IOP Conf. Series: Journal of Physics: Conf. Series **783** (2017) 012037 doi:10.1088/1742-6596/783/1/012037

Enhanced approach to PD control design for linear time-invariant descriptor systems

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Abstract. Enhanced approaches to PD controller design, adjusted for linear time-invariant descriptor systems, are proposed in the paper. Presented in the sense of the second Lyapunov method, an associated structure of linear matrix inequalities is outlined to possess the regular closed-loop system dynamic properties. A simulation example, subject to the state and output PD control, demonstrates the effectiveness of the proposed form of the design technique.

1. Introduction

Descriptor models are frequently used for modeling of industrial systems such as power system, rigid body mechanisms or chemical processes (see, e.g., [6], [7], [10], [18] and the references therein). The need for application-oriented simulation technique for such systems focused the research activities at first on the conditions of system regularity and observability [1], [3], which pave the basic way for controller and observer design algorithms, relying on the feasibility of a system of linear matrix inequalities (LMI) [12], [15], [16]. In the descriptor system context, novel bounded real lemmas for discrete and continuous-time descriptor systems are adressed in [2], [11], respectively, an extension of the positive real lemma to descriptor systems is given in [4], system output signal contraints are typically considered in the LMI synthesis of H_2 and mixed $\mathrm{H}_2/\mathrm{H}_\infty$ controllers for singular systems in [19] and the state variables of a singular system can be bounded solving singular LQ problem for descriptor systems by the way presented in [20]. From the stand point of these principles, the main problem is the canonical form, or SVD coordinate form, of the singular system description to be provided the required structural information within design [9], [14]. It seems, for singular systems, to be very attractive in controller and observer design to explore the potential of the Ljapunov matrix parametrization principle [5], [17], to improve the system performances and to simplify the design conditions.

Adapting the idea presented in [17], two methods for proportional-derivative controller (PD) design are proposed in the paper. Following examination of the PD state and output schemes, enhanced algorithms using the Ljapunov matrix parametrization approach are provided. Applied enhanced conditions in the control design develop a general framework for PD control structures for continuous-time singular systems. The approach utilizes the measurable state or output vector variables, the design conditions are based on linear matrix inequality (LMI) technique (applicable conditiond by one matrix equality), which give an effective way to calculate the PD control law parameters.

The paper is organized as follows. Ensuing the introduction given in Sec. 1, Sec. 2 presents the problem formulation focusing on assumptions about some dual singular system properties. Thus, in Sec. 3 and Sec. 4, short descriptions of the main properties of the method exploiting the Ljapunov matrix parametrization principle are presented and the enhanced conditions of the PD controller existence are analyzed and proven in Sec. 5. Finaly, Sec. 6 presents the simulation results and some concluding remarks are reached in Sec. 7.

Throughout the paper, the following notation was used: $\boldsymbol{x}^T$, $\boldsymbol{X}^T$ denotes the transpose of the vector $\boldsymbol{x}$ and the matrix $\boldsymbol{X}$, respectively, $\rho(\boldsymbol{X})$ indicates the eigenvalue spectrum of the square matrix $\boldsymbol{X}$, for a square matrix $\boldsymbol{X} \geq 0$ (> 0) means that $\boldsymbol{X}$ is a symmetric positive (semi)definite matrix, the symbol $\boldsymbol{I}_n$ indicates the n-th order unit matrix, $I\!R$ notes the set of real numbers, and $I\!R^n$, $I\!R^{n\times r}$ refer to the set of all n-dimensional real vectors and $n \times r$ real matrices, respectively.

2. Problem formulation

A linear, time-invariant descriptor multi-input, multi-output (MIMO) system in presence of an unknown disturbance can be described by the state-space equations in the following form

$$\boldsymbol{E}\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) + \boldsymbol{D}\boldsymbol{d}(t)\,, \tag{1}$$

$$\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t)\,, \tag{2}$$

where $\boldsymbol{x}(t) \in I\!R^n$, $\boldsymbol{u}(t) \in I\!R^r$, and $\boldsymbol{y}(t) \in I\!R^m$ are vectors of the system, input and output variables, respectively, $\boldsymbol{d}(t) \in I\!R^p$ is the external disturbance vector, $\boldsymbol{A} \in I\!R^{n\times n}$ is the system dynamic matrix, $\boldsymbol{B} \in I\!R^{n\times r}$ is the system input matrix $\boldsymbol{C} \in I\!R^{m\times n}$ is the system output matrix, $\boldsymbol{D} \in I\!R^{n\times p}$ is the disturbance input matrix and $\boldsymbol{E} \in I\!R^{n\times n}$ is a singular matrix.

It is considered that the pair $(\boldsymbol{A}, \boldsymbol{B})$ is controllable and the control law is of the form

$$\boldsymbol{u}(t) = -\boldsymbol{K}\boldsymbol{x}(t)\,, \tag{3}$$

where $\boldsymbol{K} \in I\!R^{r\times n}$ is the control gain matrix. This implies the closed-loop system description in the form

$$\boldsymbol{E}\dot{\boldsymbol{x}}(t) = \boldsymbol{A}_c\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{d}(t)\,, \tag{4}$$

$$\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t)\,, \tag{5}$$

where

$$\boldsymbol{A}_c = \boldsymbol{A} - \boldsymbol{B}\boldsymbol{K} \tag{6}$$

is the closed-loop system dynamic matrix.

The transfer function of the closed-loop system (4), (5) is

$$\boldsymbol{G}(s) = \boldsymbol{C}(s\boldsymbol{E} - \boldsymbol{A}_c)^{-1}\boldsymbol{D}\,, \tag{7}$$

where a complex number s is the transform variable (Laplace variable) of the Laplace transform.

Since

$$\|\boldsymbol{G}(s)\|_\infty < \gamma \Leftrightarrow \|\boldsymbol{G}_d(s)\|_\infty < \gamma\,, \tag{8}$$

where $\|\boldsymbol{G}(s)\|_\infty$ is the H_∞ norm of the transfer function (7) and $\|\boldsymbol{G}_d(s)\|_\infty$ is the H_∞ norm of the dual system transfer function

$$\boldsymbol{G}_d(s) = \boldsymbol{D}^T(s\boldsymbol{E}^T - \boldsymbol{A}_c^T)^{-1}\boldsymbol{C}^T, \tag{9}$$

it is more suitable in the control law parameter design for descriptor system (4), (5) to work with the dual state-space description of the closed-loop system

$$\boldsymbol{E}^T\dot{\boldsymbol{q}}(t) = \boldsymbol{A}_c^T\boldsymbol{q}(t) + \boldsymbol{C}^T\boldsymbol{g}(t)\,, \tag{10}$$

$$\boldsymbol{z}(t) = \boldsymbol{D}^T\boldsymbol{q}(t)\,, \tag{11}$$

where $\boldsymbol{g}(t) \in I\!R^m$, $\boldsymbol{z}(t) \in I\!R^p$ are associated input and output vector variables.

3. Full state feedback control design

These propositions give necessary and sufficient conditions for (1), (2) to be admissible.

