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Robust stabilization using LMI techniques of neutral time-delay systems subject to input saturation

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Abstract. The robust stabilization of uncertain saturated neutral systems with state delay is solved in this paper: based on a free weighting matrix approach, sufficient conditions are obtained via an LMI formulation. From these conditions, state feedback gains that ensure stability for the largest set of admissible initial conditions can be calculated solving optimization problems with LMI constraints. Some applications of this methodology to feedback control are then presented and compared with previous results in the literature.

1. Introduction

The stability analysis of time-delay systems is a topic of theoretical and practical importance, because delays appear in many areas: mechanics, physics, biology, economy, epidemics, population dynamic models, large-scale systems, automatic control systems, neural networks, chaotic systems, practical control system, and so on. It is well known that delays often lead to poor system performance, even instabilities. Therefore, stability analysis of time-delay systems has been extensively studied by many researchers (see, for example, [3, 18, 19, 28, 29]).

Moreover, almost all practical control systems are subject to input saturation, because of the existence of physical, technological or even safety constraints (see [5, 6, 7, 8, 13]). Input saturation is a also a source of performance degradation, generating also limit cycles, multiple equilibrium points and even instability. For time-delay systems, some works addressing the stability analysis and stabilization in the presence of saturating control signals can be found in the literature. In [27], conditions for stability or stabilization are proposed with state feedback. However, in that paper, the set of admissible initial conditions, for which the asymptotic stability is ensured (i.e. the domain of attraction) in the presence of control saturation, is not mentioned or explicitly defined. In [1], method for computing stabilizing state feedback control laws aiming at enlarging well defined estimates of the domain of attraction of the closed-loop system have been proposed. This method is based on the use of polytopic differential inclusions for describing the behavior of the closed-loop system with saturating inputs. In [25], the synthesis of stabilizing static anti-windup loops is addressed for the case of retarded systems presenting fixed delays. On the other hand, considering neutral systems, we can cite [11, 12, 19, 26]. In that paper, using a polytopic approach for modelling saturation effects, a method for computing stabilizing state feedback controls with the aim of maximizing the set of admissible initial conditions is proposed.

Thus, in this paper the stabilization problem for neutral systems with time-varying delay and actuator saturation subject to uncertainty is solved. By incorporating Lyapunov-Krasovskii (L-K) functional theory, free-weighting matrix technique, integral inequalities and function that corresponds to a decentralized dead-zone nonlinearity, then efficient stabilization conditions are obtained in terms of LMIs. Compared with the existing results, an optimization problem is formulated with the aim of computing stabilizing state feedback control laws. This optimization problem search the maximal delay bound for which a stabilizing control law can be found. On the other hand, when the open-loop system is unstable, the optimization objective consists in finding a control law that maximizes an estimate of the domain of attraction, or alternatively that ensures the stability for a given set of admissible initial states.

The rest of this paper is organized as follows: in Section 2, we formulate the problem of stabilization of uncertain neutral time-delay system with saturating actuator. The main results are presented in Section 3. Finally, three numerical examples are included to illustrate the results developed in this paper.

Notation: The Banach space of continuous vector functions mapping the interval $[-h_m, 0]$ into $\Re^n$ with the norm $\|\phi\|_c = \sup_{-h_m \leq t \leq 0} \|\phi(t)\|$ is denoted by $\mathcal{C}_{h_m} = \mathcal{C}([-h_m, 0], \Re^n)$. Additionally, $\overline{\lambda}(P)$ denotes the maximal eigenvalue of matrix P.

2. Problem Formulation and Preliminaries

Consider the following uncertain neutral delayed system with saturating actuators

$$\begin{aligned} &\dot{x}(t) - (C + \Delta C(t))\dot{x}(t - \tau(t)) = (A + \Delta A(t))x(t) + (A_d + \Delta A_d(t))x(t - \tau(t)) + Bu(t) \\ &x(t) = \phi(t), \ \forall t \in [-h_m, 0], \ \phi(t) \in \mathcal{C}_{h_m} \end{aligned} \tag{1}$$

where $x(t) \in \Re^n$ and $u(t) \in \Re^m$ are respectively the state and the control vectors. The matrices C, A, A_d and B are real constants of appropriate dimensions. the delay $\tau(t)$ is assumed to be unknown but bounded function of time, continuously differentiable, with their rate of change bounded as follows

$$0 \leq \tau(t) \leq h_m \ , \ \dot{\tau}(t) \leq d \tag{2}$$

we assume $0 < d < 1$ to ensure causality (see [3]).

In this paper, the uncertainties can be described as follows

$$[\Delta A(t) \quad \Delta A_d(t) \quad \Delta C(t)] = DF(t)[E_0 \quad E_1 \quad E_2] \tag{3}$$

where D, E_0, E_1 and E_2 are known real matrices of appropriate dimensions, and $F(t)$ denotes the time-varying parameter uncertainties with Lebesgue-measurable elements satisfying $F^T(t)F(t) \leq I$, $\forall t \geq 0$.

