Analytical Solution of Schrodinger Equation for Modified Anisotropic Nonquadratic with Exponential Potential Using Supersymmetric Quantum Mechanics

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Abstract. Schrodinger equation for an anisotropic nonquadratic potential that modified by exponential form in axial part is investigated using supersymmetric approach. The three dimensional Schrodinger equation for an anisotropic nonquadratic potential in cylindrical coordinate is separated into three parts that contain one dimensional Schrödinger type equation which are solved using supersymmetric operator and the idea of shape invariant potential. The energy spectrum and the total wave functions are obtained.

Keywords : Schrodinger equation, Anisotropic Nonquadratic, Supersymmetry

1. Introduction

Exact analytical solutions of the Schrodinger equation for some physical potential are important to obtain the wave function and the energy spectrum. More recently, efforts have paid considerably to explore potential solutions right from the central and non-central. There are only few potentials the Schrodinger equation can be solved exactly in the scheme of centrifugal approach.

In recent years, many studies have analysed a bound state of a charged particle moving in a potential vector and scalar potential non-central location, such as an electron moves in the Coulomb field with the same field presence Aharonov-Bohm [1-2], or monopole magnet [3], Makarov potential [4] or potential-shaped ring oscillator [9/5]. Most of these research, the eigen values and eigen functions are determined using variable separation method in the spherical coordinate system. More recently, with the idea of supersymmetric quantum mechanics invariance form [10/9], factorization method [11-12], and Nikiforov-Uvarov method [13-14] are widely used to obtain the energy spectrum and the wave function of the charged particle moved in the non-central potential.

In this paper the anisotropic nonquadratic potential which has been investigated using path integral method [15], the method of algebraic solutions [16], the semmiclassical treatment [17] and now will be investigated by using supersymmetric Quantum Mechanics (SUSY QM) [18] and the idea of the invariance form [19]. SUSY QM is a powerful tool for determining the energy spectrum and the wave functions of the form class invariance potentials [18-22].

Anisotropic nonquadratic potential given as [15-17]

$$V(\bar{r}) = V(r, \theta, z) = -\frac{\alpha}{\sqrt{x^2 + y^2}} + \frac{\beta}{y^2} + \frac{\gamma x}{y^2 \sqrt{x^2 + y^2}} + \alpha' z^2 + \frac{\beta'}{z^2} \qquad (1)$$

Modified anisotropic nonquadratic potential with exponential potential in axial part is

$$V\left(\overline{r}\right) = V\left(r,\theta,z\right) = -\frac{V_0}{\sqrt{x^2+y^2}} + \frac{V_1}{x^2} + \frac{V_2 y}{x\left(x^2+y^2\right)} + \frac{V_3 e^{-\alpha z}}{\left(1-e^{-\alpha z}\right)^2} + V_4 \frac{1+e^{-\alpha z}}{1-e^{-\alpha z}} \qquad (2)$$

Schrodinger equation in Eq. (2) is expressed in the three-dimensional type of the Schrodinger equation in cylindrical coordinate and it is solved exactly using variable separation method. Schrodinger equation of radial, angular, and axial parts are solved using SUSY operator and the idea of shape invariant.

2. Schrodinger Equation for Modified Anisotropic Nonquadratic Potential

Schrödinger equation is defined by

$$\frac{-\hbar^2}{2m}\nabla^2\psi\left(\overline{r}\right) + V\left(\overline{r}\right)\psi\left(\overline{r}\right) = E\psi\left(\overline{r}\right) \qquad (3)$$

Modified anisotropic nonquadratic potential in Eq.(2) is changed by applying cylindrical coordinates, and apply the operator ∇^2 for cylindrical coordinate, with the wave function $\psi\left(\overline{r}\right) = R(r)P(\theta)\Phi(z)$. The wave function which have variable r, θ, and z can be separated with ordinary algebra method. So the Schrodinger equation that has been separated may be written by

- Axial part

$$-\frac{\hbar^2}{2m}\frac{\partial^2 Z}{\partial z^2} + \left(\frac{V_3}{4\sinh\frac{\alpha z}{2}} + V_4\coth\frac{\alpha z}{2}\right)Z = -\lambda_1 Z \qquad (4)$$