Proposition 1 *The closed-loop singular system in (4), (5) is admissible if for given a nonzero square matrix $\boldsymbol{S} \in I\!R^{n \times n}$ satisfying the condition $\boldsymbol{SE} = 0$ there exist a symmetric positive definite matrix $\boldsymbol{T} \in I\!R^{n \times n}$ and matrices $\boldsymbol{Q} \in I\!R^{n \times n}$, $\boldsymbol{Y} \in I\!R^{r \times n}$ such that*

$$\boldsymbol{T} = \boldsymbol{T}^T > 0\,, \tag{12}$$

$$\boldsymbol{A}(\boldsymbol{TE}^T + \boldsymbol{QS}) + (\boldsymbol{TE}^T + \boldsymbol{QS})^T \boldsymbol{A}^T - \boldsymbol{BY} - \boldsymbol{Y}^T \boldsymbol{B}^T < 0\,. \tag{13}$$

When the above conditions hold, the control law gain matrix is given by

$$\boldsymbol{K} = \boldsymbol{Y}(\boldsymbol{TE}^T + \boldsymbol{QS})^{-1}. \tag{14}$$

Proof: Considering the Lyapunov function candidate in the following form

$$v(\boldsymbol{q}(t)) = \boldsymbol{q}^T(t)\boldsymbol{EP}\boldsymbol{q}(t)\,, \tag{15}$$

$$\boldsymbol{EP} = \boldsymbol{P}^T \boldsymbol{E}^T \geq 0 \tag{16}$$

and $\boldsymbol{P} \in I\!R^{n \times n}$ is a square matrix, then the time derivative of (15) for the disturbance free system (10), (11) takes the form

$$\begin{aligned} \dot{v}(\boldsymbol{q}(t)) &= \dot{\boldsymbol{q}}^T(t)\boldsymbol{EP}\boldsymbol{q}(t) + \boldsymbol{q}^T(t)\boldsymbol{P}^T\boldsymbol{E}^T\dot{\boldsymbol{q}}(t) \\ &= \boldsymbol{q}^T(t)(\boldsymbol{A}_c\boldsymbol{P} + \boldsymbol{P}^T\boldsymbol{A}_c^T)\boldsymbol{q}(t) < 0\,. \end{aligned} \tag{17}$$

Evidently, this implies

$$\boldsymbol{A}_c\boldsymbol{P} + \boldsymbol{P}^T\boldsymbol{A}_c^T < 0\,. \tag{18}$$

Inserting in (18) the matrix (6) and the matrix $\boldsymbol{P}$ parametrized as [17]

$$\boldsymbol{P} = \boldsymbol{TE}^T + \boldsymbol{QS}\,, \tag{19}$$

where $\boldsymbol{T} \in I\!R^{n \times n}$ is a positive definite matrix, $\boldsymbol{Q} \in I\!R^{n \times n}$ is a square matrix and the matrix $\boldsymbol{S} \in I\!R^{n \times n}$ is a non-zero square matrix such that

$$\boldsymbol{SE} = 0\,, \tag{20}$$

then (18) gives

$$(\boldsymbol{A} - \boldsymbol{BK})(\boldsymbol{TE}^T + \boldsymbol{QS}) + (\boldsymbol{TE}^T + \boldsymbol{QS})^T(\boldsymbol{A} - \boldsymbol{BK})^T < 0\,. \tag{21}$$

Using the notation

$$\boldsymbol{Y} = \boldsymbol{K}(\boldsymbol{TE}^T + \boldsymbol{QS})\,, \tag{22}$$

then (21) implies (13). This concludes the proof. ■

Proposition 2 *The closed-loop singular system in (4) and (5) is admissible if for given a nonzero square matrix $\boldsymbol{S} \in I\!R^{n \times n}$ satisfying the condition $\boldsymbol{SE} = 0$ there exist a symmetric positive definite matrix $\boldsymbol{T} \in I\!R^{n \times n}$, matrices $\boldsymbol{Q} \in I\!R^{n \times n}$, $\boldsymbol{Y} \in I\!R^{r \times n}$ and a positive scalar $\gamma \in I\!R$ such that*

$$\boldsymbol{T} = \boldsymbol{T}^T > 0\,, \quad \gamma > 0\,, \tag{23}$$

$$\begin{bmatrix} \boldsymbol{A}(\boldsymbol{TE}^T + \boldsymbol{QS}) + (\boldsymbol{TE}^T + \boldsymbol{QS})^T\boldsymbol{A}^T - \boldsymbol{BY} - \boldsymbol{Y}^T\boldsymbol{B}^T & * & * \\ \boldsymbol{C}(\boldsymbol{TE}^T + \boldsymbol{QS}) & -\gamma\boldsymbol{I}_m & * \\ \boldsymbol{D}^T & \boldsymbol{0} & -\gamma\boldsymbol{I}_p \end{bmatrix} < 0\,. \tag{24}$$

When the above conditions hold, the control law gain matrix is given by

$$\boldsymbol{K} = \boldsymbol{Y}(\boldsymbol{TE}^T + \boldsymbol{QS})^{-1}. \tag{25}$$

Hereafter, $*$ denotes the symmetric item in a symmetric matrix.

Proof: Considering the Lyapunov function candidate in the form

$$v(\boldsymbol{q}(t)) = \boldsymbol{q}^T(t)\boldsymbol{E}\boldsymbol{P}\boldsymbol{q}(t) + \gamma^{-1}\int_0^t (\boldsymbol{z}^T(\tau)\boldsymbol{z}(\tau) - \gamma^2\boldsymbol{g}^T(\tau)\boldsymbol{g}(\tau))\mathrm{d}\tau > 0\,, \tag{26}$$

where

$$\boldsymbol{E}\boldsymbol{P} = \boldsymbol{P}^T\boldsymbol{E}^T \geq 0 \tag{27}$$

and $\boldsymbol{P} \in I\!R^{n\times n}$ is a square matrix, then the time derivative of (26) for the system (10), (11) takes the form

$$\begin{aligned}\dot{v}(\boldsymbol{q}(t)) &= \dot{\boldsymbol{q}}^T(t)\boldsymbol{E}\boldsymbol{P}\boldsymbol{q}(t) + \boldsymbol{q}^T(t)\boldsymbol{P}^T\boldsymbol{E}^T\dot{\boldsymbol{q}}(t) + \gamma^{-1}\boldsymbol{z}^T(t)\boldsymbol{z}(t) - \gamma\boldsymbol{g}^T(t)\boldsymbol{g}(t)\\ &= \boldsymbol{q}^T(t)(\boldsymbol{A}_c\boldsymbol{P} + \boldsymbol{P}^T\boldsymbol{A}_c^T + \gamma^{-1}\boldsymbol{D}\boldsymbol{D}^T)\boldsymbol{q}(t)\\ &+ \boldsymbol{g}^T(t)\boldsymbol{C}\boldsymbol{P}\boldsymbol{q}(t) + \boldsymbol{q}^T(t)\boldsymbol{P}^T\boldsymbol{C}^T\boldsymbol{g}(t) - \gamma\boldsymbol{g}^T(t)\boldsymbol{g}(t) < 0\,.\end{aligned} \tag{28}$$

Defining the composed vector

$$\boldsymbol{q}_c^T(t) = \begin{bmatrix} \boldsymbol{q}^T(t) & \boldsymbol{g}^T(t) \end{bmatrix}, \tag{29}$$

(28) can be written as

$$\dot{v}(\boldsymbol{q}(t)) = \boldsymbol{q}_c^T(t)\boldsymbol{P}_c\boldsymbol{q}_c(t) < 0\,, \tag{30}$$

where

$$\boldsymbol{P}_c = \begin{bmatrix} \boldsymbol{A}_c\boldsymbol{P} + \boldsymbol{P}^T\boldsymbol{A}_c^T + \gamma^{-1}\boldsymbol{D}\boldsymbol{D}^T & \boldsymbol{P}^T\boldsymbol{C}^T \\ \boldsymbol{C}\boldsymbol{P} & -\gamma\boldsymbol{I}_m \end{bmatrix} < 0\,. \tag{31}$$

Then, the Schur complement implies

$$\begin{bmatrix} \boldsymbol{A}_c\boldsymbol{P} + \boldsymbol{P}^T\boldsymbol{A}_c^T & \boldsymbol{P}^T\boldsymbol{C}^T & \boldsymbol{D} \\ \boldsymbol{C}\boldsymbol{P} & -\gamma\boldsymbol{I}_m & \boldsymbol{0} \\ \boldsymbol{D}^T & \boldsymbol{0} & -\gamma\boldsymbol{I}_p \end{bmatrix} < 0 \tag{32}$$

and, substituting in (32) the matrices (6) and (19), the following inequality is obtained

$$\begin{bmatrix} (\boldsymbol{A}-\boldsymbol{B}\boldsymbol{K})(\boldsymbol{T}\boldsymbol{E}^T+\boldsymbol{Q}\boldsymbol{S}) + (\boldsymbol{T}\boldsymbol{E}^T+\boldsymbol{Q}\boldsymbol{S})^T(\boldsymbol{A}-\boldsymbol{B}\boldsymbol{K})^T & (\boldsymbol{T}\boldsymbol{E}^T+\boldsymbol{Q}\boldsymbol{S})^T\boldsymbol{C}^T & \boldsymbol{D} \\ \boldsymbol{C}(\boldsymbol{T}\boldsymbol{E}^T+\boldsymbol{Q}\boldsymbol{S}) & -\gamma\boldsymbol{I}_m & \boldsymbol{0} \\ \boldsymbol{D}^T & \boldsymbol{0} & -\gamma\boldsymbol{I}_p \end{bmatrix} < 0\,. \tag{33}$$

Using the notation (22) then (33) implies (24). This concludes the proof. ■

Note, a pair $(\boldsymbol{E};\boldsymbol{A})$ is admissible if is regular and has neither impulsive modes nor unstable finite modes. In this sense, the proposition can be used to design the gain matrix $\boldsymbol{K}$.

