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Energy analysis of four dimensional extended hyperbolic Scarf I plus three dimensional separable trigonometric non-central potentials using SUSY QM approach

A. Suparmi, C. Cari, U. A. Deta, J. Handhika
Physics Department, Graduate Program, Sebelas Maret University, Jl.Ir. Sutami 36A Kentingan Surakarta 57126, Indonesia

Email: soeparmi@staff.uns.ac.id

Abstract. The non-relativistic energies and wave functions of extended hyperbolic Scarf I plus separable non-central shape invariant potential in four dimensions are investigated using Supersymmetric Quantum Mechanics (SUSY QM) Approach. The three dimensional separable non-central shape invariant angular potential consists of trigonometric Scarf II, Manning Rosen and Poschl-Teller potentials. The four dimensional Schrodinger equation with separable shape invariant non-central potential is reduced into four one dimensional Schrodinger equations through variable separation method. By using SUSY QM, the non-relativistic energies and radial wave functions are obtained from radial Schrodinger equation, the orbital quantum numbers and angular wave functions are obtained from angular Schrodinger equations. The extended potential means there is perturbation terms in potential and cause the decrease in energy spectra of Scarf I potential.

1. Introduction

It is well known that the exact solution of non-relativistic Quantum Mechanic for some physical potentials are very important since they provide all necessary information for certain quantum system. Finding an exact solution of Schrodinger equation for some real potentials, that have been applied in molecular physics, solid state and chemistry areas, have been explored intensively by using various methods. Recently, conciderably efforts have been paid to obtain the exact solution of mixture potentials include Killingbeck [1] Sextic and octic [2], extended Cornell [3-4], and Killingbeck in external field [5-7,10] potentials. The bound state energy spectra of these potentials have been investigated by various techniques such as SUSY QM [11-12,16-18], wave function ansatz method [2,6-7,9-10], Nikiforov-Uvarov method [3-4,8,13], and Lie Algebra approach [1]. For l wave, the Schrodinger equation is only solved approximately with suitable approximation scheme for centrifugal term [14].

Furthermore, the extension in higher dimensional spaces for some physical problems is very important in some physics area. The D dimensional system has been constructed to explain the unification of gravitation and electromagnetic fields [9]. It is suspected that the dimensional system is applicable for gravitation field since it is involved in such huge universe. The D-dimensional non-relativistic and relativistic physical systems have been investigated by many authors, Killingbeck potential[1], Poschl-Teller and Manning Rosen non-central potential [12], hyperbolic tangent [13], three anharmonics potential [8].

In this paper we will attempt to solve the four dimensional Schrodinger equation for a charged particle moving in a field governed by an extended radial Scar I potential with simultaneous presence

trigonometric Scarf II, Manning Rosen, and Pöschl-Teller non-central potentials using SUSY QM with the idea of shape invariance. SUSY QM was developed based on Witten's proposal [14], while the idea of shape invariant potentials was proposed by Gendenshtein [15], since then SUSY QM became a powerful tool to determine the energy spectrum and wave function of a class of shape invariant potentials as in Sukumar, Dutt et al., and Gangopadhyaya [16-18]. This potential can be applied to study the non-relativistic effect of the complex vibration-rotation energy structure of multi-electron atoms.
This paper is organized as follows. A brief review of SUSY QM is presented in Section 2. Derivation of hyper-spherical Laplacian is briefly introduced in section 3, Solutions of radial and polar Dirac equations are presented in Section 4. The result and discussion in section 5. The conclusion is presented in Section 6.