We suppose that the input vector u is subject to amplitude limitations defined as follows

$$|u_i| \leq u_{0i}, \ u_{0i} > 0; \ i = 1, \ldots, m \tag{4}$$

The control input is $u(t) = Kx(t)$. Due to the control bounds defined in (4), the effective control signal to be applied to the system is $u(t) = sat(Kx(t))$ where $u_i(t) = sat(K_i x(t)) = sign(K_i x(t)) min\{u_{0i}, |K_i x(t)|\}$.

Hence, the closed-loop system (1) reads

$$\begin{aligned} \dot{x}(t) - (C + \Delta C(t))\dot{x}(t - \tau(t)) \ = \ & (A + \Delta A(t))x(t) + (A_d + \Delta A_d(t))x(t - \tau(t)) \\ & + Bsat(Kx(t)) \end{aligned} \tag{5}$$

In this paper, it is assumed that $\phi(t)$ is continuously differentiable over $[-h_m, 0]$, and one of our interests is to estimate the domain of attraction of the following form

$$\Xi = \{\phi(t) \in \mathcal{C}_{h_m} : \|\phi\|_c \leq \delta_1, \ \ \|\dot{\phi}\|_c \leq \delta_2\}$$

where δ_1 and δ_2 are some scalars to be determined.

Define the following function $\psi(Kx(t)) = Kx(t) - sat(Kx(t))$ where $\psi(Kx(t))$ corresponds to a decentralized dead-zone nonlinearity. Thus, the closed-loop system (5) can be written as

$$\begin{aligned} \dot{x}(t) - (C + \Delta C(t))\dot{x}(t - \tau(t)) &= (A + \Delta A(t) + BK)x(t) + (A_d + \Delta A_d(t))x(t - \tau(t)) \\ &\quad -B\psi(Kx(t)) \end{aligned} \tag{6}$$

Furthermore, the following useful lemmas are used in this paper

Lemma 2.1: [24] Let Ω, D, and E be real matrices of appropriate dimensions. Then for any $F(t)$

$$\Omega + DF(t)E + E^T F^T(t) D^T < 0$$

if and only if there exists some $\varepsilon > 0$ such that

$$\Omega + \varepsilon DD^T + \varepsilon^{-1} E^T E < 0.$$

Lemma 2.2: [18] Jensen Inequality: For any scalar $b > a$, the following inequality holds

$$(b-a)\int_a^b x^T(s)Rx(s)ds \geq \Big(\int_a^b x(s)ds\Big)^T R\Big(\int_a^b x(s)ds\Big)$$

Considering a matrix G and defining the following polyhedral set

$$\mathcal{S} = \{x(t) \in \Re^n; |(K_{(i)} - G_{(i)})x(t)| \leq u_{0(i)}\}$$

Lemma 2.3: [25] If $x(t) \in \mathcal{S}$, then the following relation

$$\psi^T(Kx(t))T_0\Big[\psi(Kx(t)) - Gx(t)\Big] \leq 0$$

is verified for any diagonal positive matrix $T_0 \in \Re^{m\times m}$.

Consider that $x(t) \in \mathcal{S}$ and the following L-K functional which will be used throughout the paper

$$\begin{aligned} V(t) &= x^T(t)Px(t) + \int_{t-\tau(t)}^t x^T(s)Qx(s)ds + \int_{-h_m}^0 \int_{t+\theta}^t \dot{x}^T(s)R\dot{x}(s)dsd\theta \\ &\quad + \int_{t-\tau(t)}^t \dot{x}^T(s)W\dot{x}(s)ds \end{aligned} \tag{7}$$

where $P, Q, R, W > 0$ need to be determined.

Finally, for a positive scalar β, the ellipsoid D_e is defined as follows

$$D_e = \{x(t) \in \Re^n; \ \ x^T(t)Px(t) \leq \beta^{-1}\}$$

The result in Lemma 2.3 can be seen as a generalized sector condition. As will be seen in the sequel, differently from the classical sector condition (used for instance in [25]), this condition will allow to obtain stability conditions directly in an LMI form.

The stabilization problems that we are interested in studying can be summarized as follows

Problem 2.1: Given d, maximize h_m in order to ensure the robust stability of the closed-loop system for some set of admissible initial conditions.

Problem 2.2: Given h_m and d, find K and a set of admissible initial conditions, as large as possible, for which the stability of the closed-loop system is ensured.

3. Main Results

Some results are derived to ensure the robust stabilization of neutral system with saturating actuators. These results will be applied to a real system control problem.

3.1. Stability results

Some results are established to ensure the asymptotic stabilization of the nominal system $(DF(t)E_0 = DF(t)E_1 = DF(t)E_2 = 0)$. Next the robust stabilization of the uncertain system is derived for any initial conditions in an estimated domain of attraction.