- Angular part

$$-\frac{\hbar^2}{2m}\frac{\partial^2 P}{\partial\theta^2} + \left(\frac{V_1}{\cos^2\theta} + V_2\tan\theta\right)P = -\lambda_2 P \qquad (5)$$

- Radial part

$$-\frac{\hbar^2}{2m}\left[r\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right)\right] - \left(E + \lambda_1 + \frac{V_0}{r}\right)r^2 R = \lambda_2 R \qquad (6)$$

3. Solution to The Three Dimensional Schrodinger Equations with SUSY QM Method

Supersymmetry quantum mechanics is used to solve one dimensional Schrödinger equation with any shape invariance potentials. After has been separated into three parts, each part of Schrodinger equation would be solved using SUSY QM to find the wave functions and energy spectra .

3.1. Solution of Axial Part

The Schrödinger equation for axial part (Eq. 23) is changed into hyperbolic function. Assume that $\gamma = \frac{\alpha}{2}$, and suppose that $V_3 = \frac{\hbar^2}{2m}\alpha^2 v(v-1)$, $V_4 = -\frac{\hbar^2}{m}q$, and $E' = -\lambda_1$ so Eq.(4) can be resolved into Schrodinger equation with the effective potential is Manning Rosen-like potential

$$-\frac{\hbar^2}{2m}\frac{\partial^2 Z}{\partial z^2} + \frac{\hbar^2}{2m}\left(\gamma^2\frac{v(v-1)}{\sinh^2\gamma z} - 2q\coth\gamma z\right)Z = E'Z \qquad (7)$$

We hypothesize that the superpotential is given by $\phi_o(z,a_0) = A\coth\gamma z + \frac{B}{A}$, by applying

$$V(z) = V_-(z;a_0) + E_0 = \phi^2(z;a_0) - \frac{\hbar}{\sqrt{2m}}\phi'(z;a_0) + E_0$$

$$V_-(z,a_0) = \phi^2(z) - \frac{\hbar}{\sqrt{2m}}\phi'(z); \qquad V_+(z,a_0) = \phi^2(z) + \frac{\hbar}{\sqrt{2m}}\phi'(z) \qquad (8)$$

Where $V(z)$ is the effective potential. So we get the superpotential, super-partner potentials and ground state energy are

$$\phi_o(\chi, a_0) = \frac{\hbar}{\sqrt{2m}} \gamma(\nu-1)\coth\gamma z - \frac{\hbar}{\sqrt{2m}} \frac{q}{\gamma(\nu-1)} \tag{9}$$

$$V_-(\chi, a_0) = \frac{\hbar^2}{2m}\left(\gamma^2 \frac{\nu(\nu-1)}{\sinh^2\gamma z} - 2q\coth\gamma z\right) + \frac{\hbar^2}{2m}\left[\gamma^2(\nu-1)^2 + \frac{q^2}{\gamma^2(\nu-1)^2}\right] \tag{10}$$

$$V_+(\chi, a_0) = \frac{\hbar^2}{2m}\left(\gamma^2 \frac{(\nu-2)(\nu-1)}{\sinh^2\gamma z} - 2q\coth\gamma z\right) + \frac{\hbar^2}{2m}\left[\gamma^2(\nu-1)^2 + \frac{q^2}{\gamma^2(\nu-1)^2}\right] \tag{11}$$

$$E_0{}' = -\frac{\hbar^2}{2m}\left[\gamma^2(\nu-1)^2 + \frac{q^2}{\gamma^2(\nu-1)^2}\right] \tag{12}$$

A pair of potentials $V_\pm(z)$ in Eqs.(8) are said to be shape invariant if they are similar in shape but different in the parameters and this condition is given as,

$$V_+(z; a_j) = V_-(z; a_{j+1}) + R(a_{j+1}) \tag{13}$$

where $j = 0,1,2,..$, and a is a parameter in our original potential. The energy eigenvalue of the Hamiltonian H_- is given by

$$E_n^{(-)} = \sum_{j=1}^{n} R(a_j)$$
$$E_n = E_n^{(-)} + E_0 \tag{14}$$

and by using Eqs.(14) and (13) we get the energy spectra of the system given as,

$$E_{n_z}{}' = -\frac{\hbar^2}{2m}\left[\gamma^2(\nu-n_z+1)^2 + \frac{q^2}{\gamma^2(\nu-n_z+1)^2}\right] \tag{15}$$