2. Review of the SUSY Quantum Mechanics Approach Using Operator and Shape Invariance

According to the definition proposed by Witten [14], in a SUSY quantum system there are super charge operators Q_i that commute with the SUSY Hamiltonian H_{ss} and they also obey to the anti commutation algebra as

$$[Q_i, H_{ss}] = 0 \quad i = 1, 2, 3, \ldots N \quad ; \quad \{Q_i, Q_j\} = \delta_{ij} H_s \tag{1}$$

SUSY QM is a one-dimensional model of SUSY field theory with N = 2, where

$$Q_1 = \frac{1}{\sqrt{2}}\left(\sigma_1 \frac{p}{\sqrt{2m}} + \sigma_2 \phi(x)\right) \text{and} \quad Q_2 = \frac{1}{\sqrt{2}}\left(\sigma_2 \frac{p}{\sqrt{2m}} + \sigma_1 \phi(x)\right) \tag{2}$$

with σ_i are the usual Pauli spin matrices, $p = -i\hbar \frac{\partial}{\partial x}$ is the usual momentum operator, and $\phi(x)$ is the super-potential. By using equations (3) and (2) we get,

$$H_{ss} = \begin{pmatrix} -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \varphi^2(x) + \frac{\hbar}{\sqrt{2m}}\varphi'(x) & 0 \\ 0 & -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \varphi^2(x) - \frac{\hbar}{\sqrt{2m}}\varphi'(x) \end{pmatrix} = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix} \tag{3}$$

with $$H_- = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_-(x) = A^+ A \text{ and } H_+ = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_+(x) = AA^+ \tag{4}$$

where $V_-(x)$ and $V_+(x)$ are the SUSY partner potentials, A^+ and A are raising and lowering operators, and

$$A^+ = -\frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + \phi(x) \text{ and } A = \frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + \phi(x) \tag{5}$$

By constructing super-potential by using the condition of usual Hamiltonian H and SUSY Hamiltinian H_-

$$H = H_- + E_0 = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_-(x; a_0) + E_0 \tag{6}$$

and by applying the condition of shape invariant potential which is defined as

$$V_+(x; a_j) = V_-(x; a_{j+1}) + R(a_{j+1}) \tag{8}$$

$$V_+(x; a_j) = \phi^2(x; a_j) + \frac{\hbar}{\sqrt{2m}}\phi'(x; a_j) \text{and } V_-(x; a_{j+1}) = \phi^2(x; a_{j+1}) - \frac{\hbar}{\sqrt{2m}}\phi'(x; a_{j+1}) \tag{9}$$

the energy eigen value of H_- [15] and H are obtained from equation (6) given as

$$E_n^{(-)} = \sum_{k=1}^{n} R(a_k) \quad \text{and} \quad E_n = E_n^{(-)} + E_0 \tag{10}$$

with j = 0,1,2,..; a is a mapping parameter. Based on the characteristics of the lowering and raising operators, the ground state and excited state wave functions are obtained from the condition that

$$A\psi_0^{(-)} = 0, \quad \psi_1^{(-)}(x; a_0) \approx A^+(x; a_0)\psi_0^{(-)}(x; a_1), \quad \psi_2^{(-)}(x; a_0) \approx A^+(x; a_0)\psi_1^{(-)}(x; a_1) \tag{11}$$

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The energy spectra are obtained from equations. (8-10), while the wave functions is obtained from equation (11).

3. Coordinate System in D Dimension

If x_i s are the coordinate components of Cartesian coordinates then they can be expressed in hyperspherical coordinate components as $r,\ \theta_1, \theta_2, ..., \theta_{n-1}$ in D dimensional system given as

$$x_1 = r\cos\theta_1 \sin\theta_2 ... \sin\theta_{D-1},\quad x_2 = r\sin\theta_1 \sin\theta_2 ... \sin\theta_{D-1}, ..., x_b = r\cos\theta_{b-1}\sin\theta_b ... \sin\theta_{D-1},\ \dots$$
$$x_{D-1} = r\cos\theta_{D-2}\sin\theta_{D-1},\ \text{and}\ x_D = r\cos\theta_{D-1} \tag{12}$$

with $b \in [3, D-1]$.The unit vector $\hat{x}$ along x axes is usually expressed as $\hat{x} = x/r$, $r^2 = \sum_{i=1}^{D} x_i^2$ $0 \le \theta_1 \le 2\pi$, $\theta_i \in (0, \pi)$ for $i = 2, 3, ..., D-1$. The D dimensional Laplacian in polar coordinates is defined as

$$\nabla_D^2 = \frac{1}{h}\sum_{j=0}^{D-1}\frac{\partial}{\partial\theta_j}\left(\frac{h}{h_j^2}\frac{\partial}{\partial\theta_j}\right)\ ;\quad \theta_0 = r\ ;\quad h = \prod_{j=0}^{D-1} h_j\ \text{ and }\ h_j^2 = \sum_{i=1}^{D}\left(\frac{\partial x_i}{\partial\theta_j}\right)^2 \tag{13}$$

