Output feedback model matching in linear impulsive systems with control feedthrough: a structural approach

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Abstract. This paper investigates the problem of structural model matching by output feedback in linear impulsive systems with control feedthrough. Namely, given a linear impulsive plant, possibly featuring an algebraic link from the control input to the output, and given a linear impulsive model, the problem consists in finding a linear impulsive regulator that achieves exact matching between the respective forced responses of the linear impulsive plant and of the linear impulsive model, for all the admissible input functions and all the admissible sequences of jump times, by means of a dynamic feedback of the plant output. The problem solvability is characterized by a necessary and sufficient condition. The regulator synthesis is outlined through the proof of sufficiency, which is constructive.

1. Introduction

Linear impulsive systems form a special class of hybrid dynamical systems, featuring a continuous-time linear behavior (governed by the so-called flow dynamics) subject to state discontinuities occurring at isolated point of the time axis (governed by the so-called jump dynamics). Linear impulsive systems have recently attracted a huge amount of research interest, mainly because they are particularly effective in modeling complex phenomena as well as composite systems — see, e.g., [1]. Nowadays, several control and observation problems have been formulated and investigated in the context of linear impulsive systems: i.e., state estimation [2], linear quadratic control [3,4], disturbance decoupling [5,6], and output regulation [7–13].

As to model matching, this is a classical problem of control theory, which still attracts noticeable attention not only for its intrinsic theoretical interest, but also because it provides powerful tools to solve more general problems — see, e.g., [14–16]. In this context, the contribution of this work lies in considering — for the first time (to the best of the author knowledge) — the problem of model matching for linear impulsive systems with a possible feedthrough term from the control input to the output and in proving a necessary and sufficient condition for its structural solution. In particular, the linear impulsive systems dealt with are allowed to have jump time instants not a-priori known and (possibly) not uniformly spaced in time. However, it is assumed that the number of jump times in any finite time interval is finite, so as to avoid Zeno behaviors.

As to the underlying methodology, this ensues from the geometric approach to linear control theory [17, 18]. This approach has been extended to cope with linear impulsive systems, e.g., in [5, 6, 13]. However, the presence of possible feedthrough terms requires further generalizations of the notion of hybrid controlled invariance, which were not given in the previous papers. Actually, the geometric

approach has proven to be quite powerful in handling also control problems stated for other classes of hybrid systems, like, in particular, switching linear systems — see, e.g., [19–36].

Notation: The symbols $\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{Z}^+$ stand for the sets of real numbers, nonnegative real numbers, and nonnegative integer numbers, respectively. Matrices and linear maps are denoted by slanted upper-case letters, like A. The image and the kernel of A are denoted by $\operatorname{Im} A$ and $\operatorname{Ker} A$, respectively. The transpose of A is denoted by $A^\top$. The inverse of a nonsingular square matrix A is denoted by A^{-1}. Vector spaces and subspaces are denoted by calligraphic letters, like $\mathcal{V}$. The symbol I denotes an identity matrix of appropriate dimensions.

2. Output feedback model matching in linear impulsive systems — problem statement

The definition of a linear impulsive system requires that the time domain be described by an interlaced sequence of continuous time intervals and isolated time instants. In particular, in this work, the time domain is specified as follows. The set $\mathcal{T} = \{t_0, t_1, \ldots\}$ represents a finite or countably infinite ordered set of strictly increasing elements of $\mathbb{R}^+$. The symbol t_f denotes the last element of $\mathcal{T}$ when the cardinality of $\mathcal{T}$ is finite. The set $\mathcal{T}$ is assumed to have no accumulation points: i.e., the number of elements of $\mathcal{T}$ is finite in any finite interval of $\mathbb{R}^+$. The symbol $\mathscr{T}$ denotes the set of all $\mathcal{T}$ satisfying this constraint. The nonnegative real axis without the elements of $\mathcal{T}$ is denoted by $\mathbb{R}^+ \setminus \mathcal{T}$.