4. Static output control

The following theorems give necessary and sufficient conditions for system (1), (2) under static output control to be admissible.

Theorem 1 *The closed-loop singular system in (4) and (5) is admissible if for given a nonzero square matrix $\boldsymbol{S} \in I\!R^{n\times n}$ satisfying the condition $\boldsymbol{S}\boldsymbol{E} = 0$ there exist a symmetric positive definite matrix $\boldsymbol{T} \in I\!R^{n\times n}$ and matrices $\boldsymbol{H} \in I\!R^{m\times m}$, $\boldsymbol{Q} \in I\!R^{n\times n}$, $\boldsymbol{Y}_o \in I\!R^{r\times m}$ such that*

$$\boldsymbol{T} = \boldsymbol{T}^T > 0\,, \tag{34}$$

$$\boldsymbol{A}(\boldsymbol{T}\boldsymbol{E}^T+\boldsymbol{Q}\boldsymbol{S}) + (\boldsymbol{T}\boldsymbol{E}^T+\boldsymbol{Q}\boldsymbol{S})^T\boldsymbol{A}^T - \boldsymbol{B}\boldsymbol{Y}_o\boldsymbol{C} - \boldsymbol{C}^T\boldsymbol{Y}_o^T\boldsymbol{B}^T < 0\,, \tag{35}$$

$$\boldsymbol{C}(\boldsymbol{T}\boldsymbol{E}^T+\boldsymbol{Q}\boldsymbol{S}) = \boldsymbol{H}\boldsymbol{C}\,. \tag{36}$$

When the above conditions hold, the control law gain matrix is given by

$$\boldsymbol{K}_o = \boldsymbol{Y}_o\boldsymbol{H}^{-1}. \tag{37}$$

IOP Conf. Series: Journal of Physics: Conf. Series **783** (2017) 012037 doi:10.1088/1742-6596/783/1/012037

Proof: The static output feedback control law is defined as

$$\boldsymbol{u}(t) = -\boldsymbol{K}_o \boldsymbol{y}(t) = -\boldsymbol{K}_o \boldsymbol{C}\boldsymbol{x}(t)\,, \tag{38}$$

where $\boldsymbol{K}_o \in I\!R^{r\times m}$ is the control law gain matrix. This implies the closed-loop system state-space equations

$$\boldsymbol{E}\dot{\boldsymbol{x}}(t) = \boldsymbol{A}_{co}\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{d}(t)\,, \tag{39}$$

$$\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t)\,, \tag{40}$$

where

$$\boldsymbol{A}_{co} = \boldsymbol{A} - \boldsymbol{B}\boldsymbol{K}_o\boldsymbol{C} \tag{41}$$

is the closed-loop system matrix. Thus, the dual state-space description of the closed-loop system takes the form

$$\boldsymbol{E}^T\dot{\boldsymbol{q}}(t) = \boldsymbol{A}_{co}^T\boldsymbol{q}(t) + \boldsymbol{C}^T\boldsymbol{g}(t)\,, \tag{42}$$

$$\boldsymbol{z}(t) = \boldsymbol{D}^T\boldsymbol{q}(t)\,. \tag{43}$$

Considering the Lyapunov function candidate as in (15), (16) then, analogously to (18), it can obtain for the disturbance free system that

$$\boldsymbol{A}_{co}\boldsymbol{P} + \boldsymbol{P}^T\boldsymbol{A}_{co}^T < 0 \tag{44}$$

and substituting (19) and (41) it yields

$$(\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K}_o\boldsymbol{C})(\boldsymbol{T}\boldsymbol{E}^T + \boldsymbol{Q}\boldsymbol{S}) + (\boldsymbol{T}\boldsymbol{E}^T + \boldsymbol{Q}\boldsymbol{S})^T(\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K}_o\boldsymbol{C})^T. \tag{45}$$

Writing here

$$\boldsymbol{B}\boldsymbol{K}_o\boldsymbol{C}(\boldsymbol{T}\boldsymbol{E}^T + \boldsymbol{Q}\boldsymbol{S}) = \boldsymbol{B}\boldsymbol{K}_o\boldsymbol{H}\boldsymbol{H}^{-1}\boldsymbol{C}(\boldsymbol{T}\boldsymbol{E}^T + \boldsymbol{Q}\boldsymbol{S}) = \boldsymbol{B}\boldsymbol{K}_o\boldsymbol{H}\boldsymbol{C} = \boldsymbol{B}\boldsymbol{Y}_o\boldsymbol{C}, \tag{46}$$

where $\boldsymbol{H} \in I\!R^{m\times m}$ is a regular matrix and

$$\boldsymbol{H}^{-1}\boldsymbol{C} = \boldsymbol{C}(\boldsymbol{T}\boldsymbol{E}^T + \boldsymbol{Q}\boldsymbol{S})^{-1}, \tag{47}$$

$$\boldsymbol{Y}_o = \boldsymbol{K}_o\boldsymbol{H}\,, \tag{48}$$

then (45) implies (35) and (47) gives (36), This concludes the proof. ■

Theorem 2 *The closed-loop square singular system in (4) and (5) is admissible if for given a nonzero square matrix $\boldsymbol{S} \in I\!R^{n\times n}$ satisfying the condition $\boldsymbol{S}\boldsymbol{E} = 0$ there exist a symmetric positive definite matrix $\boldsymbol{T} \in I\!R^{n\times n}$, matrices $\boldsymbol{H} \in I\!R^{m\times m}$, $\boldsymbol{Q} \in I\!R^{n\times n}$, $\boldsymbol{Y}_o \in I\!R^{r\times m}$ and a positive scalar $\gamma \in I\!R$ such that*

$$\boldsymbol{T} = \boldsymbol{T}^T > 0\,, \quad \gamma > 0\,, \tag{49}$$

$$\begin{bmatrix} \boldsymbol{A}(\boldsymbol{T}\boldsymbol{E}^T + \boldsymbol{Q}\boldsymbol{S}) + (\boldsymbol{T}\boldsymbol{E}^T + \boldsymbol{Q}\boldsymbol{S})^T\boldsymbol{A}^T - \boldsymbol{B}\boldsymbol{Y}_o\boldsymbol{C} - \boldsymbol{C}^T\boldsymbol{Y}_o^T\boldsymbol{B}^T & * & * \\ \boldsymbol{C}(\boldsymbol{T}\boldsymbol{E}^T + \boldsymbol{Q}\boldsymbol{S}) & -\gamma\boldsymbol{I}_m & * \\ \boldsymbol{D}^T & \boldsymbol{0} & -\gamma\boldsymbol{I}_p \end{bmatrix} < 0\,, \tag{50}$$

$$\boldsymbol{C}(\boldsymbol{T}\boldsymbol{E}^T + \boldsymbol{Q}\boldsymbol{S}) = \boldsymbol{H}\boldsymbol{C}\,. \tag{51}$$