By manipulating equations (12-13) it is obtained D dimensional Laplacian in hyper-spherical coordinates as

$$\nabla_D^2 = \frac{1}{r^{D-1}}\frac{\partial}{\partial r}\left(r^{D-1}\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\sum_{j=1}^{D-2}\frac{1}{\sin^2\theta_{j+1}\sin^2\theta_{j+2}...\sin^2\theta_{D-1}} \times \left\{\frac{1}{\sin^{j-1}\theta_j}\frac{\partial}{\partial\theta_j}\left(\sin^{j-1}\theta_j\frac{\partial}{\partial\theta_j}\right)\right\} \tag{14}$$
$$+\frac{1}{r^2}\left\{\frac{1}{\sin^{D-2}\theta_{D-1}}\frac{\partial}{\partial\theta_{D-1}}\left(\sin^{D-2}\theta_{D-1}\frac{\partial}{\partial\theta_{D-1}}\right)\right\}$$

The four dimensional Schrodinger equation with separable four dimensional non-central potential in hyperspherical coordinates which is obtained from equation (14) is given as

$$-\frac{\hbar^2}{2\mu}\left\{\begin{array}{l}\frac{1}{r^3}\frac{\partial}{\partial r}\left(r^3\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin^2\theta_2\sin^2\theta_3}\left(\frac{\partial^2}{\partial\theta_1^2}\right) \\ +\frac{1}{r^2}\left\{\frac{1}{\sin^2\theta_3}\left\{\frac{1}{\sin\theta_2}\frac{\partial}{\partial\theta_2}\left(\sin\theta_2\frac{\partial}{\partial\theta_2}\right)\right\} + \left[\frac{1}{\sin^2\theta_3}\frac{\partial}{\partial\theta_3}\left(\sin^2\theta_3\frac{\partial}{\partial\theta_3}\right)\right]\right\}\end{array}\right\}\psi(r,\theta_1,\theta_2,.\theta_3) \tag{15}$$
$$+\left[\frac{\hbar^2}{2\mu}\left\{V(r) + \frac{1}{r^2}\left\{\frac{V_1(\theta_1)}{\sin^2\theta_2\sin^2\theta_3} + \frac{V_2(\theta_2)}{\sin^2\theta_3} + V_3(\theta_3)\right\}\right\} - E\right]\psi(r,\theta_1,\theta_2,\theta_3) = 0$$

where the three dimensional separable trigonometric angular potential is defined as

$$V(\theta_1,\theta_2,\theta_3) = \frac{1}{r^2}\left\{\frac{V_1(\theta_1)}{\sin^2\theta_2\sin^2\theta_3} + \frac{V_2(\theta_2)}{\sin^2\theta_3} + V_3(\theta_3)\right\} \tag{16}$$

with

$$V_1 = \left(\frac{b^2 + c(c-1) + 2b(c-1/2)\sin\theta_1}{\cos^2\theta_1}\right);\ V_2 = \left(\frac{\upsilon(\upsilon+1)}{\sin^2\theta_2} - 2\zeta\cot\theta_2\right),\ \ V_3 = \left(\frac{\rho(\rho-1)}{\sin^2\theta_3} + \frac{\gamma(\gamma-1)}{\cos^2\theta_3}\right) \tag{17}$$

and an extended (perturbed) radial Scarf I potential is

$$V(r) = \left(\alpha^2\left\{\left(\frac{f^2 - g(g+1)}{\cosh^2\alpha r} + \frac{2f(g+1/2)\sinh\alpha r}{\cosh^2\alpha r}\right) + \left(\frac{\delta(\delta-1)}{\sinh^2\alpha r}\right) + 2d\,\mathrm{csc}h\alpha r\right\}\right)$$
(18)