The linear impulsive system Σ_S is defined by

$$\Sigma_S \equiv \begin{cases} \dot{x}_S(t) &= A_S\,x_S(t) + B_S\,u(t), \quad t \in \mathbb{R}^+ \setminus \mathcal{T}, \\ x_S(t_k) &= J_S\,x_S^-(t_k), \quad t_k \in \mathcal{T}, \\ y_S(t) &= C_S\,x_S(t) + D_S\,u(t), \quad t \in \mathbb{R}^+, \end{cases}$$

where $x_S \in \mathcal{X}_S = \mathbb{R}^{n_S}$ is the state, $u \in \mathbb{R}^p$ is the control input, and $y_S \in \mathbb{R}^q$ is the output, with $p, q \le n_S$. A_S, B_S, J_S, C_S, and D_S are constant real matrices of suitable dimensions. The direct algebraic link from the control input to the output established by the matrix D_S is called *control feedthrough*. The matrices

$$\begin{bmatrix} B_S \\ D_S \end{bmatrix}, \quad \begin{bmatrix} C_S & D_S \end{bmatrix}$$

are assumed to have full rank. The set of the admissible control input functions $u(t)$, with $t \in \mathbb{R}^+$, is assumed to be the set of all piecewise-continuous functions with values in $\mathbb{R}^p$. The so-called *flow dynamics* is governed by the differential state equation. Instead, the algebraic state equation rules the so-called *jump dynamics*. Thus, according to the linear impulsive structure of Σ, the state evolution $x_S(t)$ in the time interval $[0, t_0)$ satisfies the differential equation, with given initial state $x_S(0) = x_{S,0}$ and input function $u(t)$, with $t \in [0, t_0)$. The state $x_S(t_k)$, with $t_k \in \mathcal{T}$, is the image through J_S of $x_S^-(t_k) = \lim_{\tau \to 0^+} x_S(t_k - \tau)$. The state evolution $x_S(t)$ in the time interval $[t_k, t_{k+1})$, with $t_k, t_{k+1} \in \mathcal{T}$, satisfies the differential equation, given the initial state $x_S(t_k)$ and the input function $u(t)$, with $t \in [t_k, t_{k+1})$.

The linear impulsive model Σ_M is defined by

$$\Sigma_M \equiv \begin{cases} \dot{x}_M(t) &= A_M\,x_M(t) + B_M\,d(t), \quad t \in \mathbb{R}^+ \setminus \mathcal{T}, \\ x_M(t_k) &= J_M\,x_M^-(t_k), \quad t_k \in \mathcal{T}, \\ y_M(t) &= C_M\,x_M(t), \quad t \in \mathbb{R}^+, \end{cases}$$

where $x_M \in \mathbb{R}^{n_M}$ is the state, $d \in \mathbb{R}^q$ is the input, and $y_M \in \mathbb{R}^q$ is the output. The set of the admissible input functions $d(t)$, with $t \in \mathbb{R}^+$, is assumed to be the set of all piecewise-continuous functions with values in $\mathbb{R}^q$.

Hence, the problem of model matching by output feedback in linear impulsive systems can be stated as follows.
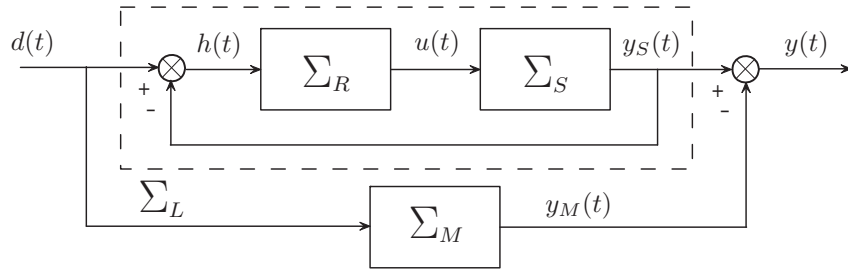
Figure 1. Block diagram for model matching by output feedback

