Diffusion in dense non-isothermal Coulomb plasma

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Abstract. We generalize our previous work on diffusion of ions in isothermal plasmas to handle non-isothermal systems. This approach, combined with the method of effective potentials for computing diffusion coefficients, allows us to formulate the diffusion problem in Coulomb systems of arbitrary coupling. The results are applied to study heat-blanketing envelopes of neutron stars in diffusive equilibrium.

1. Introduction

Diffusion in ion mixtures is an important problem of stellar physics. Here we focus on the diffusion problem in dense stellar plasmas where the ions can be moderately or strongly coupled by Coulomb forces. Such plasmas are characteristic for compact stars, i.e. for neutron stars and white dwarfs.

Consider a strongly coupled and non-magnetized multicomponent plasma consisting of several ion species ($\alpha = j$, $j = 1, 2, \ldots$) and neutralizing electron background ($\alpha = e$). Let A_j and Z_j be the mass and charge numbers of ion species j, and n_α be the number density of particles α, with $n_e = \sum_j Z_j n_j$ due to electric neutrality.

Coulomb coupling parameter of ions is proportional to a dimensionless parameter $\Gamma_0 = e^2/(ak_\mathrm{B}T)$ (see, e.g., [1–4], for more details). Here $e > 0$ is an elementary charge, $a = (4\pi n/3)^{-1/3}$ is the average ion sphere radius, $n = \sum_j n_j$ is the total number density of the ions, k_B is the Boltzmann constant and T is the temperature. It is convenient to introduce the average Coulomb coupling parameter defined as $\overline{\Gamma} = \Gamma_0 \overline{Z^{5/3}}\,\overline{Z}^{1/3}$, where the average $\overline{f}$ of any quantity f is given by $\overline{f} = n^{-1}\sum_j n_j f_j$. If $\overline{\Gamma} \gg 1$ the ions are strongly coupled (i.e. highly non-ideal), whereas at $\overline{\Gamma} \ll 1$ they are weakly coupled; $\overline{\Gamma} \sim 1$ refers to the intermediate coupling.

2. General expressions for the diffusive flux

The general approach for deriving the diffusive flux is the same as for an isothermal plasma [5, 6]. Now we need to include properly a temperature gradient. To this aim, we introduce an additional term to the generalized thermodynamic force acting on particles α,

$$\widetilde{\boldsymbol{F}_\alpha} = \boldsymbol{F}_\alpha - \left(\boldsymbol{\nabla}\mu_\alpha - \left.\frac{\partial \mu_\alpha}{\partial T}\right|_P \boldsymbol{\nabla}T \right). \tag{1}$$

Here $\boldsymbol{F}_\alpha$ is a total force, acting on particles α, and μ_α is a chemical potential. In particular, we consider $\boldsymbol{F}_\alpha = e_\alpha \boldsymbol{E} + m_\alpha \boldsymbol{g}$, where e_α and m_α are charge and mass of particles α, respectively;

$\boldsymbol{E}$ is an electric field due to plasma polarization in an external gravitational field (to maintain plasma electric neutrality) and $\boldsymbol{g}$ is a gravitational acceleration. Note that $\partial\mu_\alpha/\partial T$ has to be calculated at constant pressure P.

The quantities $\boldsymbol{d}_\alpha$, which characterize deviations from diffusive equilibrium, are introduced in the same way as in [5, 6],

$$\boldsymbol{d}_\alpha = \frac{\rho_\alpha}{\rho}\sum_\beta n_\beta \widetilde{\boldsymbol{F}_\beta} - n_\alpha \widetilde{\boldsymbol{F}_\alpha}. \tag{2}$$

They have the same properties $\sum_\alpha \boldsymbol{d}_\alpha = 0$ ($\rho_\alpha = m_\alpha n_\alpha$ being a mass density of particles α and ρ a total mass density). Using the Gibbs-Duhem relation $\sum_\alpha n_\alpha \boldsymbol{\nabla}\mu_\alpha = \boldsymbol{\nabla}P - S\boldsymbol{\nabla}T$ (S being the entropy density) we obtain

$$\sum_\alpha n_\alpha \widetilde{\boldsymbol{F}_\alpha} = \rho\boldsymbol{g} - \boldsymbol{\nabla}P. \tag{3}$$