with α is related to the width of radial potential $f, g, \delta, d, b, c, \upsilon, \zeta, \rho, \gamma$ are parameters related with the depth of potential. By setting $\psi(r,\theta_1,\theta_2,\theta_3) = \mathrm{R(r)}\,\mathrm{Y}(\theta_1,.\theta_2,\theta_3) = \mathrm{R(r)}\,\mathrm{P}_1(\theta_1)P_2(\theta_2)P_3(\theta_3)$ in equation (15) we get four one dimensional Schrodinger equation given as

$$\left\{r^2\frac{1}{r^3}\frac{\partial}{\partial r}\left(r^3\frac{\partial}{\partial r}\right)-r^2\left(V(r)-\frac{2m}{\hbar^2}E\right)\right\}R=\lambda_3 R \quad , \quad \frac{\partial^2 P_1}{\partial\theta_1^2}-V_1(\theta_1)P_1+\lambda_1 P_1=0 \tag{17}$$

$$\left\{\frac{1}{\sin\theta_2}\frac{\partial}{\partial\theta_2}\left(\sin\theta_2\frac{\partial P_2}{\partial\theta_2}\right)\right\}-V_2(\theta_2)P_2-\frac{\lambda_1 P_2}{\sin^2\theta_2}+\lambda_2 P_2=0 \tag{18}$$

$$\left[\frac{1}{\sin^2\theta_3}\frac{\partial}{\partial\theta_3}\left(\sin^2\theta_3\frac{\partial P_3(\theta_3)}{\partial\theta_3}\right)\right]-V_3(\theta_3)P_3(\theta_3)-\frac{\lambda_2}{\sin^2\theta_3}P_3(\theta_3)+\lambda_3 P_3(\theta_3)=0 \tag{19}$$

with $\lambda_1 . \lambda_2$, and λ_3 are variable separation constant.

4. Solution of Schrodinger equation with Extended Hyperbolic Scarf I Radial Potential Plus Trigonometric Non-central Potentials

4.1. Solution of radial part of Schrodinger equation

By using equations (13-14), setting $R=\frac{U}{r^{3/2}}$, and applying centrifugal approximation [19] in equation (17) we get

$$-\frac{\hbar^2}{2\mu}\frac{d^2U}{dr^2}+\left(\frac{\hbar^2\alpha^2}{2\mu}\left\{\left(\frac{f^2-g(g+1)}{\cosh^2\alpha r}+\frac{2f(g+1/2)\sinh\alpha r}{\cosh^2\alpha r}\right)+\left(\frac{\delta(\delta-1)+\lambda_3+3/4}{\sinh^2\alpha r}\right)+2d\,\mathrm{csc}h\alpha r\right\}\right)U=EU \tag{20}$$

By setting the hypothetical super potential for effective potential expressed in equation (20) as

$$\phi(r)=\alpha\left(B\ tanh\ \alpha r+C\,\mathrm{sec}h\ \ \alpha r+D\coth\alpha r\right) \tag{21}$$

and by using equation (6-10) we obtain the superpotential $\phi(r)$, the ground state energy ε_0, the energy spectra E_n and ground state radial wave function R_0 as

$$\phi(r)=\frac{\hbar g\alpha}{\sqrt{2\mu}}\ tanh\ \alpha r+\frac{\hbar f\alpha}{\sqrt{2\mu}}\mathrm{sec}h\ \ \alpha r+\frac{\hbar(\delta'-1)\alpha}{\sqrt{2\mu}}\coth\alpha r \tag{22}$$

$$\varepsilon_0=-\frac{\hbar^2\alpha^2}{2\mu}\left(g+\delta'-1\right)^2,\quad E_n=-\frac{\hbar^2\alpha^2}{2\mu}(g+(\delta'-1)-2n)^2,\quad \delta'=\frac{1}{2}+\sqrt{\delta(\delta-1)+\lambda_3+1} \tag{23}$$

$$R_0=C_0\left[\left(\cosh\alpha r\right)^{-\frac{\hbar g\alpha}{\sqrt{2\mu}}}\left(\sinh\alpha r\right)^{-\frac{\hbar(\delta'-1)\alpha}{\sqrt{2\mu}}-3/2}\left(-\coth\frac{\alpha r}{2}\right)^{-iC}\right]^{-\frac{\sqrt{2\mu}}{\hbar}} \tag{24}$$