Problem 1 (Model matching by output feedback in linear impulsive systems) *Let the linear impulsive system Σ_S and the linear impulsive model Σ_M be given. Find a linear impulsive regulator Σ_R, defined by*

$$\Sigma_R \equiv \begin{cases} \dot{x}_R(t) &= A_R\, x_R(t) + B_R\, h(t), \quad t \in \mathbb{R}^+ \setminus \mathcal{T}, \\ x_R(t_k) &= J_R\, x_R^-(t_k), \quad t_k \in \mathcal{T}, \\ u(t) &= C_R\, x_R(t), \quad t \in \mathbb{R}^+, \end{cases}$$

where $h(t) = d(t) - y_S(t)$, such that the closed-loop linear impulsive system Σ_L, defined by

$$\Sigma_L \equiv \begin{cases} \dot{x}_L(t) &= A_L\, x_L(t) + D_L\, d(t), \quad t \in \mathbb{R}^+ \setminus \mathcal{T}, \\ x_L(t_k) &= J_L\, x_L^-(t_k), \quad t_k \in \mathcal{T}, \\ y_S(t) &= C_L\, x_L(t), \quad t \in \mathbb{R}^+, \end{cases}$$

where

$$A_L = \begin{bmatrix} A_S & B_S\, C_R \\ -B_R\, C_S & A_R - B_R\, D_S\, C_R \end{bmatrix}, \quad D_L = \begin{bmatrix} 0 \\ B_R \end{bmatrix},$$

$$J_L = \begin{bmatrix} J_S & 0 \\ 0 & J_R \end{bmatrix},$$

$$C_L = \begin{bmatrix} C_S & D_S\, C_R \end{bmatrix},$$

satisfies the requirement that the output $y_S(t)$ is equal to the model output $y_M(t)$, for all $t \in \mathbb{R}^+$, when the respective initial states are zero, for all the admissible input functions $d(t)$, with $t \in \mathbb{R}^+$, and all the admissible sequences of jump times $\mathcal{T} \in \mathscr{T}$.

The block diagram in Fig. 1 illustrates the system interconnection considered in Problem 1.

3. Feedforward disturbance decoupling for the extended linear impulsive system — problem statement

As will be shown in the remainder of this work, the solution to the problem stated in Section 2 can be obtained through the solution of the problem tackled in this section. Namely, this section is focused on a problem of feedforward disturbance decoupling formulated for a new linear impulsive system — henceforth referred to as the *extended linear impulsive system* — which consists of the output-difference connection between the given linear impulsive plant Σ_S and a new linear impulsive model — from now on denoted by $\tilde{\Sigma}_M$ — obtained by suitably modifying the original linear impulsive model Σ_M.

The modified linear impulsive model $\tilde{\Sigma}_M$ is obtained from the original model Σ_M by closing a unit positive output feedback on the flow dynamics of Σ_M, so that

$$\tilde{\Sigma}_M \equiv \begin{cases} \dot{x}_M(t) = (A_M + B_M\, C_M)\, x_M(t) + B_M\, h(t), \quad t \in \mathbb{R}^+ \setminus \mathcal{T}, \\ x_M(t_k) = J_M\, x_M^-(t_k), \quad t_k \in \mathcal{T}, \\ y_M(t) = C_M\, x_M(t), \quad t \in \mathbb{R}^+. \end{cases}$$

The set of the admissible input functions to the modified model $\tilde{\Sigma}_M$ is defined as the set of all piecewise-continuous functions $h(t)$, with $t \in \mathbb{R}^+$, taking their values in $\mathbb{R}^q$.

Hence, the extended linear impulsive system — briefly denoted by Σ — is defined as the connection of the given linear impulsive system Σ_S with the modified linear impulsive model $\tilde{\Sigma}_M$, such that the control input, the disturbance input, and the output of Σ respectively are the control input of Σ_S, the input of $\tilde{\Sigma}_M$, and the difference between the outputs of Σ_S and $\tilde{\Sigma}_M$. Therefore,

$$\Sigma \equiv \begin{cases} \dot{x}(t) = A\,x(t) + B\,u(t) + H\,h(t), & t \in \mathbb{R}^+ \setminus \mathcal{T}, \\ x(t_k) = J\,x^-(t_k), & t_k \in \mathcal{T}, \\ y(t) = C\,x(t) + D\,u(t), & t \in \mathbb{R}^+, \end{cases}$$

where

$$A = \begin{bmatrix} A_S & 0 \\ 0 & A_M + B_M\,C_M \end{bmatrix}, \quad B = \begin{bmatrix} B_S \\ 0 \end{bmatrix}, \quad H = \begin{bmatrix} 0 \\ B_M \end{bmatrix}, \tag{1}$$

$$J = \begin{bmatrix} J_S & 0 \\ 0 & J_M \end{bmatrix}, \tag{2}$$

$$C = \begin{bmatrix} C_S & -C_M \end{bmatrix}, \quad D = D_S. \tag{3}$$

The state space of Σ will be denoted by $\mathcal{X}$: i.e., $\mathcal{X} = \mathbb{R}^n$, where $n = n_S + n_M$.