So we get the first separation variable constant is

$$\lambda_1 = \left[\gamma^2(\nu-n_z+1)^2 + \frac{q^2}{\gamma^2(\nu-n_z+1)^2}\right] \tag{16}$$

The raising and lowering operator for axial part are given by

$$A^+ = -\frac{\hbar}{\sqrt{2m}}\frac{d}{dz} + \frac{\hbar}{\sqrt{2m}}\gamma(\nu-1)\coth\gamma z - \frac{\hbar}{\sqrt{2m}}\frac{q}{\gamma(\nu-1)} \tag{17}$$

$$A = \frac{\hbar}{\sqrt{2m}}\frac{d}{dz} + \frac{\hbar}{\sqrt{2m}}\gamma(\nu-1)\coth\gamma z - \frac{\hbar}{\sqrt{2m}}\frac{q}{\gamma(\nu-1)} \tag{18}$$

Using the lowering operator Eq.(18) and applying

$$H_-\psi_0^{(-)} = 0 \rightarrow A\psi_0^{(-)} = 0 \tag{19}$$

So we get the ground state of wave function is

$$Z_0 = C'\sinh^{(\nu-1)}\gamma z\, e^{\frac{q}{\gamma(\nu-1)}z} \tag{20}$$

And applying

$$\psi_n^{(-)}(z; a_0) \approx A^+(z; a_0)A^+(z; a_1)....A^+(z; a_{n-1})\psi_0^{(-)}(z; a_n) \tag{21}$$

The first excited wave function is given by

$$Z_1(z, a_0) = C\sinh^\nu\gamma z\, e^{\frac{q}{\gamma\nu}z}\left[\coth\gamma z + \frac{q}{\gamma^2\nu(\nu-1)}\right] \tag{22}$$

With $a_0 = \nu, a_1 = \nu-1,..., a_n = \nu-n$ so we can get the next excited wave functions of axial part by using raising Eq.(17) and lowering Eq.(18) operator and applying Eqs.(14) and (21).

3.2. Solution of Angular Part

The potential of Schrodinger equation for angular part is changed by $V_1 = \frac{\hbar^2}{2m} v(v+1)$ and $V_2 = -\frac{\hbar^2}{m} q$, so the effective potential in angular part becomes Kepler problem in hypersphere potential-like and we assumed that $E'' = -\lambda_2$. Then the angular part becomes,

$$-\frac{\hbar^2}{2m}\frac{\partial^2 P}{\partial \theta^2} + \frac{\hbar^2}{2m}\left(\frac{v(v+1)}{\cos^2\theta} - 2q\tan\theta\right)P = E''P \tag{23}$$

To find solution of angular part, we put the same method like in the subsection 4.1. We have to define the superpotentials, $\phi_0(\theta) = A\tan\theta + B$, by using Eqs.(8) and (13-14) we get the ground state energy for angular part and the energy spectra are given by

$$E_0'' = -\frac{\hbar^2}{2m}\left((v+1)^2 + \frac{q^2}{(v+1)^2}\right) \tag{24}$$

$$E_{n_\theta}'' = -\frac{\hbar^2}{2m}\left((v+n_\theta+1)^2 + \frac{q^2}{(v+n_\theta+1)^2}\right) \tag{25}$$

So the second variable constant is

$$\lambda_2 = \left(\frac{q^2}{(v+n_\theta+1)^2} + (v+n_\theta+1)^2\right) \tag{26}$$