We are mainly interested in the outer heat-blanketing envelopes of neutron stars which are in hydrostatic equilibrium as a whole. Then $\sum_\alpha n_\alpha \widetilde{\boldsymbol{F}_\alpha} = \rho\boldsymbol{g} - \boldsymbol{\nabla}P = 0$ and we can rewrite (2) in the form

$$\boldsymbol{d}_\alpha = -\frac{\rho_\alpha}{\rho}\boldsymbol{\nabla}P - Z_\alpha n_\alpha e\boldsymbol{E} + n_\alpha\left(\boldsymbol{\nabla}\mu_\alpha - \left.\frac{\partial\mu_\alpha}{\partial T}\right|_P \boldsymbol{\nabla}T\right), \tag{4}$$

with $Z_e = -1$ for the electrons. The electric field can be calculated assuming the electron quasi-equilibrium (with respect to ion motion) which occurs mostly because the electrons are much lighter than the ions (e.g. [5]). Then $\boldsymbol{d}_e = 0$ and $m_e \to 0$, leading to $\widetilde{\boldsymbol{F}_e} = 0$, and to

$$e\boldsymbol{E} = -\left(\boldsymbol{\nabla}\mu_e - \left.\frac{\partial\mu_e}{\partial T}\right|_P \boldsymbol{\nabla}T\right). \tag{5}$$

This expression can be written in alternate forms using standard thermodynamic relations.

Chemical potentials are usually known as functions of temperature and number densities, but not the pressure. It is useful to rewrite $\partial\mu_\alpha/\partial T$ at constant P in terms of $\partial\mu_\alpha/\partial T$ at constant n_α. In the following relation, written for a binary ionic mixture (BIM), the pressure is assumed to be known as a function of T and n_α (in other words, the equation of state – EOS – of the system is known),

$$\left.\frac{\partial\mu}{\partial T}\right|_P = \left.\frac{\partial\mu}{\partial T}\right|_{n_1,n_2} - \frac{\partial P}{\partial T}\left(\frac{n_1}{n_2}\frac{\partial\mu}{\partial n_1} + \frac{\partial\mu}{\partial n_2}\right)\left(\frac{n_1}{n_2}\frac{\partial P}{\partial n_1} + \frac{\partial P}{\partial n_2}\right)^{-1}. \tag{6}$$

The general expression for the diffusive flux reads (also see [5])

$$\boldsymbol{J}_\alpha = \Phi\sum_{\beta\neq\alpha} m_\alpha m_\beta D_{\alpha\beta}\boldsymbol{d}_\beta + D_{\alpha,T}\frac{\boldsymbol{\nabla}T}{T}. \tag{7}$$

Here Φ is a normalization function, $\Phi = n/(\rho k_B T)$ for a BIM; $D_{\alpha\beta}$ is a diffusion coefficient for particles α with respect to particles β; $D_{\alpha,T}$ is a thermal diffusion coefficient of particles α. Writing the diffusive flux in such a form guaranties that the diffusion and thermal diffusion coefficients introduced here coincide with their standard definitions in physical kinetics of weakly coupled plasma [7, 8]. For the diffusive currents one always have $\sum_\alpha \boldsymbol{J}_\alpha = 0$.