Consequently, the disturbance decoupling problem, by a feedforward action, for the extended linear impulsive system Σ can be cast as follows.

Problem 2 (Feedforward disturbance decoupling for the extended linear impulsive system) *Let the extended linear impulsive system Σ be given. Find a linear impulsive regulator Σ_R such that the compensated linear impulsive system*

$$\tilde{\Sigma} \equiv \begin{cases} \dot{\tilde{x}}(t) = \tilde{A}\,\tilde{x}(t) + \tilde{H}\,h(t), & t \in \mathbb{R}^+ \setminus \mathcal{T}, \\ \tilde{x}(t_k) = \tilde{J}\,\tilde{x}^-(t_k), & t_k \in \mathcal{T}, \\ y(t) = \tilde{C}\,\tilde{x}(t), & t \in \mathbb{R}^+, \end{cases}$$

where

$$\tilde{A} = \begin{bmatrix} A & B\,C_R \\ 0 & A_R \end{bmatrix}, \quad \tilde{H} = \begin{bmatrix} H \\ B_R \end{bmatrix},$$

$$\tilde{J} = \begin{bmatrix} J & 0 \\ 0 & J_R \end{bmatrix},$$

$$\tilde{C} = \begin{bmatrix} C & D\,C_R \end{bmatrix},$$

satisfies the requirement that the output $y(t)$ is zero, for all $t \in \mathbb{R}^+$, when the initial state is zero, for all the admissible input functions $h(t)$, with $t \in \mathbb{R}^+$, and all the admissible sequences of jump times $\mathcal{T} \in \mathscr{T}$.

The block diagram in Fig. 2 shows the system interconnection taken into consideration in Problem 2.

4. Geometric approach to linear impulsive systems with control feedthrough
As is shown by the broad literature available on disturbance decoupling, probably the most powerful tools to deal with this kind of control problems are those provided by the geometric approach [17, 18]. Over the time, the basic notions originally set forth to handle linear time-invariant systems have been generalized and adapted to deal with more general classes of dynamical systems. In particular, as far as linear impulsive systems are concerned, some fundamental notions, like those of invariance and controlled invariance, have been extended, so as to fit this class of dynamical systems, in some previous
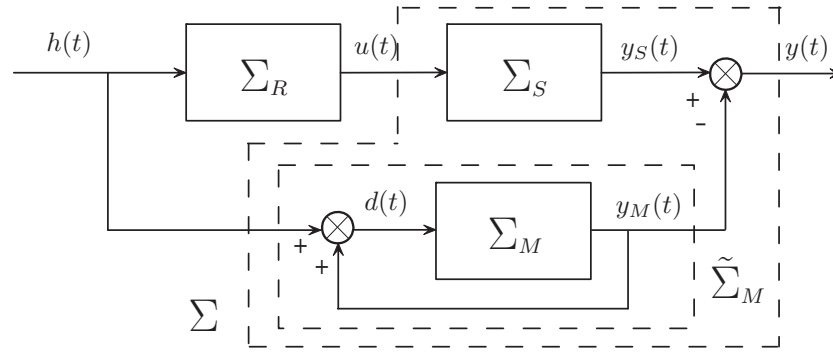
Figure 2. Block diagram for feedforward disturbance decoupling

papers [5, 6, 13]. However, in the specific case dealt with in this work, the extended linear impulsive system Σ shows a direct feedthrough term from the control input to the output. Consequently, the notion of hybrid controlled invariance must be completed by the new notion of output-nulling hybrid controlled invariance, as is illustrated below.