Lemma 3.1: If there exist symmetric positive definite matrices P, Q, R, W, appropriately sized matrices T_1, T_2, Y_1, Y_2, T_1, T_2 and a positive definite diagonal matrix T_0 satisfying

$$\Psi = \begin{bmatrix} \Psi_{11} & * & * & * & * & * \\ \Psi_{21} & \Psi_{22} & * & * & * & * \\ \Psi_{31} & \Psi_{32} & \Psi_{33} & * & * & * \\ \Psi_{41} & \Psi_{42} & 0 & \Psi_{44} & * & * \\ \Psi_{51} & \Psi_{52} & 0 & 0 & \Psi_{55} & * \\ \Psi_{61} & \Psi_{62} & 0 & 0 & 0 & \Psi_{66} \end{bmatrix} < 0 \tag{8}$$

where

$$\begin{aligned}
\Psi_{11} &= T_1A + A^TT_1^T + T_1BK + K^TB^TT_1^T + Y_1 + Y_1^T + Q, \;\; \Psi_{31} = A_d^TT_1^T - Y_1^T \\
\Psi_{21} &= -T_1^T + T_2A + T_2BK + Y_2 + P, \;\; \Psi_{22} = -T_2 - T_2^T + h_mR + W, \;\; \Psi_{32} = A_d^TT_2^T - Y_2^T \\
\Psi_{33} &= -(1-d)Q, \;\; \Psi_{41} = -Y_1^T, \;\; \Psi_{42} = -Y_2^T, \;\; \Psi_{44} = -\frac{R}{h_m}, \;\; \Psi_{51} = C^TT_1^T, \;\; \Psi_{52} = C^TT_2^T \\
\Psi_{55} &= -(1-d)W, \;\; \Psi_{61} = T_0G - B^TT_1^T, \;\; \Psi_{62} = -B^TT_2^T, \;\; \Psi_{66} = -2T_0
\end{aligned}$$

Then, the nominal system (6) is asymptotically stable.

Proof 1: Using (2), the time derivative of the functional (7) along the trajectory of the system (6) is given by

$$\begin{aligned}
\dot{V}(t) &= 2x^T(t)P\dot{x}(t) + x^T(t)Qx(t) - (1-d)x^T(t-\tau(t))Qx(t-\tau(t)) + h_m\dot{x}^T(t)R\dot{x}(t) \\
&\quad - \int_{t-\tau(t)}^{t} \dot{x}^T(s)R\dot{x}(s)ds + \dot{x}^T(t)W\dot{x}(t) - (1-d)\dot{x}^T(t-\tau(t))W\dot{x}(t-\tau(t))
\end{aligned}$$

Using the free weighting matrix approach, for appropriately dimensioned matrices T_1, T_2, Y_1 and Y_2, we have

$$\begin{aligned}
&2\Big[x^T(t)T_1 + \dot{x}^T(t)T_2\Big]\Big[-\dot{x}(t) + C\dot{x}(t-\tau(t)) + (A+BK)x(t) + A_dx(t-\tau(t)) \\
&\qquad\qquad -B\psi(Kx(t))\Big] = 0 \\
&2\Big[x^T(t)Y_1 + \dot{x}^T(t)Y_2\Big]\Big[x(t) - x(t-\tau(t)) - \int_{t-\tau(t)}^{t}\dot{x}(s)ds\Big] = 0
\end{aligned} \tag{9}$$

From Lemma 2.3, it follows that

$$\dot{V}(t) \leq \dot{V}(t) - 2\psi^T(Kx(t))T_0[\psi(Kx(t)) - Gx(t)] \tag{10}$$

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Then, applying Lemma 2.2, adding the terms on the left of (9) and (10) to $\dot{V}(t)$ allows us to express $\dot{V}(t)$ as

$$\begin{aligned}\dot{V}(t) \;\le\; & 2x^T(t)P\dot{x}(t) + x^T(t)Qx(t) - (1-d)\Big(x^T(t-\tau(t))Qx(t-\tau(t)) + \dot{x}^T(t-\tau(t)) \\ & \times W\dot{x}(t-\tau(t))\Big) + \dot{x}^T(t)\Big(h_m R + W\Big)\dot{x}(t) - \Big(\int_{t-\tau(t)}^{t}\dot{x}(s)ds\Big)^T \frac{R}{h_m}\Big(\int_{t-\tau(t)}^{t}\dot{x}(s)ds\Big) \\ & +2\Big[x^T(t)T_1 + \dot{x}^T(t)T_2\Big]\Big[-\dot{x}(t) + C\dot{x}(t-\tau(t)) + (A+BK)x(t) + A_d x(t-\tau(t)) \\ & -B\psi(Kx(t))\Big] + 2\Big[x^T(t)Y_1 + \dot{x}^T(t)Y_2\Big]\Big[x(t) - x(t-\tau(t)) - \int_{t-\tau(t)}^{t}\dot{x}(s)ds\Big] \\ & -2\psi^T(Kx(t))T_0\Big[\psi(Kx(t)) - Gx(t)\Big] \end{aligned} \qquad (11)$$

By simple manipulation, (11) can be rewritten as

$$\dot{V}(t) \le \eta^T(t)\Psi\eta(t)$$

where Ψ is defined in (8) and

$$\eta^T(t) = \begin{bmatrix} x^T(t) & \dot{x}^T(t) & x^T(t-\tau(t)) & \int_{t-\tau(t)}^{t}\dot{x}^T(s)ds & \dot{x}^T(t-\tau(t)) & \psi^T(Kx(t)) \end{bmatrix}$$

The condition (8) holds implies that $\dot{V}(t) < 0$. It follows that the trajectories of nominal system (6) converge asymptotically to the origin. This result gives a general solution for testing stability. In the following, we provide a new result that permits a robust stabilizing controller to be calculated. Thus, based on Lemma 3.1, some results are now derived to ensure robust stabilization of saturated neutral state-delayed system subject to uncertainty (3).