When the above conditions hold, the control law gain matrix is given by

$$\boldsymbol{K}_o = \boldsymbol{Y}_o\boldsymbol{H}^{-1}. \tag{52}$$

Proof: Considering the Lyapunov function candidate in the form (26), (27) then, analogously to (31), it can obtain that

$$\begin{bmatrix} \boldsymbol{A}_{co}\boldsymbol{P} + \boldsymbol{P}^T\boldsymbol{A}_{co}^T & \boldsymbol{P}^T\boldsymbol{C}^T & \boldsymbol{D} \\ \boldsymbol{C}\boldsymbol{P} & -\gamma\boldsymbol{I}_m & \boldsymbol{0} \\ \boldsymbol{D}^T & \boldsymbol{0} & -\gamma\boldsymbol{I}_p \end{bmatrix} < 0 \,. \tag{53}$$

Substituting in (53) the matrices (19) and (41), the following inequality is obtained

$$\begin{bmatrix} (\boldsymbol{A}-\boldsymbol{B}\boldsymbol{K}_o\boldsymbol{C})(\boldsymbol{T}\boldsymbol{E}^T+\boldsymbol{Q}\boldsymbol{S})+(\boldsymbol{T}\boldsymbol{E}^T+\boldsymbol{Q}\boldsymbol{S})^T(\boldsymbol{A}-\boldsymbol{B}\boldsymbol{K}_o\boldsymbol{C})^T & (\boldsymbol{T}\boldsymbol{E}^T+\boldsymbol{Q}\boldsymbol{S})^T\boldsymbol{C}^T & \boldsymbol{D} \\ \boldsymbol{C}(\boldsymbol{T}\boldsymbol{E}^T+\boldsymbol{Q}\boldsymbol{S}) & -\gamma\boldsymbol{I}_m & \boldsymbol{0} \\ \boldsymbol{D}^T & \boldsymbol{0} & -\gamma\boldsymbol{I}_p \end{bmatrix} < 0 \tag{54}$$

and using the substitutions (47), (48) then (54) implies (50). This concludes the proof. ∎

5. PD feedback control design

The results of the previous sections can be extended to LMI-based control design conditions for linear time-invariant descriptor systems.

Lemma 1 *The generalized dual state-space description of the descriptor system with full state PD feedback control takes the form*

$$\boldsymbol{E}^{\bullet T}\dot{\boldsymbol{q}}^{\bullet}(t) = \boldsymbol{A}^{\bullet T}\boldsymbol{q}^{\bullet}(t) + \boldsymbol{C}^{\bullet T}\boldsymbol{g}(t)\,, \tag{55}$$

$$\boldsymbol{z}(t) = \boldsymbol{D}^{\bullet T}\boldsymbol{q}^{\bullet}(t)\,, \tag{56}$$

where

$$\boldsymbol{q}^{\bullet T}(t) = \begin{bmatrix} \boldsymbol{q}^T(t) & \dot{\boldsymbol{q}}^T(t) \end{bmatrix}, \tag{57}$$

$$\boldsymbol{A}_c = \boldsymbol{A} - \boldsymbol{B}\boldsymbol{K}\,, \quad \boldsymbol{W}_c = \boldsymbol{E} + \boldsymbol{B}\boldsymbol{L}\,, \tag{58}$$

$$\boldsymbol{E}^{\bullet T} = \begin{bmatrix} \boldsymbol{I}_n & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}, \; \boldsymbol{A}^{\bullet T} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I}_n \\ \boldsymbol{A}_c^T & -\boldsymbol{W}_c^T \end{bmatrix}, \; \boldsymbol{C}^{\bullet T} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{C}^T \end{bmatrix}, \; \boldsymbol{D}^{\bullet T} = \begin{bmatrix} \boldsymbol{D}^T & \boldsymbol{0} \end{bmatrix}. \tag{59}$$

Proof: The full state PD feedback control law is defined as

$$\boldsymbol{u}(t) = -\boldsymbol{K}\boldsymbol{x}(t) - \boldsymbol{L}\dot{\boldsymbol{x}}(t)\,, \tag{60}$$

where $\boldsymbol{K}, \boldsymbol{L} \in I\!R^{r \times n}$ are the control law gain matrices. This implies the closed-loop system state-space equations

$$(\boldsymbol{E} + \boldsymbol{B}\boldsymbol{L})\dot{\boldsymbol{x}}(t) = (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K})\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{d}(t), \tag{61}$$

$$\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t)\,. \tag{62}$$

Using the notations (58), the dual state-space description of the closed-loop system with the full state PD feedback control is

$$\boldsymbol{W}_c^T\dot{\boldsymbol{q}}(t) = \boldsymbol{A}_c^T\boldsymbol{q}(t) + \boldsymbol{C}^T\boldsymbol{g}(t), \tag{63}$$

$$\boldsymbol{z}(t) = \boldsymbol{D}^T\boldsymbol{q}(t). \tag{64}$$

Considering the equality

$$\dot{\boldsymbol{q}}(t) = \dot{\boldsymbol{q}}(t) \tag{65}$$

and rewriting (64) as

$$\boldsymbol{A}_c^T\boldsymbol{q}(t) + \boldsymbol{C}^T\boldsymbol{g}(t) - \boldsymbol{W}_c^T\dot{\boldsymbol{q}}(t) = 0\,, \tag{66}$$

then (65) and (66) can be written compactly as

$$\begin{bmatrix} \boldsymbol{I}_n & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{q}}(t) \\ \ddot{\boldsymbol{q}}(t) \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I}_n \\ \boldsymbol{A}_c^T & -\boldsymbol{W}_c^T \end{bmatrix} \begin{bmatrix} \boldsymbol{q}(t) \\ \dot{\boldsymbol{q}}(t) \end{bmatrix} + \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{C}^T \end{bmatrix} \boldsymbol{g}(t)\,, \tag{67}$$

while (65) implies

$$\boldsymbol{z}(t) = \begin{bmatrix} \boldsymbol{D}^T & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{q}(t) \\ \dot{\boldsymbol{q}}(t) \end{bmatrix}. \tag{68}$$

Using the notations (57), (59), then (67), (68) implies (55), (56), respectively. This concludes the proof. ∎

Lemma 2 *The generalized dual state-space description of the descriptor system with output PD feedback control takes the form*

$$\boldsymbol{E}^{\bullet T} \dot{\boldsymbol{q}}^{\bullet}(t) = \boldsymbol{A}_o^{\bullet T} \boldsymbol{q}^{\bullet}(t) + \boldsymbol{C}^{\bullet T} \boldsymbol{g}(t)\,, \tag{69}$$

$$\boldsymbol{z}(t) = \boldsymbol{D}^{\bullet T} \boldsymbol{q}^{\bullet}(t)\,, \tag{70}$$

where

$$\boldsymbol{q}^{\bullet T}(t) = \begin{bmatrix} \boldsymbol{q}^T(t) & \dot{\boldsymbol{q}}^T(t) \end{bmatrix}, \tag{71}$$

$$\boldsymbol{A}_{co} = \boldsymbol{A} - \boldsymbol{B}\boldsymbol{K}_o\boldsymbol{C}\,, \quad \boldsymbol{W}_{co} = \boldsymbol{E} + \boldsymbol{B}\boldsymbol{L}_o\boldsymbol{C}\,, \tag{72}$$

$$\boldsymbol{E}^{\bullet T} = \begin{bmatrix} \boldsymbol{I}_n & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}, \ \boldsymbol{A}_o^{\bullet T} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I}_n \\ \boldsymbol{A}_{co}^T & -\boldsymbol{W}_{co}^T \end{bmatrix}, \ \boldsymbol{C}^{\bullet T} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{C}^T \end{bmatrix}, \ \boldsymbol{D}^{\bullet T} = \begin{bmatrix} \boldsymbol{D}^T & \boldsymbol{0} \end{bmatrix}. \tag{73}$$