4.2. Solution of Schrodinger for the first angular part θ_1

By using equation (16-17) we get one dimensional Schrodinger equation for θ_1

$$-\frac{\hbar^2}{2\mu}\frac{\partial^2 P_1(\theta_1)}{\partial\theta_1^2}+\frac{\hbar^2}{2\mu}\left(\frac{b^2+c(c-1)+2b(c-1/2)\sin\theta_1}{\cos^2\theta_1}\right)P_1(\theta_1)=\frac{\hbar^2}{2\mu}\lambda_1 P_1(\theta_1) \tag{25}$$

By applying SUSY QM method and the idea of shape invariant given in equations (7-11) and (25) we obtain the superpotential $\phi(\theta_1)$, variable separation constant that related to angular quantum number λ_1 and ground state wave function P_{10} as

$$\phi(\theta_1)=\frac{\hbar}{\sqrt{2\mu}}c\tan\theta_1+\frac{\hbar}{\sqrt{2\mu}}b\sec\theta_1 \tag{26}$$

$$\lambda_1=(c+n_1)^2 \ .\ n_1=0,1,2,...\ P_{10}=\left(\cos\theta_1\right)^{c+b}\left(1+\sin\theta_1\right)^{-b} \tag{27}$$

4.3. Solution of Schrodinger for angular part for θ_2 and θ_3

By setting $\mathrm{P}_j(\theta_j)=\dfrac{Q_j(\theta_j)}{\sin^{(j-1)/2}\theta_j}$ in equations(18-19) for $j = 2,3$, and together with equation (16) we get

$$Q_2''(\theta_j)+\left\{\lambda_2+(1/4)\right\}Q_2(\theta_2)-\left(\frac{\upsilon(\upsilon+1)+\lambda_1-(1/4)}{\sin^2\theta_2}-2\zeta\cot\theta_2\right)\mathrm{Q}_2(\theta_2)=0 \quad (28)$$

$$Q_3''(\theta_3)+\left\{\lambda_3+1\right\}Q_3(\theta_3)-\left(\frac{\rho(\rho-1)+\lambda_2}{\sin^2\theta_3}+\frac{\gamma(\gamma-1)}{\cos^2\theta_3}\right)\mathrm{Q}_3(\theta_3)=0 \quad (29)$$

By setting $\upsilon(\upsilon+1)+\lambda_1-(1/4)=\upsilon'(\upsilon'+1)$ and $\dfrac{\hbar^2}{2M}\left\{\lambda_2+\dfrac{1}{4}\right\}=E_2$ in equation (28) and applying equation (7-11) we obtain the superpotential $\varphi(\theta_2)$, the variable separation constant λ_2 which is related to angular quantum number and ground state wave function P_{20} as

$$\varphi(\theta_2)=-\frac{\hbar}{\sqrt{2M}}\left((\upsilon'+1)\cot\theta_2-\frac{\zeta'}{(\upsilon'+1)}\right) \quad (30)$$

$$\lambda_2=\left((\upsilon'+(n_2+1))^2-\frac{\zeta^2}{(\upsilon'+(n_2+1))^2}\right)-1/4 \quad n_2=0,1,2,... \quad \upsilon'=-\frac{1}{2}+\sqrt{\upsilon(\upsilon+1)+\lambda_1} \quad (31)$$

$$P_{20}=\left(\sin\theta_2\right)^{\upsilon'+1/2}e^{-\zeta\theta_2/(\upsilon'+1)} \quad (32)$$

