Analysis of D Dimensional Dirac equation for *q*-deformed Posch-Teller combined with *q*-deformed trigonometric Manning Rosen Non-Central potential using Asymptotic Iteration Method (AIM)

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Abstract. In this study, we used asymptotic iteration method (AIM) to obtain the relativistic energy spectra and wavefunctions for D Dimensional Dirac equation. Solution of the D Dimensional Dirac equation using asymptotic iteration method was done by four steps. The first step, we substitutied q deformed Poschl-Teller potential plus q-deformed Manning Rosen Non-Central potential into D dimensional Dirac equation. And then, general term of D dimensioanl Dirac equation for q deformed Poschl-Teller potential plus q-deformed Manning Rosen Non-Central potential was reduced into one dimensioanal Dirac equation, consist of radial part and angular part. The second step, both of one dimensional part must be reduced to hypergeometric type differential equation by suitable parameter change. And then, hypergeometric type differential equation was transformed into AIM type differential equation. For the last step, AIM type differential equation can be solved to obtain the relativistic energy and wavefunctions of Dirac equation. Relativistic energy and wavefunctions were visualized by using Matlab software.

1. Introduction

P. A. M Dirac submit the equation known as the Dirac equation. The Dirac equation has a probability density which is always a positive value, but the solution still provide information of a free particle of negative energy. The Dirac equation describes the basic correspondent for particle spin ½ in electron[1]. Dirac found that a particle that has mass and charge as an electron must have an intrinsic angular momentum and magnetic moment[2]. The Dirac equation is used when a particle is exposed to potential field strong relativistic effects must be considered that emberikan corrections to the nonrelativistic quantum mechanics[3]. In a spin ½ particle, there is the concept of spin symmetry and symmetry pseudospin. Spin symmetry and pseudospin symmetry occurs vector potential $V(r)$ and a scalar potential $S(r)$ is a constant. Spin symmetry occurs when $V(r) = S(r)$ and pseudosopi symmetry occurs when $V(r) = - S(r)$. The concept of spin symmetry has been applied to a spectrum of machinery and antinukleon, and the concept of symmetry pseudospin used to explain quasi-degeneration of double nucleons, superdepormasi in nuclei[6], exotic nuclei[7], and to establish an affective nuclear shell-model scheme[8].

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This study presents the completion of D-Dimensional Dirac equation in the case of spin symmetry for q-deformed Posch-Teller potential and q-deformed non-**c**entral Manning Rosen using asymptotic iteration method. completion of the D-Dimensional Dirac equation by reduction into hipergeometri equations with variable substitution. Energy spectrum and the wave functions obtained from asymptotic iteration method.

2. Asymptotic Iteration Method (AIM)

This method is used to solve differensial equetion in the following form:

$$y_n''(x)-\lambda_0(x)y_n'(x)-s_o(x)y_n(x)=0 \quad (1)$$

The one-dimensional Dirac equation can be reduced into hypergeometric or confluent hypergeometric type differential equation by suitable changes of variables, and then changes it into the differential equation which has the form in Eq.(1). The solution of Eq.(1) can be obtained by using iteration of λ_i and s_i,

$$\lambda_i(x)=\lambda_{i-1}'+\lambda_{i-1}\lambda_0+s_{i-1}$$
$$s_i(x)=s_{i-1}'+s_o\lambda_{i-1}$$
$$i=1,2,3,\ldots \quad (2)$$

Eigenvalues can be obtained using equation:[9]

$$\lambda_i(x)s_{i-1}(x)-\lambda_{i-1}(x)s_i(x)=0=\Delta_i, i=1,2,3\ldots. \quad (3)$$

On the other hand, Eq.(1) can be written in term:

$$y''(x)=2\left(\frac{tx^{N+1}}{1-bx^{N+2}}-\frac{c+1}{x}\right)y'(x)-\frac{Wx^N}{1-bx^{N+2}} \quad (4)$$