Theorem 3.1: If there exist symmetric positive definite matrices $\overline{P}$, $\overline{Q}$, $\overline{R}$, $\overline{W}$, appropriately sized matrices X, $\overline{Y_1}$, $\overline{Y_2}$, M, U, a diagonal matrix S of appropriate dimension, a real scalar α and positive scalars $\varepsilon, \beta, \delta$ satisfying the conditions (12)-(14)

$$\begin{bmatrix} \Omega_{11} + \varepsilon DD^T & * & * & * & * & * & * \\ \Omega_{21} + \alpha\varepsilon DD^T & \Omega_{22} + \alpha^2\varepsilon DD^T & * & * & * & * & * \\ \Omega_{31} & \Omega_{32} & \Omega_{33} & * & * & * & * \\ \Omega_{41} & \Omega_{42} & 0 & \Omega_{44} & * & * & * \\ \Omega_{51} & \Omega_{52} & 0 & 0 & \Omega_{55} & * & * \\ \Omega_{61} & \Omega_{62} & 0 & 0 & 0 & \Omega_{66} & * \\ E_0X^T & 0 & E_1X^T & 0 & E_2X^T & 0 & -\varepsilon I \end{bmatrix} < 0 \qquad (12)$$

$$\begin{bmatrix} \overline{P} & * \\ U_i - M_i & \beta u_{0i}^2 \end{bmatrix} \ge 0, \; i = 1, ..., m \qquad (13)$$

$$\begin{aligned} & \Big(\overline{\lambda}(X^{-1}\overline{P}X^{-T}) + h_m\overline{\lambda}(X^{-1}\overline{Q}X^{-T})\Big)\|\phi\|_c^2 \\ & +\Big(\frac{h_m^2}{2}\overline{\lambda}(X^{-1}\overline{R}X^{-T}) + h_m\overline{\lambda}(X^{-1}\overline{W}X^{-T})\Big)\|\dot{\phi}\|_c^2 \le \beta^{-1} \end{aligned} \qquad (14)$$

where

$$\begin{aligned} \Omega_{11} &= AX^T + XA^T + BU + U^TB^T + \overline{Y_1} + \overline{Y_1}^T + \overline{Q}, \; \Omega_{21} = -X + \alpha AX^T + \alpha BU + \overline{P} + \overline{Y_2} \\ \Omega_{22} &= -\alpha X - \alpha X^T + h_m\overline{R} + \overline{W}, \; \Omega_{31} = XA_d^T - \overline{Y_1}^T, \; \Omega_{32} = \alpha XA_d^T - \overline{Y_2}^T \\ \Omega_{33} &= -(1-d)\overline{Q}, \; \Omega_{41} = -\overline{Y_1}^T, \; \Omega_{42} = -\overline{Y_2}^T, \; \Omega_{44} = -\frac{\overline{R}}{h_m}, \; \Omega_{51} = XC^T, \; \Omega_{52} = \alpha XC^T \\ \Omega_{55} &= -(1-d)\overline{W}, \; \Omega_{61} = M - SB^T, \; \Omega_{62} = -\alpha SB^T, \; \Omega_{66} = -2S^T \end{aligned}$$

then, the uncertain saturated delayed system is robustly stable and the trajectories of $x(t)$ remain within the ellipsoid D_e when the state feedback control law is used, with $K = UX^{-T}$.

Proof 2: Set $T_2 = \alpha T_1$, $\alpha > 0$. From (8), it is easily seen that T_2 is nonsingular and consequently T_1 is inversible. Multiplying (8) by $diag\{T_1^{-1}, T_1^{-1}, T_1^{-1}, T_1^{-1}, T_1^{-1}, T_0^{-1}\}$ on the left and by its transpose on the right. Then, introduce a new change of variables such that $X = T_1^{-1}$, $U = KX^T$, $M = GX^T$ $S = T_0^{-1}$ and $\overline{\Pi} = X\Pi X^T$ where $\Pi = P, Q, R, W, Y_1, Y_2$. Thus, we obtain the following equation

$$\Omega = \begin{bmatrix} \Omega_{11} & * & * & * & * & * \\ \Omega_{21} & \Omega_{22} & * & * & * & * \\ \Omega_{31} & \Omega_{32} & \Omega_{33} & * & * & * \\ \Omega_{41} & \Omega_{42} & 0 & \Omega_{44} & * & * \\ \Omega_{51} & \Omega_{52} & 0 & 0 & \Omega_{55} & * \\ \Omega_{61} & \Omega_{62} & 0 & 0 & 0 & \Omega_{66} \end{bmatrix} < 0 \tag{15}$$