Proof: The output PD feedback control law is defined as

$$\boldsymbol{u}(t) = -\boldsymbol{K}_o\boldsymbol{y}(t) - \boldsymbol{L}_o\dot{\boldsymbol{y}}(t) = -\boldsymbol{K}_o\boldsymbol{C}\boldsymbol{x}(t) - \boldsymbol{L}_o\boldsymbol{C}\dot{\boldsymbol{x}}(t)\,, \tag{74}$$

where $\boldsymbol{K}_o, \boldsymbol{L}_o \in I\!\!R^{r \times m}$ are the control law gain matrices. This implies the closed-loop system state-space equations

$$(\boldsymbol{E} + \boldsymbol{B}\boldsymbol{L}_o\boldsymbol{C})\dot{\boldsymbol{x}}(t) = (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K}_o\boldsymbol{C})\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{d}(t)\,, \tag{75}$$

$$\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t)\,. \tag{76}$$

Applying the notations (72), the dual state-space description of the closed-loop system with the output PD feedback control is

$$\boldsymbol{W}_{co}^T \dot{\boldsymbol{q}}(t) = \boldsymbol{A}_{co}^T \boldsymbol{q}(t) + \boldsymbol{C}^T \boldsymbol{g}(t), \tag{77}$$

$$\boldsymbol{z}(t) = \boldsymbol{D}^T \boldsymbol{q}(t). \tag{78}$$

Following the same way as before, it can obtain

$$\begin{bmatrix} \boldsymbol{I}_n & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{q}}(t) \\ \ddot{\boldsymbol{q}}(t) \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I}_n \\ \boldsymbol{A}_{co}^T & -\boldsymbol{W}_{co}^T \end{bmatrix} \begin{bmatrix} \boldsymbol{q}(t) \\ \dot{\boldsymbol{q}}(t) \end{bmatrix} + \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{C}^T \end{bmatrix} \boldsymbol{g}(t)\,, \tag{79}$$

$$\boldsymbol{z}(t) = \begin{bmatrix} \boldsymbol{D}^T & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{q}(t) \\ \dot{\boldsymbol{q}}(t) \end{bmatrix} \tag{80}$$

and using the notations (71), (73), then (79), (80) implies (69), (70), respectively. This concludes the proof. ∎

Theorem 3 *The closed-loop singular system in (4) and (5) is admissible if for given positive $\delta \in I\!R$ and a nonzero square matrix $\boldsymbol{S} \in I\!R^{n\times n}$ satisfying the condition $\boldsymbol{S}\boldsymbol{E} = 0$ there exist symmetric positive definite matrices $\boldsymbol{P}, \boldsymbol{T} \in I\!R^{n\times n}$, matrices $\boldsymbol{Q} \in I\!R^{n\times n}$, $\boldsymbol{Y}, \boldsymbol{Z} \in I\!R^{r\times n}$ and a positive scalar $\gamma \in I\!R$ such that*

$$\boldsymbol{P} = \boldsymbol{P}^T > 0\,, \quad \boldsymbol{T} = \boldsymbol{T}^T > 0\,, \quad \gamma > 0\,, \tag{81}$$

$$\begin{bmatrix} \boldsymbol{\Phi}(1,1) & * & * & * \\ \boldsymbol{\Phi}(2,1) & -\boldsymbol{\Phi}(2,2) & * & * \\ \boldsymbol{C}(\boldsymbol{T}\boldsymbol{E}^T + \boldsymbol{Q}\boldsymbol{S}) & \delta \boldsymbol{C}(\boldsymbol{T}\boldsymbol{E}^T + \boldsymbol{Q}\boldsymbol{S}) & -\gamma \boldsymbol{I}_m & * \\ \boldsymbol{D}^T & \boldsymbol{0} & \boldsymbol{0} & -\gamma \boldsymbol{I}_p \end{bmatrix} < 0\,, \tag{82}$$

$$\boldsymbol{\Phi}(1,1) = \boldsymbol{A}(\boldsymbol{T}\boldsymbol{E}^T + \boldsymbol{Q}\boldsymbol{S}) + (\boldsymbol{T}\boldsymbol{E}^T + \boldsymbol{Q}\boldsymbol{S})^T\boldsymbol{A}^T - \boldsymbol{B}\boldsymbol{Y} - \boldsymbol{Y}^T\boldsymbol{B}^T, \tag{83}$$

$$\boldsymbol{\Phi}(2,1) = \delta(\boldsymbol{T}\boldsymbol{E}^T + \boldsymbol{Q}\boldsymbol{S})^T\boldsymbol{A}^T - \delta\boldsymbol{Y}^T\boldsymbol{B}^T + \boldsymbol{P} - \boldsymbol{E}(\boldsymbol{T}\boldsymbol{E}^T + \boldsymbol{Q}\boldsymbol{S}) - \boldsymbol{B}\boldsymbol{Z}\,, \tag{84}$$

$$\boldsymbol{\Phi}(2,2) = \delta\boldsymbol{E}(\boldsymbol{T}\boldsymbol{E}^T + \boldsymbol{Q}\boldsymbol{S}) + \delta(\boldsymbol{T}\boldsymbol{E}^T + \boldsymbol{Q}\boldsymbol{S})^T\boldsymbol{E}^T + \delta\boldsymbol{B}\boldsymbol{Z} + \delta\boldsymbol{Z}^T\boldsymbol{B}^T\,. \tag{85}$$

When the above conditions hold, the control law gain matrices are given by

$$\boldsymbol{K} = \boldsymbol{Y}(\boldsymbol{T}\boldsymbol{E}^T + \boldsymbol{Q}\boldsymbol{S})^{-1}, \quad \boldsymbol{L} = \boldsymbol{Z}(\boldsymbol{T}\boldsymbol{E}^T + \boldsymbol{Q}\boldsymbol{S})^{-1}. \tag{86}$$

Proof: Considering the Lyapunov function candidate in the form

$$v(\boldsymbol{q}^\bullet(t)) = \boldsymbol{q}^{\bullet T}(t)\boldsymbol{E}^\bullet\boldsymbol{P}^\bullet\boldsymbol{q}(t) + \gamma^{-1}\int_0^t (\boldsymbol{z}^T(\tau)\boldsymbol{z}(\tau) - \gamma^2\boldsymbol{g}^T(\tau)\boldsymbol{g}(\tau))\mathrm{d}\tau > 0\,, \tag{87}$$

where

$$\boldsymbol{E}^\bullet\boldsymbol{P}^\bullet = \boldsymbol{P}^{\bullet T}\boldsymbol{E}^{\bullet T} \geq 0 \tag{88}$$

and $\boldsymbol{P}^\bullet \in I\!R^{2n\times 2n}$ is a square matrix, then following the same way as above it can obtain in analogy with (32) that

$$\begin{bmatrix} \boldsymbol{A}^\bullet\boldsymbol{P}^\bullet + \boldsymbol{P}^{\bullet T}\boldsymbol{A}^{\bullet T} & \boldsymbol{P}^{\bullet T}\boldsymbol{C}^{\bullet T} & \boldsymbol{D}^\bullet \\ \boldsymbol{C}^\bullet\boldsymbol{P}^\bullet & -\gamma\boldsymbol{I}_m & \boldsymbol{0} \\ \boldsymbol{D}^{\bullet T} & \boldsymbol{0} & -\gamma\boldsymbol{I}_p \end{bmatrix} < 0\,. \tag{89}$$