By setting $\rho(\rho-1)+\lambda_2=\rho'(\rho'-1)$ in equation (29) and by manipulating equations (7-11) we obtain the superpotential $\varphi(\theta_3)$, the variable separation constant λ_3 and ground state wave function as

$$\varphi(\theta_3)=\frac{\hbar}{\sqrt{2M}}((-\rho'\cot\theta_3+\gamma\tan\theta_3) \quad (33)$$

$$\lambda_3=\left(2n_3+\rho'+\gamma\right)^2-1 \quad ; \quad P_{30}=\left(\sin\theta_3\right)^{\rho'-1}\left(\cos\theta_3\right)^{\gamma} \quad \rho'=1/2+\sqrt{(\rho-1/2)^2+\lambda_2} \quad (34)$$

5. Result and Discussion

From the solutions of four one dimensional Schrodinger equations using SUSY QM and the idea of shape invariant potential are exactly obtained in the scheme of centrifugal approximation. The non-relativistic energy spectra is obtained in the close form given as

$$E_n=-\frac{\hbar^2\alpha^2}{2\mu}(g+(\delta'-1)-2n)^2 \quad (35)$$

with $\delta'=1/2+\sqrt{\delta(\delta-1)+\lambda_3+1}$, $\lambda_3=\left(2n_3+\rho'+\gamma\right)^2-1$; $\rho'=1/2+\sqrt{(\rho-1/2)^2+\lambda_2}$

(36)

$$\lambda_2=\left((\upsilon'+(n_2+1))^2-\frac{\zeta^2}{(\upsilon'+(n_2+1))^2}\right)-1/4\,;\ \upsilon'=-1/2\pm\sqrt{\upsilon(\upsilon+1)+\lambda_1}\,;\ \lambda_1=(c+n_1)^2 \quad (37)$$

The extended Scarf I potential is Scarf potential plus Manning Rosen potential. The total ground state wave function is given as

$$\psi_{10}=C_0\left[\left(\cosh\alpha r\right)^{-g}\left(\sinh\alpha r\right)^{-(\delta'-1)-3/2}\left(-\coth\frac{\alpha r}{2}\right)^{-if}\right]^{-\frac{\alpha\sqrt{2m}}{\hbar}}\left(\cos\theta_1\right)^{c+b}\left(1+\sin\theta_1\right)^{-b} \quad (38)$$
$$\left(\sin\theta_2\right)^{\upsilon'+1/2}e^{-\zeta\theta_2/(\upsilon'+1)}\left(\sin\theta_3\right)^{\rho'-1}\left(\cos\theta_3\right)^{\gamma}$$

In Table 1 below, are non-relativistic energy spectra for some value of potential constants δ.

From Table 1 is shown that there is perturbation terms in potential and cause the decrease in energy spectra of Scarf I potential.

Table 1. Non-relativistic energy eigenvalues of particle under the influence of Scarf I potential

δ	$E_{nn_1n_2n_3}$					
	E_{0000}	E_{1000}	E_{2000}	E_{3000}	E_{4000}	E_{5000}
0	-2.4351e-37	-1.3568e-37	-5.9155e-38	-1.3947e-38	-4.9379e-41	-1.7464e-38
2	-2.5315e-37	-1.4290e-37	-6.3956e-38	-1.6326e-38	-7.0221e-42	-1.5000e-38
4	-3.0079e-37	-1.7920e-37	-8.8924e-38	-2.9956e-38	-2.2997e-39	-5.9549e-39
6	-3.8472e-37	-2.4516e-37	-1.3691e-37	-5.9972e-38	-1.4345e-38	-2.8667e-41
8	-5.0321e-37	-3.4135e-37	-2.1080e-37	-1.1156e-37	-4.3632e-38	-7.0155e-39

6. Conclusion

The non-relativistic energy spectra for non-central potential which is combination of extended hyperbolic Scarf I potential with three dimensional trigonometric angular potential is obtained in the closed form. The four dimensional energy spectra and wave functions depend on the variable separation constant, the radial and angular quantum numbers. It is worthy to explore more complicated shape invariant potential which is extension of the standard shape invariant potential.

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