Now, let us replace A, A_d and C by $A + DF(t)E_0$, $A_d + DF(t)E_1$ and $C + DF(t)E_2$, respectively. We find that equation (15) is equivalent to the following condition

$$\Omega \; + \begin{bmatrix} D \\ \alpha D \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} F(t) \begin{bmatrix} E_0X^T & 0 & E_1X^T & 0 & E_2X^T & 0 \end{bmatrix}$$
$$+ \begin{bmatrix} XE_0^T \\ 0 \\ XE_1^T \\ 0 \\ XE_2^T \\ 0 \end{bmatrix} F^T(t) \begin{bmatrix} D^T & \alpha D^T & 0 & 0 & 0 & 0 \end{bmatrix} < 0$$

According to Lemma 2.1 and by the Schur complement we obtain LMI (12). On other hand, the satisfaction of (13) guarantees that $\forall x \in D_e$, $x \in \mathcal{S}$. In fact, $D_e \subset \mathcal{S}$ is verified by the following conditions

$$\begin{bmatrix} P & * \\ K_i - G_i & \beta u_{0i}^2 \end{bmatrix} \geq 0 \tag{16}$$

pre and post multiplying (16) by $\Delta = diag\{X, I\}$ and its transpose will result in the LMI (13).

Moreover, the satisfaction of (14) can be proven as follows. From the L-K functional defined in (7) we have

$$\begin{aligned} V(0) \;&\leq\; x^T(0)Px(0) + \int_{-h_m}^{0} x^T(s)Qx(s)ds + \int_{-h_m}^{0}\int_{\theta}^{0} \dot{x}^T(s)R\dot{x}(s)dsd\theta + \int_{-h_m}^{0} \dot{x}^T(s)W\dot{x}(s)ds \\ &\leq \Big(\overline{\lambda}(P) + h_m\overline{\lambda}(Q)\Big)\|\phi(\theta)\|_c^2 + \Big(\frac{h_m^2}{2}\overline{\lambda}(R) + h_m\overline{\lambda}(W)\Big)\|\dot{\phi}(\theta)\|_c^2 = \delta \end{aligned}$$

Therefore, we have $x^T(t)Px(t) \leq V(t) \leq V(0) \leq \delta \leq \beta^{-1}$, that is for all $t \geq 0$ the trajectories of the system do not leave the set D_e for any initial functions $\phi(\theta)$ in D_e which ensures that $x(t) \in \mathcal{S}$. This completes the proof.

The above results can be applied to the real system control problem by adopting a fourth-order linearized model of the match number in a wind tunnel.

3.2. Application to a feedback control of Mach Number in a Wind Tunnel
In steady-state operating conditions (some constant fan speed, liquid nitrogen injection rate, and gaseous-nitrogen vent rate), the dynamic response of the Mach Number perturbations δM to small perturbations in the guide vane angle actuator $\delta\theta_a$ in a driving fan is described by the following equations [22, 23]

$$\begin{aligned} \frac{1}{a}\delta\dot{M}(t) + \delta M(t) &= k\delta\theta(t-\tau(t)) \\ \delta\ddot{\theta}(t) + 2\xi w\delta\dot{\theta}(t) + w^2\delta\theta(t) &= w^2\delta\theta_a(t) \end{aligned} \tag{17}$$

where $\delta\theta$ is the guide vane angle, a, k, ξ, w are parameters depending on the operating point which are presumed constant when the perturbation δM, $\delta\theta$, $\delta\theta_a$ are small and the delay $\tau(t)$ represents the time of the transport between the fan and the test section.

Rewriting (17) in state space form yields

$$\begin{aligned} &\dot{x}(t) = Ax(t) + A_d x(t-\tau(t)) + Bsat(u(t)), \ \ t > 0 \\ &x(\theta) = \phi(\theta), \ \ \forall\theta \in [-h_m, 0], \ \ \phi(\theta) \in \mathcal{C}^v_{h_m}, \end{aligned} \tag{18}$$

where

$$x = \begin{bmatrix} \delta M \\ \delta\theta \\ \delta\theta_a \end{bmatrix}, \ A = \begin{bmatrix} -a & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -w^2 & -2\xi w \end{bmatrix}, \ A_\tau = \begin{bmatrix} 0 & ka & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ 0 \\ w^2 \end{bmatrix}$$

The control $u(t)$ represents $\delta\theta_a$. This system can be seen as a particular case of nominal system (5) when $C = \Delta A(t) = \Delta A_d(t) = \Delta C(t) = 0$. Then, the following theorem gives a condition to stabilize system (18).