Eq.(4) is AIM-type differential equation which is solved by using Eq.(5)[10,11]

$$y_n(x)=(-1)^nC'(N+2)^n\,{}_2F_1(-n,p+n,\sigma,bx^{N+2}) \quad (5)$$

where

$$(\sigma)_n=\frac{\Gamma(\sigma+n)}{\Gamma(\sigma)}, \qquad \sigma=\frac{2c+N+3}{N+2} \quad (7)$$

$$p=\frac{(2c+1)b+2t}{(N+2)b} \quad (8)$$

C' is normalization constant and ${}_2F_1$ is hypergeometric function.[11]

3. Solution of Dirac Equation in D Dimension

The dirac equation with scalar potential $S(r)$ and vector potential $V(r)$ $(\hbar=1, c=1)$[16],

$$\{\vec{\alpha}.\vec{p}+\beta(M+S(\vec{r}))\}\psi(\vec{r})=\{E-V(\vec{r})\}\psi(\vec{r}) \quad (9)$$

which E is relativistic energy of system and $\vec{p}$ is momentum operator ($\vec{p}=-i\nabla$), while $\vec{\alpha}$ and β is matrix in term:

$$\hat{\alpha}_i=\begin{pmatrix}0 & \hat{\sigma}\\ \hat{\sigma} & 0\end{pmatrix}, \quad (10)$$

$$\beta_i=\begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix} \quad (11)$$

Where $\hat{\sigma}_i$ are Pauli's matrices and **1** is the 2×2 unit matrix. Here we have used the relation for the Pauli's matrices as follow

$$\hat{\sigma}_i\hat{\sigma}_j + \hat{\sigma}_j\hat{\sigma}_i = 2\delta_{ij}\mathbf{1} \tag{12}$$

The wave function of spinor Dirac can be classified in two form, spinor upper $\chi(\bar{r})$ and spinor lower $\varphi(\bar{r})$ as follows:[12,13]

$$\psi(r) = \begin{pmatrix} f_{nk}(r) \\ g_{nk}(r) \end{pmatrix} = \begin{pmatrix} \frac{F_{nk}(r)}{r^{\frac{D-1}{2}}} Y^{l}_{l_1 \ldots l_{D-1}}(\hat{x} = \theta_1, \theta_2, \ldots, \theta_{D-1}) \\ i\frac{G_{nk}(r)}{r^{\frac{D-1}{2}}} Y^{\tilde{l}}_{\tilde{l}_1 \ldots \tilde{l}_{D-1}}(\hat{x} = \theta_1, \theta_2, \ldots, \theta_{D-1}) \end{pmatrix} \tag{13}$$

we have

$$c\vec{\sigma}.\bar{p}g_{nk}(r) = \left[E - V(\vec{r}) - Mc^2 - S(\vec{r})\right]f_{nk}(r) \tag{14a}$$

$$c\vec{\sigma}.\bar{p}f_{nk}(r) = \left[E - V(\vec{r}) + Mc^2 + S(\vec{r})\right]g_{nk}(r) \tag{14b}$$

For spin symmetry, equation (14a) becomes

$$g_{nk}(r) = \frac{\vec{\sigma}.\bar{p}}{[E+M]} f_{nk}(r) \tag{15a}$$

$$\vec{\sigma}.\bar{p}g_{nk}(r) = [E - 2V(\vec{r}) - M]f_{nk}(r) \tag{15b}$$

Substituting Eq.(15a) into Eq.(15b) yields

$$[\boldsymbol{p}^2 + 2V(r)(E+M)]f_{nk}(\vec{r}) = [E^2 - M^2]f_{nk}(\vec{r}) \tag{16}$$

In spherical coordinates, modified Pöschl-Teller potential combined with trigonometric Manning Rosen non-central potential is defined as

$$V(r,\theta) = \left(\frac{\kappa(\kappa-1)}{\sinh_q^2 \alpha r} + \frac{\eta(\eta+1)}{\cosh_q^2 \alpha r}\right) + \frac{1}{r^2}\left(\frac{\nu(\nu-1)}{\sin_q^2 \theta} - 2q\cot_q \theta\right) \tag{17}$$