The raising and lowering operator for angular part are given by

$$A^+(\theta, a_0) = -\frac{\hbar}{\sqrt{2m}}\frac{d}{d\theta} + \frac{\hbar}{\sqrt{2m}}(v+1)\tan\theta - \frac{\hbar}{\sqrt{2m}}\frac{q}{(v+1)} \tag{27}$$

$$A(\theta, a_0) = \frac{\hbar}{\sqrt{2m}}\frac{d}{d\theta} + \frac{\hbar}{\sqrt{2m}}(v+1)\tan\theta - \frac{\hbar}{\sqrt{2m}}\frac{q}{(v+1)} \tag{28}$$

Using the lowering operator Eq.(28) and applying Eq.(19) we get the ground state of wave function is

$$P_0 = D'\left(\cos^{(v+1)}\theta\right)e^{\frac{q\theta}{(v+1)}} \tag{29}$$

And the first excited wave function for angular part is

$$P_1 = D\frac{\hbar}{\sqrt{2m}}\frac{e^{\frac{q\theta}{(v+2)}}}{\sec^{(v+2)}\theta}\left[(v+1)\tan\theta - (v+2)\cos\theta - q\left(\frac{1}{v+1} + \frac{1}{v+2}\right)\right] \tag{30}$$

With $a_0 = v, a_1 = v+1, ..., a_n = v+n$ so we can get the next excited wave functions by using raising Eq.(27) and lowering Eq.(28) operator and applying Eq.(21).

3.3. Solution of Radial Part

Radial part in Eq.(6) must reduce into Schrödinger equation by assumed that $R = \frac{U}{\sqrt{r}}$ and then we change the form of the potentials with $V_0 = e^2$, so the effective potential in radial part becomes Coulomb-like potential and we assumed that $E''' = E - \frac{\hbar^2}{2m}\lambda_1$. Then the radial part has been reduced into Schrödinger equation, given as

$$-\frac{\hbar^2}{2m}\frac{\partial^2 U}{\partial r^2} + \left(\frac{\hbar^2}{2m}\frac{\lambda_2 - \frac{1}{4}}{r^2} - \frac{e^2}{r}\right)U = E'''U \tag{31}$$

We hypothesize that the superpotential for the radial part is $\phi_0(r) = \frac{A}{r} + B$, by using the same method like subsections 1 and 2 so we get the ground state energy and the energy spectra for radial part are given by

$$E_0''' = -\frac{me^4}{2\hbar^2\left(\sqrt{\lambda_2}-\frac{1}{2}\right)^2} \tag{32}$$

$$E_{n_r}''' = -\frac{me^4}{2\hbar^2\left(\sqrt{\lambda_2}-n_r-\frac{1}{2}\right)^2} \tag{33}$$

So we get the energy spectra of the system is

$$E = \frac{\hbar^2}{2m}\left[\gamma^2(v-n_z+1)^2 + \frac{q^2}{\gamma^2(v-n_z+1)^2}\right] - \frac{me^4}{2\hbar^2\left(\sqrt{\frac{q^2}{(v+n_\theta+1)^2}+(v+n_\theta+1)^2}-n_r-\frac{1}{2}\right)^2} \tag{34}$$

The energy spectra of an anisotropic non-quadratic potential expressed in (Eq. 2) is given in (Eq. 57). There are two parts of the energy spectra, the first part is associated with the axial part of the potential, and the second part is associated with radial and polar parts of potential. Without the presence of axial part of potential, the energy spectra reduces to the energy spectra of Coulomb-like potential that is modified by the presence of the angular potential.

The raising and lowering operator for radial part are given by

$$A^+(r,a_0) = -\frac{\hbar}{\sqrt{2m}}\frac{d}{dr} + \frac{\hbar}{\sqrt{2m}}\frac{\sqrt{\lambda_2}-\frac{1}{2}}{r} - \frac{\sqrt{2m}}{2\hbar}\frac{e^2}{\sqrt{\lambda_2}-\frac{1}{2}} \tag{35}$$

$$A(r,a_0) = \frac{\hbar}{\sqrt{2m}}\frac{d}{dr} + \frac{\hbar}{\sqrt{2m}}\frac{\sqrt{\lambda_2}-\frac{1}{2}}{r} - \frac{\sqrt{2m}}{2\hbar}\frac{e^2}{\sqrt{\lambda_2}-\frac{1}{2}} \tag{36}$$