For the sake of immediacy, the following statements give the definitions of hybrid invariant subspace, hybrid controlled invariant subspace and output-nulling hybrid controlled invariant subspace with reference to the extended linear impulsive system Σ. However, it is understood that the special structure of the matrices of Σ shown in (1)–(3) does not play any role in these definitions. The symbol $\mathscr{H}$ will be used henceforth to denote hybrid invariance and hybrid controlled invariance. The short notations $\mathcal{B}$ and $\mathcal{H}$ are used to denote $\mathrm{Im}\, B$ and $\mathrm{Im}\, H$, respectively. A subspace $\mathcal{V} \subseteq \mathcal{X}$ is said to be an $\mathscr{H}$-invariant subspace if $A\mathcal{V} \subseteq \mathcal{V}$ and $J\mathcal{V} \subseteq \mathcal{V}$. A subspace $\mathcal{V} \subseteq \mathcal{X}$ is said to be an $\mathscr{H}$-controlled invariant subspace if $A\mathcal{V} \subseteq \mathcal{V} + \mathcal{B}$ and $J\mathcal{V} \subseteq \mathcal{V}$. Furthermore, it can be shown that a subspace $\mathcal{V} \subseteq \mathcal{X}$, with a basis matrix V, is an $\mathscr{H}$-controlled invariant subspace if and only there exist matrices L_A, L_J, and M such that $A\,V = V\,L_A + B\,M$ and $J\,V = V\,L_J$. Hence, the definition of output-nulling $\mathscr{H}$-controlled invariant subspace is introduced as follows.

Definition 1 *A subspace $\mathcal{V} \subseteq \mathcal{X}$, with a basis matrix V, is said to be an output-nulling $\mathscr{H}$-controlled invariant subspace if there exist matrices L_A, L_J, and M such that $A\,V = V\,L_A + B\,M$, $J\,V = V\,L_J$, and $C\,V = D\,M$.*

Moreover, the notion of output-nulling $\mathscr{H}$-controlled invariant subspace is characterized by the following necessary and sufficient condition, whose proof is a consequence of the properties enjoyed by simultaneous invariant and output-nulling controlled invariant subspaces in linear time-invariant systems.

Proposition 1 *A subspace $\mathcal{V} \subseteq \mathcal{X}$ is an output-nulling $\mathscr{H}$-controlled invariant subspace if and only if there exists a linear map F such that $(A + B\,F)\,\mathcal{V} \subseteq \mathcal{V}$ and $\mathcal{V} \subseteq \mathrm{Ker}\,(C + D\,F)$ hold along with $J\,\mathcal{V} \subseteq \mathcal{V}$.*

Any linear map F satisfying the conditions of Proposition 1 is said to be a *friend* of the output-nulling $\mathscr{H}$-controlled invariant subspace $\mathcal{V}$.

As can be shown by simple algebraic arguments, the set of all output-nulling $\mathscr{H}$-controlled invariant subspaces is an upper semilattice with respect to the sum and the inclusion of subspaces. The maximum of the set of all output-nulling $\mathscr{H}$-controlled invariant subspaces is henceforth denoted by $\mathcal{V}^*_{\mathscr{H}}$.

5. Feedforward disturbance decoupling for the extended linear impulsive system — problem solution

The geometric notions introduced in the previous section allow us to completely characterize the solvability of Problem 2 by a necessary and sufficient condition, which is the purpose of this section. As

will be clear from the following, such condition consists of an inclusion of subspaces, so that it can be given in coordinate-free terms. However, since the if-part of the proof is constructive, it is convenient to make some preliminary remarks, aimed at expressing that condition with respect to conveniently chosen coordinates.

Firstly, it is worth pointing out that the linear map $A + B\,F$, where F is a friend of the maximal output-nulling $\mathscr{H}$-controlled invariant subspace $\mathcal{V}_{\mathscr{H}}^*$, is represented by a matrix with a characteristic upper block-triangular structure, provided that a suitable change of coordinates is performed in the state space. More precisely, let the similarity transformation T be defined by $T = [\,T_1\ T_2\,]$, with $\mathrm{Im}\,T_1 = \mathcal{V}_{\mathscr{H}}^*$. Then, with respect to the new coordinates,

$$A' + B'\,F' = T^{-1}\,(A + B\,F)\,T = \begin{bmatrix} A'_{11} + B'_1\,F'_1 & A'_{12} + B'_1\,F'_2 \\ 0 & A'_{22} + B'_2\,F'_2 \end{bmatrix}, \tag{4}$$