Subsitute Eq.(17) into Eq.(16) ans simplify the resuling equation, and let

$$f_{nk} = \frac{F_{nk}(r)}{r}\Theta(\theta)\phi(\varphi) \tag{18}$$

Then we have,

$$\frac{r^2}{F_{nk}(r)}\frac{r^{\frac{D-1}{2}}}{r^{D-1}}\frac{\partial}{\partial r}\left(r^{D-1}\frac{\partial}{\partial r}\frac{F_{nk}(r)}{r^{\frac{D-1}{2}}}\right) + r^2\left(\frac{\kappa(\kappa-1)}{\sinh_q^2 \alpha r} + \frac{\eta(\eta+1)}{\cosh_q^2 \alpha r}\right)[E+M] - \left[E^2 + M^2\right]r^2 = l_{D-1}(l_{D-1} + D - 2) \tag{19}$$

Separating the variables in Eq.(20), we obtain

$$\frac{d^2F_{nk}(r)}{dr^2} - \frac{\left(l_{D-1} + \frac{D-1}{2}\right)\left(l_{D-1} + \frac{D-3}{2}\right)}{r^2}F_{nk}(r) + \left([E+M]\left(\frac{\kappa(\kappa-1)}{\sinh_q^2 \alpha r} + \frac{\eta(\eta+1)}{\cosh_q^2 \alpha r}\right) - \left[E^2 - M^2\right]\right)F_{nk}(r) = 0 \tag{20}$$

$$\frac{1}{Y_3(\theta_3)}\frac{1}{\sin^2\theta_3}\left(\frac{\partial}{\partial\theta_3}\sin^2\theta_3\frac{\partial}{\partial\theta_3}\right)Y_3(\theta_3) + \left(\lambda_3 - \frac{\lambda_2}{\sin^2\theta_3} - [E+M]V(\theta_3)\right) = 0 \tag{21}$$

4. Analytical solution of radial and angular parts of the dirac equation

4.1 Solution of the radial part

The radial part of the Dirac equation in Eq. (21) we use the approximation value for the centrifugal term[14,15],

$$\frac{1}{r^2} \approx \frac{\mu^2}{4\sinh^2 \mu r} \tag{22}$$

Substituting Eq.(22) into Eq.(20) and simplifying the equation by substituting variable $\cosh_q^2 \quad = z$, we have

$$z(q-z)\frac{d^2F_{nk}(z)}{dz^2} + \frac{1}{2}(q-2z)\frac{dF_{nk}(z)}{dz} - \left(\frac{A_s}{4(q-z)} - \frac{B_s}{4z} + \frac{E_s^{'}}{4}\right)F_{nk}(z) = 0 \tag{23}$$

by substituting,

$$F_{nk}(z) = z^{\delta}(q-z)^{\gamma} f \tag{24}$$

Into Eq(23), we have the second-orde differential

$$z(q-z)f^{''} + \left(\left(2\delta + \frac{1}{2}\right)q + z(2\delta + 2\gamma + 1)\right)f^{'} - \left((\delta+\gamma)^2 - \frac{E_s^{'}}{4}\right)f = 0 \tag{25}$$

Where $4\delta^2 q = 2\delta q + B_s$ and $4\gamma^2 q = 2\gamma q + A_s$

Eq.(25) can be transform to differential equation type AIM,

$$f^{''} = \left(\frac{(2\delta + 2\gamma + 1)z - \left(2\delta + \frac{1}{2}\right)q}{z(q-z)}\right)f^{'} + \left(\frac{(\delta+\gamma)^2 - \frac{E_s^{'}}{4}}{z(q-z)}\right)f \tag{26}$$

From Eq.(26), we have

$$\lambda_0 = \frac{\left((2\delta+2\gamma+1)z - \left(2\delta+\frac{1}{2}\right)q\right)}{z(q-z)} \tag{27}$$

$$s_0 = \frac{\left((\delta+\gamma)^2 - \frac{E_S'}{4}\right)}{z(q-z)} \tag{28}$$