Considering the matrix $\boldsymbol{P}^\bullet$ of the following form

$$\boldsymbol{P}^\bullet = \begin{bmatrix} \boldsymbol{P}_1^\circ & \boldsymbol{P}_2^\circ \\ \boldsymbol{P}_3^\circ & \boldsymbol{P}_4^\circ \end{bmatrix}, \tag{90}$$

then with respect to (88)

$$\begin{bmatrix} \boldsymbol{I}_n & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}\begin{bmatrix} \boldsymbol{P}_1^\circ & \boldsymbol{P}_2^\circ \\ \boldsymbol{P}_3^\circ & \boldsymbol{P}_4^\circ \end{bmatrix} = \begin{bmatrix} \boldsymbol{P}_1^{\circ T} & \boldsymbol{P}_3^{\circ T} \\ \boldsymbol{P}_2^{\circ T} & \boldsymbol{P}_4^{\circ T} \end{bmatrix}\begin{bmatrix} \boldsymbol{I}_n & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \geq 0\,, \tag{91}$$

which gives

$$\begin{bmatrix} \boldsymbol{P}_1^\circ & \boldsymbol{P}_2^\circ \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} = \begin{bmatrix} \boldsymbol{P}_1^{\circ T} & \boldsymbol{0} \\ \boldsymbol{P}_2^{\circ T} & \boldsymbol{0} \end{bmatrix} \geq 0\,. \tag{92}$$

It is evident that (92) can be satisfied only if

$$\boldsymbol{P} = \boldsymbol{P}_1^\circ = \boldsymbol{P}_1^{\circ T} > 0, \quad \boldsymbol{P}_2^\circ = \boldsymbol{P}_2^{\circ T} = \boldsymbol{0}\,. \tag{93}$$

Thus, using (93), it yields

$$\boldsymbol{A}^{\bullet}\boldsymbol{P}^{\bullet} = \begin{bmatrix} \mathbf{0} & \boldsymbol{A}_c \\ \boldsymbol{I}_n & -\boldsymbol{W}_c \end{bmatrix} \begin{bmatrix} \boldsymbol{P}_1^{\circ} & \mathbf{0} \\ \boldsymbol{P}_3^{\circ} & \boldsymbol{P}_4^{\circ} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}_c\boldsymbol{P}_3^{\circ} & \boldsymbol{A}_c\boldsymbol{P}_4^{\circ} \\ \boldsymbol{P}_1^{\circ} - \boldsymbol{W}_c\boldsymbol{P}_3^{\circ} & \boldsymbol{W}_c\boldsymbol{P}_4^{\circ} \end{bmatrix}, \tag{94}$$

$$\boldsymbol{C}^{\bullet}\boldsymbol{P}^{\bullet} = \begin{bmatrix} \mathbf{0} & \boldsymbol{C} \end{bmatrix} \begin{bmatrix} \boldsymbol{P}_1^{\circ} & \mathbf{0} \\ \boldsymbol{P}_3^{\circ} & \boldsymbol{P}_4^{\circ} \end{bmatrix} = \begin{bmatrix} \boldsymbol{C}\boldsymbol{P}_3^{\circ} & \boldsymbol{C}\boldsymbol{P}_4^{\circ} \end{bmatrix} \tag{95}$$

and, after some algebraic manipulations, (89) takes the following form

$$\begin{bmatrix} \boldsymbol{A}_c\boldsymbol{P}_3^{\circ} + \boldsymbol{P}_3^{\circ T}\boldsymbol{A}_c^T & \boldsymbol{A}_c\boldsymbol{P}_4^{\circ} + \boldsymbol{P}_1^{\circ} - \boldsymbol{P}_3^{\circ T}\boldsymbol{W}_c^T & \boldsymbol{P}_3^{\circ T}\boldsymbol{C}^T & \boldsymbol{D} \\ \boldsymbol{P}_4^{\circ T}\boldsymbol{A}_c^T + \boldsymbol{P}_1^{\circ} - \boldsymbol{W}_c\boldsymbol{P}_3^{\circ} & -\boldsymbol{W}_c\boldsymbol{P}_4^{\circ} - \boldsymbol{P}_4^{\circ T}\boldsymbol{W}_c^T & \boldsymbol{P}_4^{\circ T}\boldsymbol{C}^T & \mathbf{0} \\ \boldsymbol{C}\boldsymbol{P}_3^{\circ} & \boldsymbol{C}\boldsymbol{P}_4^{\circ} & -\gamma\boldsymbol{I}_m & \mathbf{0} \\ \boldsymbol{D}^T & \mathbf{0} & \mathbf{0} & -\gamma\boldsymbol{I}_p \end{bmatrix} < 0\,. \tag{96}$$

Applying the substitutions

$$\boldsymbol{P}_3^{\circ} = \boldsymbol{T}\boldsymbol{E}^T + \boldsymbol{Q}\boldsymbol{S}\,, \quad \boldsymbol{P}_4^{\circ} = \delta\boldsymbol{P}_3^{\circ}\,, \tag{97}$$

$$\boldsymbol{Y} = \boldsymbol{K}(\boldsymbol{T}\boldsymbol{E}^T + \boldsymbol{Q}\boldsymbol{S})\,, \quad \boldsymbol{Z} = \boldsymbol{L}(\boldsymbol{T}\boldsymbol{E}^T + \boldsymbol{Q}\boldsymbol{S})\,, \tag{98}$$

where $\delta \in I\!R$ is a positive scalar, $\boldsymbol{T} \in I\!R^{n\times n}$ is a positive definite matrix, $\boldsymbol{Q} \in I\!R^{n\times n}$ is a square matrix and the matrix $\boldsymbol{S} \in I\!R^{n\times n}$ is a non-zero square matrix such that $\boldsymbol{S}\boldsymbol{E} = 0$, then

$$\boldsymbol{\Phi}(1,1) = \boldsymbol{A}_c\boldsymbol{P}_3^{\circ} + \boldsymbol{P}_3^{\circ T}\boldsymbol{A}_c^T = \boldsymbol{A}(\boldsymbol{T}\boldsymbol{E}^T + \boldsymbol{Q}\boldsymbol{S}) + (\boldsymbol{T}\boldsymbol{E}^T + \boldsymbol{Q}\boldsymbol{S})^T\boldsymbol{A}^T - \boldsymbol{B}\boldsymbol{Y} - \boldsymbol{Y}^T\boldsymbol{B}^T, \tag{99}$$

$$\boldsymbol{\Phi}(2,2) = \boldsymbol{W}_c\boldsymbol{P}_4^{\circ} + \boldsymbol{P}_4^{\circ T}\boldsymbol{W}_c^T = \boldsymbol{E}(\boldsymbol{T}\boldsymbol{E}^T + \boldsymbol{Q}\boldsymbol{S}) + (\boldsymbol{T}\boldsymbol{E}^T + \boldsymbol{Q}\boldsymbol{S})^T\boldsymbol{E}^T + \delta\boldsymbol{B}\boldsymbol{Z} + \delta\boldsymbol{Z}^T\boldsymbol{B}^T, \tag{100}$$

$$\begin{aligned} \boldsymbol{\Phi}(2,1) &= \boldsymbol{P}_4^{\circ T}\boldsymbol{A}_c^T + \boldsymbol{P}_1^{\circ} - \boldsymbol{W}_c\boldsymbol{P}_3^{\circ} \\ &= \delta(\boldsymbol{T}\boldsymbol{E}^T + \boldsymbol{Q}\boldsymbol{S})^T\boldsymbol{A}^T - \delta\boldsymbol{Y}^T\boldsymbol{B}^T + \boldsymbol{P}_1^{\circ} - \boldsymbol{E}(\boldsymbol{T}\boldsymbol{E}^T + \boldsymbol{Q}\boldsymbol{S}) - \boldsymbol{B}\boldsymbol{Z}\,, \end{aligned} \tag{101}$$

$$\boldsymbol{C}\boldsymbol{P}_3^{\circ} = \boldsymbol{C}(\boldsymbol{T}\boldsymbol{E}^T + \boldsymbol{Q}\boldsymbol{S})\,, \tag{102}$$

$$\boldsymbol{C}\boldsymbol{P}_4^{\circ} = \delta\boldsymbol{C}(\boldsymbol{T}\boldsymbol{E}^T + \boldsymbol{Q}\boldsymbol{S})\,. \tag{103}$$

Using the notations given above, the resulting formulas to (96) take the forms (82)-(85). This concludes the proof. ∎