where the structural zero submatrix in the lower left corner — namely,

$$A'_{21} + B'_2\,F'_1 = 0, \tag{5}$$

is due to $(A + B\,F)$-invariance of $\mathcal{V}_{\mathscr{H}}^*$. A similar reasoning holds for the linear map J, which, with respect to the same coordinates, is represented by

$$J' = T^{-1}\,J\,T = \begin{bmatrix} J'_{11} & J'_{12} \\ 0 & J'_{22} \end{bmatrix}, \tag{6}$$

where the structural zero submatrix in the lower left corner is due to J-invariance of $\mathcal{V}_{\mathscr{H}}^*$. Moreover, with respect to the same coordinates, the linear map $C + D\,F$, where F is the considered friend of $\mathcal{V}_{\mathscr{H}}^*$, is represented by a matrix with a structural zero submatrix in the first block of columns. Namely, it ensues that

$$C' + D'\,F' = (C + D\,F)\,T = \begin{bmatrix} 0 & C'_2 + D\,F'_2 \end{bmatrix}, \tag{7}$$

where the structural zero submatrix

$$C'_1 + D\,F'_1 = 0, \tag{8}$$

is due to $\mathcal{V}_{\mathscr{H}}^* \subseteq \mathrm{Ker}\,(C + D\,F)$.

Secondly, it is useful to express the subspace inclusion that will be shown to be the necessary and sufficient condition for solvability of Problem 2 in a coordinate-dependent fashion and, in particular, with respect to the basis introduced above. Namely, the subspace inclusion

$$\mathcal{H} \subseteq \mathcal{V}_{\mathscr{H}}^* \tag{9}$$

holds if and only if

$$H' = T^{-1}\,H = \begin{bmatrix} H'_1 \\ 0 \end{bmatrix}. \tag{10}$$

In fact, the structural zero block in H' expresses the condition that a basis matrix of the subspace $\mathcal{H}$ is a linear combination of the column vectors of the basis matrix T_1 of $\mathcal{V}_{\mathscr{H}}^*$.

Hence, the necessary and sufficient condition for Problem 2 to have a solution can be stated as follows.

Theorem 1 *Let the linear impulsive system Σ be given. Problem 2 has a solution if and only if (9) holds.*

Proof: If. Let (9) hold. Let F be a friend of $\mathcal{V}_{\mathscr{H}}^*$. Hence, with respect to suitably chosen coordinates, (4), (6), (7), and (10) hold. The remainder of this proof of sufficiency will refer to these coordinates. Let the hybrid linear regulator Σ_R be defined by the following matrices:

$$A'_R = A'_{11} + B'_1\,F'_1, \quad B'_R = H'_1, \quad J'_R = J'_{11}, \quad C'_R = F'_1.$$

Then, it will be shown that the regulator Σ_R thus defined, with zero initial state, solves Problem 2. To this aim, note that the cascade — denoted by $\tilde{\Sigma}$ in Problem 2 — between the linear impulsive regulator Σ_R and the extended linear impulsive system Σ is described by

$$
\tilde{\Sigma} \equiv \begin{cases}
\dot{x}_1(t) = A'_{11} x_1(t) + A'_{12} x_2(t) + B'_1 F'_1 x_R(t) + H'_1 h(t), & t \in \mathbb{R}^+ \setminus \mathcal{T}, \\
\dot{x}_2(t) = A'_{21} x_1(t) + A'_{22} x_2(t) + B'_2 F'_1 x_R(t), & t \in \mathbb{R}^+ \setminus \mathcal{T}, \\
\dot{x}_R(t) = (A'_{11} + B'_1 F'_1) x_R(t) + H'_1 h(t), & t \in \mathbb{R}^+ \setminus \mathcal{T}, \\
x_1(t_k) = J'_{11} x_1^-(t_k) + J'_{12} x_2^-(t_k), & t_k \in \mathcal{T}, \\
x_2(t_k) = J'_{22} x_2^-(t_k), & t_k \in \mathcal{T}, \\
x_R(t_k) = J'_{11} x_R^-(t_k), & t_k \in \mathcal{T}, \\
y(t) = C'_1 x_1(t) + C'_2 x_2(t) + D F'_1 x_R(t), & t \in \mathbb{R}^+,
\end{cases}
$$