By using Eq.(3) and using Matlab 2011 software, energy eigenvalue can be obtained, with $\varepsilon = \frac{E_S'}{4}$

$$s_0\lambda_1 - s_1\lambda_0 = 0 \to \varepsilon_1 = (2\delta + 2\gamma + 1) + \{(\delta+\gamma)(\delta+\gamma+1)\} = (\delta+\gamma+1)^2$$
$$s_1\lambda_2 - s_2\lambda_1 = 0 \to \varepsilon_2 = (4\delta + 4\gamma + 4) + \{(\delta+\gamma)(\delta+\gamma+1)\} = (\delta+\gamma+2)^2$$
$$s_2\lambda_3 - s_3\lambda_2 = 0 \to \varepsilon_2 = (6\delta + 6\gamma + 9) + \{(\delta+\gamma)(\delta+\gamma+1)\} = (\delta+\gamma+3)^2$$

can be generalized as follow

$$\varepsilon = (\delta + \gamma + n_r)^2 \tag{29}$$

From Eq. (63) we get relativistic energy Eq. of this system is

$$\frac{\left[E^2 - M^2\right]}{\alpha^2}\frac{1}{4} = \left(\frac{1}{2}q\sqrt{B_{s1} + \frac{1}{4}q} \pm \frac{1}{2}q\sqrt{A_s + \frac{1}{4}q} + n_r + \frac{1}{2}q^2\right)^2 \tag{30}$$

where n_r is radial quantum numbers ($n_r = 0,1,2$ ...), l is orbital quantum numbers which is obtained from angular part solution. And then, radial wavefunction can be obtain by using Eq.(5), Eq.(6), Eq.(7) and Eq.(8), we have:

$c = \gamma + \frac{1}{4}, N = -1, t = \delta - \frac{3}{2}, b = 1,$
so, $\sigma = \frac{2c+N+3}{N+2} = 2\delta + \frac{1}{2}$ and $p = \frac{(2c+1)b+2t}{(N+2)b} = 2\delta + 2\gamma$

From Eq.(5), we have,

$$f_{nr}(z) = (-1)^{n_r} C_2 (1)^{n_r} \left(2\delta + \frac{1}{2}\right)_{n_r} {}_2F_1\left(-n_r, 2\delta + 2\gamma + n_r, 2\delta + \frac{1}{2}, z\right) \tag{31}$$

By substituting Eq.(31) to Eq.(24), we have radial wavefunction,

$$f_{nr}(z) = z^{\delta}(q-z)^{\gamma}(-1)^{n_r}C_2(1)^{n_r}\left(2\delta+\tfrac{1}{2}\right)_{n_r}\,{}_2F_1\left(-n_r, 2\delta+2\gamma+n_r, 2\delta+\tfrac{1}{2}, z\right) \quad (32)$$

which $z = \cosh_q^2$, so

$$f_{nr}(z) = z^{\delta}(q-z)^{\gamma}(-1)^{n_r}C_2(1)^{n_r}\left(2\delta+\tfrac{1}{2}\right)_{n_r}\,{}_2F_1\left(-n_r, 2\delta+2\gamma+n_r, 2\delta+\tfrac{1}{2}, z\right) \quad (33)$$

where $C(n_r)$ is radial normalization constant , ${}_2F_1$ is hypergeometric function.

4.2 Solution of the angular part

For angular part in Eq.(21), can be obtain by using AIM to find orbital quantum number.

$$\frac{\partial^2 Y_1(\theta_1)}{\partial\theta_1^2} - [E+M]\left(\frac{\nu(\nu-1)}{\sin_q^2\theta_1} - 2q\cot_q\theta_1\right)Y_1(\theta_1) + \lambda_1 Y_1(\theta_1) = 0 \quad (34)$$