Using the lowering operator Eq.(27) and applying Eq.(19) we get the ground state of wave function is

$$U_0 = F'\, r^{\sqrt{\lambda_2}-\frac{1}{2}}\, e^{\frac{m}{\hbar^2}\frac{e^2}{\sqrt{\lambda_2}-\frac{1}{2}}r} \tag{37}$$

And the first excited wave function for radial part is given by

$$U_1 = C\left[\left(\frac{\hbar}{\sqrt{2m}}r^{\sqrt{\lambda_2}-\frac{5}{2}}e^{\frac{m}{\hbar^2}\frac{e^2}{\sqrt{\lambda_2}-\frac{3}{2}}r}\right) - \left(\frac{\sqrt{2m}}{2\hbar}e^2\left(\frac{1}{\sqrt{\lambda_2}-\frac{5}{2}}+\frac{1}{\sqrt{\lambda_2}-\frac{3}{2}}\right)\right)r^{\sqrt{\lambda_2}-\frac{3}{2}}e^{\frac{m}{\hbar^2}\frac{e^2}{\sqrt{\lambda_2}-\frac{3}{2}}r}\right] \tag{38}$$

With $a_0 = \sqrt{\lambda_2}, a_1 = \sqrt{\lambda_2}-1,...,a_n = \sqrt{\lambda_2}-n$ so we can get the next excited wave functions by using raising Eq.(35) and lowering Eq.(36) operator and applying Eq.(19). The total wave functions, the un-normalized ground state and first excited state ones, are obtained from Eqs.(20,22), Eqs.(29-30) and Eqs.(37-38) given as

$$\psi_0(r,\theta,z) = G'\left(\cos^{(v+1)}\theta\right)\left(\sinh^{(v-1)}\gamma z\right)r^{\sqrt{\lambda_2}-\frac{1}{2}}e^{\frac{m}{\hbar^2}\frac{e^2}{\sqrt{\lambda_2}-\frac{1}{2}}r+\frac{q}{(v+1)}\theta+\frac{q}{\gamma(v-1)}z} \tag{58}$$

$$\psi_1(r,\theta,z) \cong \left[\left(\frac{\hbar}{\sqrt{2m}}r^{\sqrt{\lambda_2}-\frac{5}{2}}e^{\frac{m}{\hbar^2}\frac{e^2}{\sqrt{\lambda_2}-\frac{3}{2}}r}\right) - \left(\frac{\sqrt{2m}}{2\hbar}e^2\left(\frac{1}{\sqrt{\lambda_2}-\frac{5}{2}}+\frac{1}{\sqrt{\lambda_2}-\frac{3}{2}}\right)\right)r^{\sqrt{\lambda_2}-\frac{3}{2}}e^{\frac{m}{\hbar^2}\frac{e^2}{\sqrt{\lambda_2}-\frac{3}{2}}r}\right]$$

$$\times \frac{\hbar}{\sqrt{2m}}\frac{e^{\frac{q\theta}{(v+2)}}}{\sec^{(v+2)}\theta}\left[(v+1)\tan\theta - (v+2)\cos\theta - q\left(\frac{1}{v+1}+\frac{1}{v+2}\right)\right] \tag{59}$$

$$\times \sinh^v\gamma z\, e^{\frac{q}{\gamma v}z}\left[\coth\gamma z + \frac{q}{\gamma^2 v(v-1)}\right]$$

with λ_2 expressed in (Eq. 26) respectively.

4. Conclusion

By applying a cylindrical coordinate system, the Schrödinger equation for the anisotropic nonquadratic that modified by exponential form in axial part reduces to a perfectly variable separable Schrödinger equation. Three dimensional Schrödinger equation reduces to one radial Schrodinger equation, one angular Schrodinger equation, and one axial Schrodinger equation. These three one dimensional Schrödinger equation are exactly solvable since each of Schrödinger equation with shape invariant potential. By using SUSY Quantum Mechanics method and the idea of shape invariance the energy spectra are calculated and so the total wave functions which are combination of axial, angular and radial parts.

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