Theorem 4 *The closed-loop square singular system in (4) and (5) is admissible if for given positive $\delta \in I\!R$ and a nonzero square matrix $\boldsymbol{S} \in I\!R^{n\times n}$ satisfying the condition $\boldsymbol{S}\boldsymbol{E} = 0$ there exist symmetric positive definite matrices $\boldsymbol{P}, \boldsymbol{T} \in I\!R^{n\times n}$, matrices $\boldsymbol{Q} \in I\!R^{n\times n}$, $\boldsymbol{Y}_o, \boldsymbol{Z}_o \in I\!R^{r\times m}$, a regular matrix $\boldsymbol{H} \in I\!R^{m\times m}$ and a positive scalar $\gamma \in I\!R$ such that*

$$\boldsymbol{P} = \boldsymbol{P}^T > 0\,, \quad \boldsymbol{T} = \boldsymbol{T}^T > 0\,, \quad \gamma > 0\,, \tag{104}$$

$$\begin{bmatrix} \boldsymbol{\Phi}_o(1,1) & * & * & * \\ \boldsymbol{\Phi}_o(2,1) & -\boldsymbol{\Phi}_o(2,2) & * & * \\ \boldsymbol{C}(\boldsymbol{T}\boldsymbol{E}^T + \boldsymbol{Q}\boldsymbol{S}) & \delta\boldsymbol{C}(\boldsymbol{T}\boldsymbol{E}^T + \boldsymbol{Q}\boldsymbol{S}) & -\gamma\boldsymbol{I}_m & * \\ \boldsymbol{D}^T & \mathbf{0} & \mathbf{0} & -\gamma\boldsymbol{I}_p \end{bmatrix} < 0\,, \tag{105}$$

$$\boldsymbol{\Phi}_o(1,1) = \boldsymbol{A}(\boldsymbol{T}\boldsymbol{E}^T + \boldsymbol{Q}\boldsymbol{S}) + (\boldsymbol{T}\boldsymbol{E}^T + \boldsymbol{Q}\boldsymbol{S})^T\boldsymbol{A}^T - \boldsymbol{B}\boldsymbol{Y}_o\boldsymbol{C} - \boldsymbol{C}^T\boldsymbol{Y}_o^T\boldsymbol{B}^T, \tag{106}$$

$$\boldsymbol{\Phi}_o(2,1) = \delta(\boldsymbol{T}\boldsymbol{E}^T + \boldsymbol{Q}\boldsymbol{S})^T\boldsymbol{A}^T - \delta\boldsymbol{Y}^T\boldsymbol{B}^T + \boldsymbol{P} - \boldsymbol{E}(\boldsymbol{T}\boldsymbol{E}^T + \boldsymbol{Q}\boldsymbol{S}) - \boldsymbol{B}\boldsymbol{Z}_o\boldsymbol{C}\,, \tag{107}$$

$$\boldsymbol{\Phi}_o(2,2) = \delta\boldsymbol{E}(\boldsymbol{T}\boldsymbol{E}^T + \boldsymbol{Q}\boldsymbol{S}) + \delta(\boldsymbol{T}\boldsymbol{E}^T + \boldsymbol{Q}\boldsymbol{S})^T\boldsymbol{E}^T + \delta\boldsymbol{B}\boldsymbol{Z}_o\boldsymbol{C} + \delta\boldsymbol{C}^T\boldsymbol{Z}_o^T\boldsymbol{B}^T\,. \tag{108}$$

$$\boldsymbol{C}(\boldsymbol{T}\boldsymbol{E}^T + \boldsymbol{Q}\boldsymbol{S}) = \boldsymbol{H}\boldsymbol{C}\,, \tag{109}$$

When the above conditions hold, the control law gain matrices are given by

$$\boldsymbol{K}_o = \boldsymbol{Y}_o\boldsymbol{H}^{-1}, \quad \boldsymbol{L}_o = \boldsymbol{Z}_o\boldsymbol{H}^{-1}. \tag{110}$$

Proof: Considering the Lyapunov function candidate (87), (88) and following the same way as above, after some algebraic manipulations it can be obtained

$$\left[\begin{array}{cccc} \boldsymbol{A}_{co}\boldsymbol{P}_3^\circ + \boldsymbol{P}_3^{\circ T}\boldsymbol{A}_{co}^T & \boldsymbol{A}_{co}\boldsymbol{P}_4^\circ + \boldsymbol{P}_1^\circ - \boldsymbol{P}_3^{\circ T}\boldsymbol{W}_{co}^T & \boldsymbol{P}_3^{\circ T}\boldsymbol{C}^T & \boldsymbol{D} \\ \boldsymbol{P}_4^{\circ T}\boldsymbol{A}_{co}^T + \boldsymbol{P}_1^\circ - \boldsymbol{W}_{co}\boldsymbol{P}_3^\circ & -\boldsymbol{W}_{co}\boldsymbol{P}_4^\circ - \boldsymbol{P}_4^{\circ T}\boldsymbol{W}_{co}^T & \boldsymbol{P}_4^{\circ T}\boldsymbol{C}^T & \boldsymbol{0} \\ \boldsymbol{C}\boldsymbol{P}_3^\circ & \boldsymbol{C}\boldsymbol{P}_4^\circ & -\gamma\boldsymbol{I}_m & \boldsymbol{0} \\ \boldsymbol{D}^T & \boldsymbol{0} & \boldsymbol{0} & -\gamma\boldsymbol{I}_p \end{array}\right] < 0\,. \tag{111}$$

Using the substitutions (47), (97) it yields

$$\boldsymbol{B}\boldsymbol{K}_o\boldsymbol{C}(\boldsymbol{T}\boldsymbol{E}^T + \boldsymbol{Q}\boldsymbol{S}) = \boldsymbol{B}\boldsymbol{K}_o\boldsymbol{H}\boldsymbol{H}^{-1}\boldsymbol{C}(\boldsymbol{T}\boldsymbol{E}^T + \boldsymbol{Q}\boldsymbol{S}) = \boldsymbol{B}\boldsymbol{K}_o\boldsymbol{H}\boldsymbol{C} = \boldsymbol{B}\boldsymbol{Y}_o\boldsymbol{C}, \tag{112}$$

$$\boldsymbol{B}\boldsymbol{L}_o\boldsymbol{C}(\boldsymbol{T}\boldsymbol{E}^T + \boldsymbol{Q}\boldsymbol{S}) = \boldsymbol{B}\boldsymbol{L}_o\boldsymbol{H}\boldsymbol{H}^{-1}\boldsymbol{C}(\boldsymbol{T}\boldsymbol{E}^T + \boldsymbol{Q}\boldsymbol{S}) = \boldsymbol{B}\boldsymbol{L}_o\boldsymbol{H}\boldsymbol{C} = \boldsymbol{B}\boldsymbol{Z}_o\boldsymbol{C}, \tag{113}$$

and, consequently,

$$\boldsymbol{\Phi}_o(1,1) = \boldsymbol{A}_{co}\boldsymbol{P}_3^\circ + \boldsymbol{P}_3^{\circ T}\boldsymbol{A}_{co}^T = \boldsymbol{A}(\boldsymbol{T}\boldsymbol{E}^T + \boldsymbol{Q}\boldsymbol{S}) + (\boldsymbol{T}\boldsymbol{E}^T + \boldsymbol{Q}\boldsymbol{S})^T\boldsymbol{A}^T - \boldsymbol{B}\boldsymbol{Y}_o\boldsymbol{C} - \boldsymbol{C}^T\boldsymbol{Y}_o^T\boldsymbol{B}^T, \tag{114}$$

$$\boldsymbol{\Phi}(2,2) = \boldsymbol{W}_{co}\boldsymbol{P}_4^\circ + \boldsymbol{P}_4^{\circ T}\boldsymbol{W}_{co}^T = \boldsymbol{E}(\boldsymbol{T}\boldsymbol{E}^T + \boldsymbol{Q}\boldsymbol{S}) + (\boldsymbol{T}\boldsymbol{E}^T + \boldsymbol{Q}\boldsymbol{S})^T\boldsymbol{E}^T + \delta\boldsymbol{B}\boldsymbol{Z}_o\boldsymbol{C} + \delta\boldsymbol{C}^T\boldsymbol{Z}_o^T\boldsymbol{B}^T, \tag{115}$$