Equation (65) can be written as, $cot_q\theta_1 = i(1-2z)$

$$z_1(1-z_1)\frac{d^2Y_1(z_1)}{d\theta_1^2} + (1-2z_1)\frac{dY_1(z_1)}{d\theta_1} + \left([E+M]\nu(\nu-1) - \frac{(-2qi[E+M]+\lambda_1)}{4z_1} - \frac{(2qi[E+M]+\lambda_1)}{4(1-z_1)}\right)Y_1(z_1) = 0 \quad (35)$$

by using

$$Y_1(z_1) = z_1^{\alpha}(1-z_1)^{\beta}\, f_s(z_1) \quad (36)$$

and simplying it, Eq.(35) can be transform to hypergeometric differential equation:

$$z_1(1-z_1)f_s''(z_1) + [(2\alpha+1) - z_1(2\alpha+2\beta+2)]f_s'(z_1) + ([E+M]\nu(\nu+1) - (\alpha+\beta)(\alpha+\beta+1))f_s(z_1) = 0 \quad (37)$$

Eq. (37) is hypergeometry type Eq. and we can solve it by AIM as follow

$$\lambda_0 = \frac{(z_1(2\alpha+2\beta+2)) - (2\alpha+1)}{z_1(1-z_1)} \quad (38)$$

$$s_0 = \frac{((\alpha+\beta)(\alpha+\beta+1) - [E+M]v(v+1))}{z_1(1-z_1)} \quad (39)$$

one can generalize Eq. (39)

$$L_1^{'} = \pm\sqrt{\frac{-(qi[E+M])^2}{\left(\sqrt{[E+M]\nu(\nu+1)+\frac{1}{4}} - n_l - \frac{1}{2}\right)^4} - \left(\sqrt{[E+M]\nu(\nu+1)+\frac{1}{4}} - n_l - \frac{1}{2}\right)^2} \quad (40)$$

Where L is orbital quantum number and n_l is angular quantum number. From Eq.(5), we have

$$Y(z) = (-1)^{n_l}C_2(1)^{n_l}(\sigma)_{n_l}\,{}_2F_1(-n_l, \rho+n_l, \sigma, bz^{N+2}) \quad (41)$$

C_z is angular normalization constant.

Solution for θ ,and θ_3 can be determined by the same way with solution for θ_1, And we get solution for orbital quantum number for θ_2,and θ_3 respectively as follows:

$$L_2^{'} = \pm\sqrt{\frac{-(qi[E+M])^2}{\left(\sqrt{\left(\lambda_1-\frac{1}{4}\right) - \frac{[E+M]}{q}\nu(\nu-1)+\frac{1}{4}} - n_l - \frac{1}{2}\right)^4} - \left(\sqrt{\left(\lambda_1-\frac{1}{4}\right) - \frac{[E+M]}{q}\nu(\nu-1)+\frac{1}{4}} - n_l - \frac{1}{2}\right)^2 - \frac{1}{2}} \quad (42)$$

$$L_3^{'} = \pm\sqrt{\frac{-(qi[E+M])^2}{\left(\sqrt{\lambda_2 q - [E+M]\nu(\nu-1)+\frac{1}{4}} - n_l - \frac{1}{2}\right)^4} - \left(\sqrt{\lambda_2 q - [E+M]\nu(\nu-1)+\frac{1}{4}} - n_l - \frac{1}{2}\right)^2 - 1} \quad (43)$$

5. Result and Discussion

In this section, we discuss several results which were obtained in the previous section. From relativistic energy equation in Eq.(30) and orbital quantum number equation in Eq.(41), and by using Matlab software we have numeric solution of relativistic energy are listed in Table 1 with parameters $\kappa = 2, \eta = 2, v = 3, q = 2$ and $M = 5\ fm^{-1}$, the negative value of relativistic energy is taken due to

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the pseudospin symmetric limit[19]. By inspecting Table 1, show that increase of value α and n_r in the same quantum state causes decrease energy eigenvalue.