where the state of Σ has been partitioned as $x = \begin{bmatrix} x_1^\top & x_2^\top \end{bmatrix}^\top$ according to (4), (6), (7), and (10). By applying the change of variables $\eta(t) = x_1(t) - x_R(t)$, with $t \in \mathbb{R}^+$, the system $\tilde{\Sigma}$ can also be written as

$$
\tilde{\Sigma} \equiv \begin{cases}
\dot{\eta}(t) = A'_{11} \eta(t) + A'_{12} x_2(t), & t \in \mathbb{R}^+ \setminus \mathcal{T}, \\
\dot{x}_2(t) = A'_{21} \eta(t) + A'_{22} x_2(t), & t \in \mathbb{R}^+ \setminus \mathcal{T}, \\
\dot{x}_R(t) = (A'_{11} + B'_1 F'_1) x_R(t) + H'_1 h(t), & t \in \mathbb{R}^+ \setminus \mathcal{T}, \\
\eta(t_k) = J'_{11} \eta^-(t_k) + J'_{12} x_2^-(t_k), & t_k \in \mathcal{T}, \\
x_2(t_k) = J'_{22} x_2^-(t_k), & t_k \in \mathcal{T}, \\
x_R(t_k) = J'_{11} x_R^-(t_k), & t_k \in \mathcal{T}, \\
y(t) = C'_1 \eta(t) + C'_2 x_2(t), & t \in \mathbb{R}^+,
\end{cases}
$$

where (5) and (8) have been taken into account. Hence, the assumption of zero initial state implies $\eta(t) = 0$ and $x_2(t) = 0$, for all $t \in \mathbb{R}^+$, which also implies $y(t) = 0$, for all $t \in \mathbb{R}^+$, for all the admissible input functions $h(t)$, with $t \in \mathbb{R}^+$, and all the admissible jump time sequences $\mathcal{T} \in \mathscr{T}$.

Only if. If (9) does not hold, no other output-nulling $\mathscr{H}$-controlled invariant subspace containing $\mathcal{H}$ exists, since the set of all output-nulling $\mathscr{H}$-controlled invariant subspaces is an upper semilattice and $\mathcal{V}^*_{\mathscr{H}}$ is the maximum. ∎

6. Model matching by output feedback — problem solution

The aim of this section is to show that the problem of feedforward disturbance decoupling for the extended linear impulsive system solved in Section 5 is equivalent to the output feedback model matching problem stated in Section 2. In other words, a linear impulsive regulator solves any of these problems if and only if it also solves the other one. The following theorem formalizes this result.

Theorem 2 *A linear impulsive regulator Σ_R solves Problem 2 if and only if it solves Problem 1.*

Proof: If. Let the linear impulsive regulator Σ_R solve Problem 1. Hence, the overall linear impulsive system with measurement feedback — henceforth denoted by $\tilde{\Sigma}'$ — is described by

$$
\tilde{\Sigma}' \equiv \begin{cases}
\dot{x}_S(t) = A_S x_S(t) + B_S C_R x_R(t), & t \in \mathbb{R}^+ \setminus \mathcal{T}, \\
\dot{x}_R(t) = -B_R C_S x_S(t) + (A_R - B_R D_S C_R) x_R(t) + B_R d(t), & t \in \mathbb{R}^+ \setminus \mathcal{T}, \\
\dot{x}_M(t) = A_M x_M(t) + B_M d(t), & t \in \mathbb{R}^+ \setminus \mathcal{T}, \\
x_S(t_k) = J_S x_S^-(t_k), & t_k \in \mathcal{T}, \\
x_R(t_k) = J_R x_R^-(t_k), & t_k \in \mathcal{T}, \\
x_M(t_k) = J_M x_M^-(t_k), & t_k \in \mathcal{T}, \\
y(t) = C_S x_S(t) + D_S C_R x_R(t) - C_M x_M(t), & t \in \mathbb{R}^+.
\end{cases}
\tag{11}
$$