Theorem 3.2: If there exist symmetric positive definite matrices $\overline{P}$, $\overline{Q}$, $\overline{R}$, appropriately sized matrices X, $\overline{Y_1}$, $\overline{Y_2}$, M, U, a diagonal matrix S of appropriate dimension and a real scalar α satisfying the condition (19)-(21)

$$\begin{bmatrix} \Omega_{11} & * & * & * & * \\ \Omega_{21} & \Omega_{22} - \overline{W} & * & * & * \\ \Omega_{31} & \Omega_{32} & \Omega_{33} & * & * \\ \Omega_{41} & \Omega_{42} & 0 & \Omega_{44} & * \\ \Omega_{61} & \Omega_{62} & 0 & 0 & \Omega_{66} \end{bmatrix} < 0 \tag{19}$$

$$\begin{bmatrix} \overline{P} & * \\ U_i - M_i & \beta u_{0i}^2 \end{bmatrix} \geq 0 \tag{20}$$

$$\Big(\overline{\lambda}(X^{-1}\overline{P}X^{-T}) + h_m\overline{\lambda}(X^{-1}\overline{Q}X^{-T})\Big)\|\phi(\theta)\|_c^2 + \Big(\frac{h_m^2}{2}\overline{\lambda}(X^{-1}\overline{R}X^{-T})\Big)\|\dot{\phi}(\theta)\|_c^2 \leq \beta^{-1} \tag{21}$$

then, the saturated state-delayed system (18) with the state feedback control law and $K = UX^{-T}$ is robustly stable and the trajectories of $x(t)$ remain within the ellipsoid D_e.

Proof 3: It suffices to follow the same steps of the Proof 1 and 2 considering $W = 0$, $C = 0$ and $\Delta A(t) = \Delta A_d(t) = \Delta C(t) = 0$.

Remark 3.1: In deriving Lemma 3.1, the slack variable T_1, T_2, Y_1, Y_2 are introduced in order to reduce the conservatism of the asymptotic stability conditions. It can be seen from the Proof 1 that $\dot{V}(t)$ remains unaffected by the introduction equation (9) i.e. the slack variables T_1, T_2,

Y_1, Y_2. So these matrices leads to a more flexible LMI condition in (8) and consequently reduce the conservatism of Lemma 3.1 and consequently that of the next results. This advantage can be seen from the numerical examples.

Remark 3.2: In the above proof, the free-weighting matrices T_1, T_2, Y_1, Y_2 are employed to offer more flexibility to our results which may lead to some improvements. Specifically, the first equation of the (9) which is equal to zero, was added to the derivative of the L-K functional. Moreover, the Leibniz-Newton formula was employed to obtain a delay-dependent criterion, and the relationships between those terms was also taken into account. That is, the second equation of the (9) which is equal to zero, was added to the derivative of the L-K functional, as well.

Remark 3.3: The tuning parameter α is designed to improve results and gives a better performance to system. An adequate choice of this parameter gives improved result for which the system is robustly stable.

4. Optimization Problems

In this section we show how the theoretical conditions can be casted into LMI-based optimization problems to determine a suitable stabilizing gain K and a domain of attraction which ensure that the state trajectory of the closed-loop system (6) starting from any initial functions $\phi(\theta)$ in D_e will remain within D_e for all $t > 0$.

The idea is to develop a methodology to estimate the largest possible domain of initial conditions for which it can be ensured that the closed-loop system trajectories remain bounded. As in [5], we impose the following conditions

$$\begin{bmatrix} \sigma_1 I & \widetilde{X} \\ \widetilde{X}^T & \widetilde{P} \end{bmatrix} \geq 0, \begin{bmatrix} \sigma_2 I & \widetilde{X} \\ \widetilde{X}^T & \widetilde{Q} \end{bmatrix} \geq 0, \begin{bmatrix} \sigma_3 I & \widetilde{X} \\ \widetilde{X}^T & \widetilde{R} \end{bmatrix} \geq 0, \begin{bmatrix} \sigma_4 I & \widetilde{X} \\ \widetilde{X}^T & \widetilde{W} \end{bmatrix} \geq 0 \tag{22}$$

It follows that condition (14) is satisfied if the following LMI holds

$$\left[\sigma_1 + h_m\sigma_2 + \frac{h_m^2}{2}\sigma_3 + h_m\sigma_4\right]\delta^2 \leq \beta^{-1} \tag{23}$$

where $\delta^2 = \max(\|\phi(\theta)\|_c^2, \|\dot{\phi}(\theta)\|_c^2)$, $X^{-1} = \widetilde{X}$ and $\overline{\Pi}^{-1} = \widetilde{\Pi}$ with $\Pi = P, Q, R, W$ and the *stability radius* $\delta > 0$ a scalar to be determined.