Table 1. Relativistic energy corresponding to several sates of a particle under the influence of modified Pöschl-Teller potential and trigonometric Scarf II potential.

n_r	n_l	k	$E_{n_r n_{lk}}(fm^{-1})$ $\alpha = 0,1fm^{-1}$	$\alpha = 0,2fm^{-1}$	$\alpha = 0,3fm^{-1}$
0	0	0	-5.0805	-5.1083	-5.1411
1	1	0	-5.0715	-5.1181	-5.1351
2	2	0	-5.0798	-5.1289	-5.1738
3	0	0	-5.0887	-5.1358	-5.2678

By varying parameter which corresponding value δ and γ, some of the radial wavefunctions are listed in Table 2. Radial wavefunctions for particle under the influence of modified Pöschl-Teller potential and Manning Rosen potential are affected by potential constants κ, eta, q, v and by α .The parameter α has a dimension inverse of distance in space that describes the reach of Pöschl-Teller potential. If α is enlarged, physically means that the potential reach is smaller in a space. By inspecting Table 2, due to the increase in the value of α causes particles move further away from the nucleus and show that change in radial wavefunctions are affected of potential constants κ, eta, q and v.

Table 2. Energy eigenvalue in fm^{-1} with $n_r = 2$ $n_l = 2, v = 2, q = 4, M = 5, \alpha = 0.05$ for particle under the influence of modified Pöschl-Teller potential and Manning-Rosen non-central potential variation k.

Nr	*Enr*	F_{n_r}
0	-5.14162	$\left(\cos_q^2 \alpha r\right)^{\delta}\left(\sin_q^2 \alpha r\right)^{\gamma} C^{'}\left(2\delta+\frac{1}{2}\right)$
1	-5.09810	$\left(\cos_q^2 \alpha r\right)\left(\sin_q^2 \alpha r\right) C^{'}\left(-\left(2\delta+\frac{1}{2}\right)\right)\left(1+\frac{(-1)_n(2\delta+2\gamma+`)}{\left(2\delta+\frac{1}{2}\right)}\left(\cos_q^2 \alpha r\right)\right)$
2	-5.05656	$\left(\cos_q^2 \alpha r\right)\left(\sin_q^2 \alpha r\right) C^{'}\left(2\delta+\frac{1}{2}\right)\left(1+\frac{(-(4\delta+4\gamma+4))}{\left(2\delta+\frac{1}{2}\right)}\cos_q^2 \alpha r+\frac{(2\delta+2\gamma+2)(2\delta+2\gamma+3)}{\left(2\delta+\frac{1}{2}\right)\left(2\delta+\frac{3}{2}\right)}\left(\cos_q^2 \alpha r\right)^2\right)$
3	-5.03558	$\left(\cos_q^2 \alpha r\right)\left(\sin_q^2 \alpha r\right) C^{'}\left(-\left(2\delta+\frac{1}{2}\right)\right)\left(1-\frac{(6\delta+6\gamma+9)}{\left(2\delta+\frac{1}{2}\right)n_r}\cos_q^2 \alpha r+\frac{(3)(2\delta+2\gamma+3)(2\delta+2\gamma+4)}{\left(2\delta+\frac{1}{2}\right)\left(2\delta+\frac{3}{2}\right)}\left(\cos_q^2 \alpha r\right)^2-\frac{(2\delta+2\gamma+3)(2\delta+2\gamma+4)(2\delta+2\gamma+5)}{\left(2\delta+\frac{1}{2}\right)\left(2\delta+\frac{3}{2}\right)\left(2\delta+\frac{5}{2}\right)}\left(\cos_q^2 \alpha r\right)^2\right)$

6. Conclusion

The Dirac equation in D dimensions of q-deformed trigonometric Poschl-Teller potential combined with Manning-Rosen non-central potential using Asymptotic Iteration Method (AIM). The radial part of D- dimensions of the Dirac equation reduces to one dimensional Schrodinger type equation in centrifugal approximation scheme. In the exact spin symmetric case, the relativistic energy equation reduces to the non-relativistic energy in the non-relativistic condition. The radial part of the wavefunction is obtained approximately from Eq.(33) and the angular part in Eq.(41). The results show that the disturbance of modified Pöschl-Teller Potential and trigonometric Manning-Rosen non-central potential change in the wave function of the radial part and the angular part. Relativistic energy equatione can be obtained via AIM in Eq.(30) and equation of orbital quantum number l in Eq.(40), Eq.(42) and Eq.(43), where both are interrelated between quantum numbers. Relativistic energy also is solved numerically using Matlab software, where the increase in the radial quantum number n_r causes a decrease in the energy spectrum.

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