Note that, since Σ_R solves Problem 1, on the assumption of zero initial state, the output of $\tilde{\Sigma}'$ satisfies the condition that $y(t) = 0$, for all $t \in \mathbb{R}^+$, for all the admissible input functions $d(t)$, with $t \in \mathbb{R}^+$. Hence,

one can replace $y_S(t) = C_S\,x_S(t) + D_S\,C_R\,x_R(t)$ with $y_M(t) = C_M\,x_M(t)$ in the state equations of $\tilde{\Sigma}'$. Thus, the following equations for the new system henceforth denoted by $\tilde{\Sigma}''$ are obtained:

$$\tilde{\Sigma}'' \equiv \begin{cases} \dot{x}_S(t) = A_S\,x_S(t) + B_S\,C_R\,x_R(t), \quad t \in \mathbb{R}^+ \setminus \mathcal{T}, \\ \dot{x}_R(t) = A_R\,x_R(t) - B_R\,C_M\,x_M(t) + B_R\,d(t), \quad t \in \mathbb{R}^+ \setminus \mathcal{T}, \\ \dot{x}_M(t) = A_M\,x_M(t) + B_M\,d(t), \quad t \in \mathbb{R}^+ \setminus \mathcal{T}, \\ x_S(t_k) = J_S\,x_S^-(t_k), \quad t_k \in \mathcal{T}, \\ x_R(t_k) = J_R\,x_R^-(t_k), \quad t_k \in \mathcal{T}, \\ x_M(t_k) = J_M\,x_M^-(t_k), \quad t_k \in \mathcal{T}, \\ y(t) = C_S\,x_S(t) + D_S\,C_R\,x_R(t) - C_M\,x_M(t), \quad t \in \mathbb{R}^+. \end{cases} \quad (12)$$

Moreover, since $y(t) = 0$ for all $t \in \mathbb{R}^+$, for all the admissible $d(t)$, with $t \in \mathbb{R}^+$, such condition holds, in particular, by picking $d(t) = h(t) + C_M\,x_M(t)$, where $h(t)$, with $t \in \mathbb{R}^+$, denotes any admissible input function. Then, the resulting system is the linear impulsive system $\tilde{\Sigma}$ considered in Problem 2, as is shown by the following equations, derived from those of $\tilde{\Sigma}''$ with the replacement mentioned above:

$$\tilde{\Sigma} \equiv \begin{cases} \dot{x}_S(t) = A_S\,x_S(t) + B_S\,C_R\,x_R(t), \quad t \in \mathbb{R}^+ \setminus \mathcal{T}, \\ \dot{x}_R(t) = A_R\,x_R(t) + B_R\,h(t), \quad t \in \mathbb{R}^+ \setminus \mathcal{T}, \\ \dot{x}_M(t) = (A_M + B_M\,C_M)\,x_M(t) + B_M\,h(t), \quad t \in \mathbb{R}^+ \setminus \mathcal{T}, \\ x_S(t_k) = J_S\,x_S^-(t_k), \quad t_k \in \mathcal{T}, \\ x_R(t_k) = J_R\,x_R^-(t_k), \quad t_k \in \mathcal{T}, \\ x_M(t_k) = J_M\,x_M^-(t_k), \quad t_k \in \mathcal{T}, \\ y(t) = C_S\,x_S(t) + D_S\,C_R\,x_R(t) - C_M\,x_M(t), \quad t \in \mathbb{R}^+. \end{cases} \quad (13)$$

The equations of $\tilde{\Sigma}$, which hold with $y(t) = 0$ for all $t \in \mathbb{R}^+$, for all the admissible $h(t)$, with $t \in \mathbb{R}^+$, show that the linear impulsive regulator Σ_R also solves Problem 2: i.e., the problem of decoupling the signal $h(t)$, with $t \in \mathbb{R}^+$, in the extended linear impulsive system Σ, including the modified linear impulsive model $\tilde{\Sigma}_M$.

Only if. Let the linear impulsive regulator Σ_R solve Problem 2. Then, in order to show that Σ_R also solves Problem 1, the same reasoning presented in the proof of sufficiency can be followed backward — i.e., from $\tilde{\Sigma}$ to $\tilde{\Sigma}'$. ∎

7. Conclusions
In this work, the problem of finding a linear impulsive regulator ensuring that the forced response of a given linear impulsive system matches that of a linear impulsive model, for all the admissible inputs and all the admissible sequences of jump time instants, has been formulated. The solvability of the problem has been characterized through a necessary and sufficient condition. A computational procedure for the synthesis of the linear impulsive regulator has been illustrated through the if-part of the proof.

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