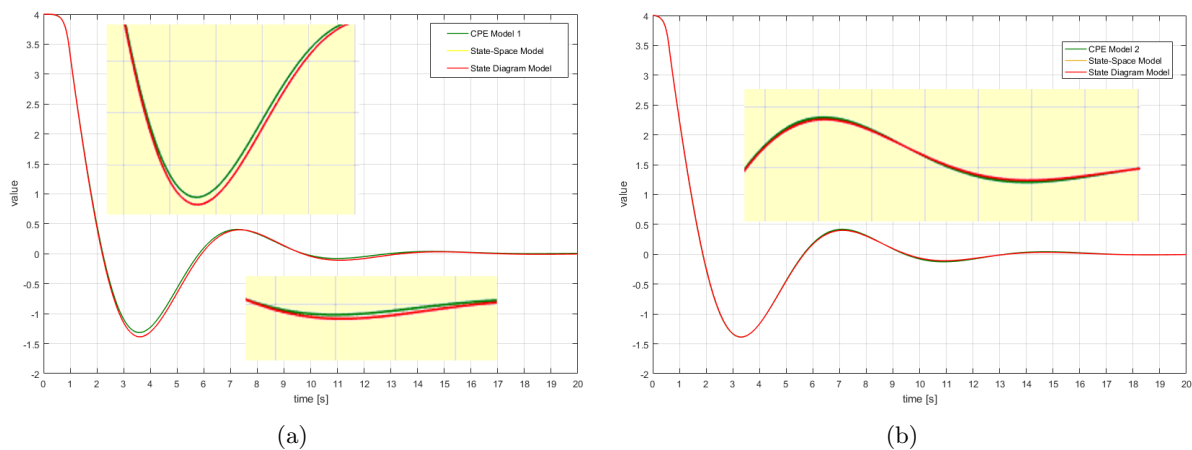


Figure 10: Step response of the state-space model, state diagram model and (a) the first CPE model (b) the second CPE model.

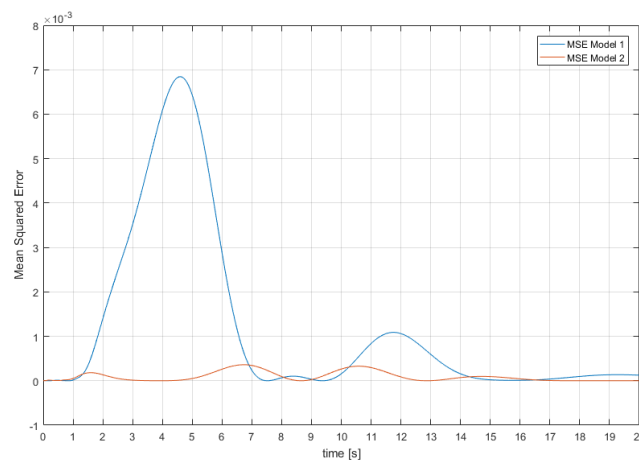


Figure 11: Mean square error

5. Concluding Remarks

The paper presents a method, based on the one-dimensional digraph theory, for finding the realisations of a given proper transfer function of all-pole continuous-time fractional linear systems. For the obtained realisation, we have constructed two models based on constant phase element method. The first model consists from 20 lines and the second model 10 lines. Each line consists of properly selected resistors and capacitors. Prepared theoretical models (state-space model and space diagram model) and practical model were compared in terms of the quality of the model and its fit.

Further work includes extension of the digraph based algorithm to find a broader class of electrical circuits corresponding to the transfer function and practical construction of the analogue models for different values of α .

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