$$\begin{aligned} \boldsymbol{\Phi}(2,1) &= \boldsymbol{P}_4^{\circ T}\boldsymbol{A}_{co}^T + \boldsymbol{P}_1^\circ - \boldsymbol{W}_{co}\boldsymbol{P}_3^\circ \\ &= \delta(\boldsymbol{T}\boldsymbol{E}^T + \boldsymbol{Q}\boldsymbol{S})^T\boldsymbol{A}^T - \delta\boldsymbol{C}^T\boldsymbol{Y}^T\boldsymbol{B}^T + \boldsymbol{P}_1^\circ - \boldsymbol{E}(\boldsymbol{T}\boldsymbol{E}^T + \boldsymbol{Q}\boldsymbol{S}) - \boldsymbol{B}\boldsymbol{Z}\boldsymbol{C}\,, \end{aligned} \tag{116}$$

where

$$\boldsymbol{Y}_o = \boldsymbol{K}_o\boldsymbol{H}\,, \quad \boldsymbol{Z}_o = \boldsymbol{L}_o\boldsymbol{H}\,. \tag{117}$$

Using (102), (103) together with (114)-(116), the resulting formulas to (111) take the forms (105)-(108). This concludes the proof. ■

6. Illustrative example

In the example, there is considered the system (1), (2) in the state-space representation, where the system matrices are

$$\boldsymbol{A} = \left[\begin{array}{rrr} -1.0522 & -1.8666 & 0.5102 \\ -0.4380 & -5.4335 & 0.9205 \\ -0.5522 & 0.1334 & -0.4898 \end{array}\right], \ \boldsymbol{E} = \left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right], \ \boldsymbol{B} = \left[\begin{array}{rr} 3 & 1 \\ 1 & -1 \\ 3 & 0 \end{array}\right], \ \boldsymbol{D} = \left[\begin{array}{r} 0.5 \\ 0.0 \\ -0.4 \end{array}\right],$$

$$\boldsymbol{S} = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right], \ \boldsymbol{C} = \left[\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 1 & 1 \end{array}\right].$$

Solving (81), (82) using Self-Dual-Minimization (SeDuMi) package [13] for MATLAB, the PD control parameter design is feasible for $\delta = 8.5$ and

$$\boldsymbol{P} = \left[\begin{array}{rrr} 3.7029 & 0.4317 & -0.2470 \\ 0.4317 & 4.3529 & -0.6160 \\ -0.2470 & -0.6160 & 3.7778 \end{array}\right], \ \boldsymbol{T} = \left[\begin{array}{rrr} 2.3214 & -0.1005 & 0.0306 \\ -0.1005 & 0.1029 & -0.0287 \\ 0.0306 & -0.0287 & 0.2532 \end{array}\right],$$

$$\boldsymbol{Y} = \left[\begin{array}{rrr} 0.0154 & -0.0040 & 0.1063 \\ 0.3175 & -0.0202 & -0.2021 \end{array}\right], \ \boldsymbol{Z} = \left[\begin{array}{rrr} 0.0154 & -0.0040 & 0.1063 \\ 0.3175 & -0.0202 & -0.2021 \end{array}\right],$$

$$\boldsymbol{Q} = \left[\begin{array}{rcc} -0.0568 & 0 & 0 \\ 0.0586 & 0 & 0 \\ -0.0638 & 0 & 0 \end{array}\right], \ \gamma = 3.1113\,,$$

The PD state control gain matrices are computed using (86) as follows

$$\boldsymbol{K} = \begin{bmatrix} 42.7410 & 41.6846 & -0.0198 \\ 324.6550 & 315.5865 & -4.2538 \end{bmatrix}, \ \boldsymbol{L} = \begin{bmatrix} 25.6406 & 25.1733 & -0.2372 \\ 125.1027 & 120.9441 & -1.5855 \end{bmatrix},$$

and because the matrix $\boldsymbol{W}_c = \boldsymbol{E} + \boldsymbol{B}\boldsymbol{L}$ is regular, the closed-loop of given singular system is regular and the control law guaranties the stable matrix $\boldsymbol{A}_{cr} = \boldsymbol{W}_c^{-1}\boldsymbol{A}_c$, where the regular closed-loop system matrix eigenvalues spectrum is

$$\rho(\boldsymbol{A}_{cr}) = \{-1.6619 \ \ -2.1503 \ \ -3.2236\}\,.$$

Setting the tuning parameter $\delta = 8.5$ and solving the conditions (104)-(106), the LMI matrix variables are

$$\boldsymbol{P} = \begin{bmatrix} 2.5516 & 1.7656 & -1.2073 \\ 1.7656 & 5.4100 & 0.0585 \\ -1.2073 & 0.0585 & 2.1754 \end{bmatrix}, \ \boldsymbol{T} = \begin{bmatrix} 2.7041 & -0.1202 & 0.0228 \\ -0.1202 & 0.1329 & -0.0131 \\ 0.0228 & -0.0131 & 0.2250 \end{bmatrix},$$

$$\boldsymbol{Y}_o = \begin{bmatrix} -0.0161 & 0.0499 \\ 0.2263 & -0.2072 \end{bmatrix}, \ \boldsymbol{Z}_o = \begin{bmatrix} 0.0284 & 0.0019 \\ 0.0867 & -0.1082 \end{bmatrix}, \ \boldsymbol{H} = \begin{bmatrix} 0.0029 & 0.0097 \\ -0.0921 & 0.2119 \end{bmatrix},$$

$$\boldsymbol{Q} = \begin{bmatrix} -0.0546 & 0 & 0 \\ 0.0575 & 0 & 0 \\ -0.1496 & 0 & 0 \end{bmatrix}, \ \gamma = 2.9852\,.$$

The obtained output PD controller gain matrices are

$$\boldsymbol{K}_o = \begin{bmatrix} 0.7739 & 0.1999 \\ 19.0529 & -1.8541 \end{bmatrix}, \ \boldsymbol{L}_o = \begin{bmatrix} 4.0836 & -0.1787 \\ 5.5590 & -0.7660 \end{bmatrix}$$

and because the matrix $\boldsymbol{W}_{co} = \boldsymbol{E} + \boldsymbol{B}\boldsymbol{L}_o\boldsymbol{C}$ is regular, the closed-loop with the singular system under the output PD controller is regular and guaranties the stable matrix $\boldsymbol{A}_{cro} = \boldsymbol{W}_{co}^{-1}\boldsymbol{A}_{co}$, where the regular closed-loop system matrix eigenvalues spectrum is

$$\rho(\boldsymbol{A}_{cro}) = \{-0.9701 \ \ -1.3546 \ \ -4.5882\}\,.$$

Thus, one can see that the assymptotic stability is guarantied.

7. Concluding remarks

In the paper, new enhanced formulations of the PD control parameter design for continuous-time descriptor linear systems are given in the strict LMI forms to retain admissibility condition and eliminate the impulsive modes of the plant. Applying linear feedback PD control laws to the system (1), (2), the triple ($\boldsymbol{A}$,$\boldsymbol{E}$,$\boldsymbol{B}$) is finite dynamics stabilizable. The proposed new formulations are fully adapted to the H_∞control problem design by introducing a tuning slack variable. The design approach can be preferable in computations, and simple in a reference model control, since gives the regular nominal closed-loop system dynamic properties. A simulation example, subject to given type of controller, demonstrates the effectiveness of the proposed LMI-based design forms.

Acknowledgment

The work presented in the paper was supported by VEGA, the Grant Agency of the Ministry of Education and the Academy of Science of Slovak Republic, under Grant No. 1/0348/14. This support is very gratefully acknowledged.

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