Therefore, let us construct a feasibility problem as follows

$$\begin{aligned}
&\text{Minimize Trace}\Big(\overline{P}\widetilde{P} + \overline{Q}\widetilde{Q} + \overline{R}\widetilde{R} + \overline{W}\widetilde{W} + (X + X^T)(\widetilde{X} + \widetilde{X}^T)\Big) \\
&\text{subject to } \overline{\Pi} > 0, \widetilde{\Pi} > 0, \beta > 0, \delta > 0, \sigma_{i=1,\ldots,4} > 0, (12),(13),(22),(23), \\
&\begin{bmatrix} \overline{P} & * \\ I & \widetilde{P} \end{bmatrix} \geq 0, \begin{bmatrix} \overline{Q} & * \\ I & \widetilde{Q} \end{bmatrix} \geq 0, \begin{bmatrix} \overline{R} & * \\ I & \widetilde{R} \end{bmatrix} \geq 0, \begin{bmatrix} \overline{W} & * \\ I & \widetilde{W} \end{bmatrix} \geq 0, \\
&\begin{bmatrix} X + X^T & * \\ I & \widetilde{X} + \widetilde{X}^T \end{bmatrix} \geq 0.
\end{aligned} \tag{24}$$

The new LMIs problem can be solved by using the following like cone complementarity algorithm

<u>*Step 1:*</u> Given h_m, β, fix initial values $\alpha = \alpha_0$ and choose a sufficiently large initial δ such that there exits a feasible solution to LMI conditions in (24). Set $\delta_0 = \delta$ and $\alpha_0 = \alpha$.

<u>*Step 2:*</u> Find a set of feasible matrices $\left(\overline{\Pi}, X, \widetilde{\Pi}, \widetilde{X}, \sigma_{i=1,\ldots,4}\right)_0$ that satisfies (24).

Step 3: Solve the following LMI minimization problem

$$\text{Minimize Trace}\Big(\overline{P}\widetilde{P}_0+\overline{Q}\widetilde{Q}_0+\overline{R}\widetilde{R}_0+\overline{W}\widetilde{W}_0+(X+X^T)(\widetilde{X}_0+\widetilde{X}_0^T)+\overline{P}_0\widetilde{P}+\overline{Q}_0\widetilde{Q}+\overline{R}_0\widetilde{R}$$
$$+\overline{W}_0\widetilde{W}+(X_0+X_0^T)(\widetilde{X}+\widetilde{X}^T)\Big)$$

subject to LMIs in (24)

Step 4: Substitute the new matrix variables from the previous step into (24). If the result is feasible, then set $\delta_0=\delta$ and $\alpha_0=\alpha$. If it is not feasible, then set the new matrices to be $\left(\overline{\Pi},X,\widetilde{\Pi},\widetilde{X},\sigma_{i=1,\ldots,4}\right)_0$ and go to step 3.

5. Illustrative Examples

In this section we illustrate our methodology over three examples borrowed from [2, 4, 20].
Example 1 [2]. Consider the nominal system of (1), where

$$A=\begin{bmatrix}0.5 & -1\\ 0.5 & -0.5\end{bmatrix},\ A_d=\begin{bmatrix}0.6 & 0.4\\ 0 & -0.5\end{bmatrix},\ B=\begin{bmatrix}1\\1\end{bmatrix},\ u_0=5,\ d=0,\ C=0,\ W=0$$

We can apply the stability results presented in Theorem 3.1 with $\Delta A(t)=\Delta A_d(t)=\Delta C(t)=0$. Taking $\beta=1$, we obtain a stability radius $\delta=271$ when a tuning parameter is $\alpha=143$. For the comparison with other approaches we give the Table 1

Table 1. Comparison of h_m with obtained K.

Approach	h_m	K
[1]	0.35	Not Reported
[15]	1.854	$(-25.8809\quad -4.9315)$
[30]	2.248	Not Reported
[10]	2	$(-\ 5.7702\quad -0.9754)$
[9]	2	$(-\ 5.6104\quad -0.9147)$
Th. 3 [4]	1.854	$(-\ 1.7008\quad 0.2776)$
[2]	1.854	$(-\ 2.2346\quad 0.0580)$
Th. 3.1	3.262	$(-\ 3.9263\quad -2.5783)\times 10^3$

Note that as pointed in Section 2, the effective control signal applied to the system is $u(t)=sat(Kx(t))$. in this case, $u(t)$ is a scalar so we have $u(t)=sat(Kx(t))=sign(Kx(t))min\{u_0,|Kx(t)|\}$ and the amplitude limitation of the input is always satisfied.

From the numerical example it is clear that our approach is less conservative in stabilizing the system with larger time-delay bound than those of [1, 15, 30, 10, 9, 4, 2].
Example 2 [4]. The system is described by (1) with

$$A=\begin{bmatrix}1 & 1.5\\ 0.3 & -2\end{bmatrix},\ A_d=\begin{bmatrix}0 & -1\\ 0 & 0\end{bmatrix},\ B=\begin{bmatrix}10\\1\end{bmatrix},\ C=cI,\ u_0=15,\ h_m=1,\ D=E_{0,1,2}=0$$

Taking $\beta=1$ and applying the proposed algorithm, we obtain a stability radius of $\delta=344$ and $\delta=404$. Their results are listed in Table 2 and Table 3 along with the results obtained by Theorem 3.1 for $\alpha=111$ and $\alpha=94$, respectively.

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The domain of initial condition obtained for neutral system in which $c = 0.2$ and $d = 0.1$ is given by $\delta = 344$, while for retarded system (i.e. $c = 0$), it is given by $\delta = 404$. For the comparison with other approaches, we give Table 2 for neutral system and Table 3 for time delay system.

It is clear that the obtained δ is significantly larger than those obtained in [17, 16, 2, 1, 15, 4].

To see graphically the improvements of our approach, we represent the domains of attraction of Table 2 and Table 3 in Figure 1 and Figure 2.

Table 2. Comparison of δ and K for $c = 0.2$ and $d = 0.1$.

Approach	δ	K
[17]	12.88	$(-0.2780 \quad -0.1390)$
[16]	70.74	$(-0.1325 \quad 0.0153)$
[2]	76.2262	$(-0.2359 \quad -0.0453)$
Th. 3.1	344	$(-0.1032 \quad -0.0272)$

Table 3. Comparison of δ and K for $c = d = 0$.

Approach	δ	K
[1]	67.0618	Not Reported
[15]	79.43	$(-7.9130 \quad 0.7323)$
[16]	83.55	$(-0.1950 \quad 0.0649)$
[2]	84.6074	$(-0.2223 \quad -0.0246)$
Th. 3 [4]	106.2856	$(-0.6646 \quad -0.0239)$
Th. 3.1	404	$(-0.1013 \quad -0.0495)$

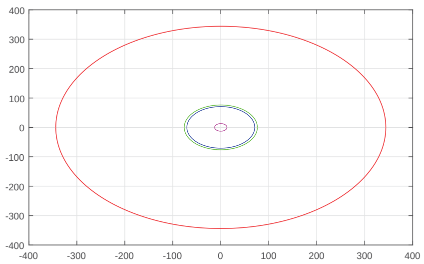

Figure 1. The domain of attraction in Table 2.

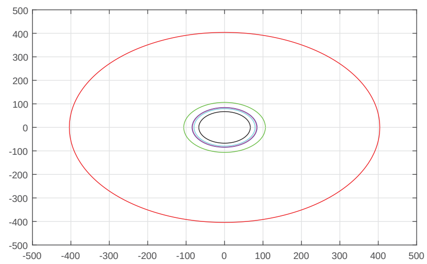

Figure 2. The domain of attraction in Table 3.

Example 3 In this example, we consider the linearized model of the Mach Number in a Wind Tunnel described by (17)-(18), all the numerical values of the parameters are borrowed from [22]. The same parameters have been used in [20] and are as follows

$$\frac{1}{a} = 1.964s,\ k = -0.0117deg^{-1},\ \xi = 0.8\, and\, w = 6rad/s$$

Using Theorem 3.2, letting, $u_0 = 1$ and $d = 0$. The following state feedback control law

$$K = \begin{pmatrix} 0.0000 & 0.7308 & 0.1398 \end{pmatrix}$$

stabilizes the system (18), hence the system is stabilizable for any time delay less than the upper bound 26, and the stability radius is 389 where $\alpha = 2$ and $\beta = 1$. For comparison with the results of others papers see Table 4.

Table 4. Comparison of h_m.

method	[22]	[20]	this paper
h_m	0.33	0.9685	26

From Table 4, the proposed method gives much larger time delay bound than the results of [22, 20].

On other hand, the values of the control parameters are adjusted to 0.75, 1 and 0.5, respectively. Thus, our results are less conservative than those of [21].

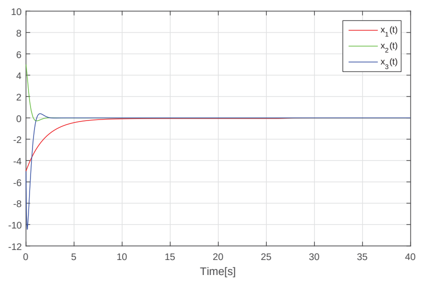

Figure 3. The response of the state variables using Runge-Kutta Method.

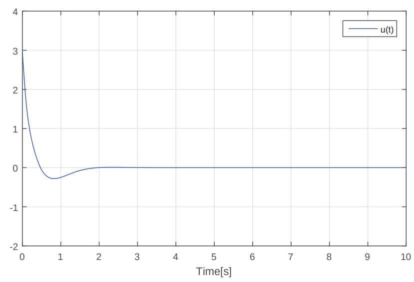

Figure 4. The response of the control input variable using Runge-Kutta Method.

Note that even though the control input is initially saturated, the states, in due course, are driven to the origin with $x_0(t) = [-5 \quad 5 \quad -5]^T$. The numerical simulation presented in this example shows that very good transient responses can be obtained by using control law compared to [21], which can be implemented in a controlling microprocessor as a simple feedback, using state variable $x(t)$ and delayed state variable $x(t - \tau(t))$ and control with a saturation at its input.

6. Conclusions

We have presented a methodology for robust stabilization of uncertain neutral time-delay system with saturating actuator, with the derived conditions given as LMIs that depend of the maximum value of the delay. These conditions then guarantee the stability of the closed loop system when the initial states are taken within a calculated region of attraction. The proposed conditions have being shown to be less conservative than those previously proposed in the literature by numerical examples, that have also illustrated the feasibility of the proposed approach.

Since the class of systems investigated in this paper appear in many process control applications, one can expect that the feedback design method presented here may have a wide range of applicability. The results were applied to the feedback control of Mach Number in a Wind Tunnel for illustration.

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