3. Examples

As an example let us study diffusive equilibrium configurations of BIMs in heat-blanketing envelopes of neutron stars. Although the thermal diffusion term can affect an equilibrium configuration, it is usually small compared to ordinary diffusion. This allows us to neglect thermal diffusion. In addition, electrons weakly affect the ion transport [9] so that the ion subsystem can be studied (quasi-)independently. This simplifies the diffusive flux,

$$\boldsymbol{J}_2 = -\boldsymbol{J}_1 = \frac{nm_1m_2}{\rho k_{\rm B}T} D_{12}\boldsymbol{d}_1, \tag{8}$$

where D_{12} is the interdiffusion coefficient. According to (8) the diffusion equilibrium $\boldsymbol{J}_2 = 0$ is equivalent to the condition $\boldsymbol{d}_1 = 0$. The latter equation can then be used to calculate the equilibrium configuration. In fact, it represents the chemical equilibrium approach of [10] modified for non-isothermal systems. Note that in our particular case we do not need an explicit expression for D_{12}. However, generally, one needs both the diffusion and thermal diffusion coefficients to find the equilibrium configuration.

Let us stress the importance of the last (temperature) term in (1). For strongly coupled ions and degenerate electrons (described in [5]) all temperature derivatives exactly cancel out and the resulting expressions for the diffusive currents are the same as in [5], with the equilibrium configuration equivalent to chemical equilibrium [10]. However, for high enough surface temperatures and light elements like hydrogen, helium or carbon in the heat-blanketing envelope (especially in the outermost parts of the envelope, at $\rho \lesssim 10^7$ g cm^{-3}) the ions can be coupled not strongly but moderately, and the temperature term becomes important.

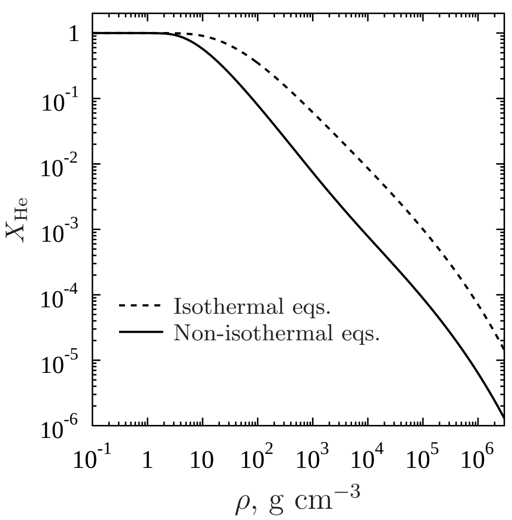

Figure 1: Comparison of isothermal (dashed curve) and non-isothermal (solid curve) approaches to diffusive equilibrium in a BIM in the heat-blanketing envelope of a neutron star. $X_{\rm He}$ is the number fraction of He in He − C mixture shown versus the total mass density. The neutron star has the mass $M = 1.4$ M$_\odot$, radius $R = 10$ km and the surface temperature $T_s = 0.755$ MK.

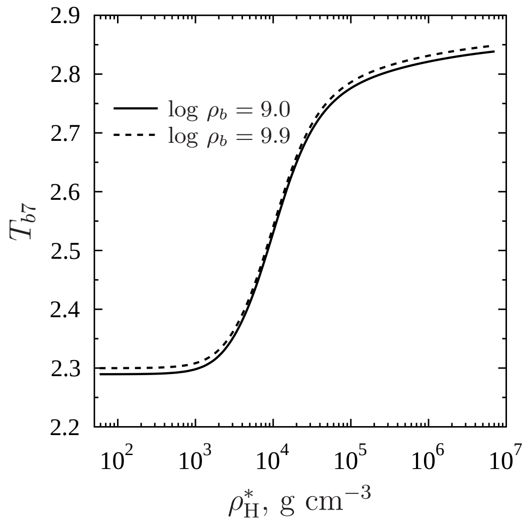

Figure 2: Internal temperature of a neutron star $T_{b7} = T_b/10^7$K in an H − He mixture as a function of $\rho^*_{\rm H}$. One can see the transition from pure He (small amount of H, low $\rho^*_{\rm H}$) to pure H (a lot of H, large $\rho^*_{\rm H}$). Neutron star parameters are the same as in Fig. 1; $\rho^*_{\rm H}$ is an effective density of the transition from H to He. Solid and dashed curves correspond to different densities ρ_b at the bottom of the envelope.

Fig. 1 compares the fraction of helium in a helium − carbon mixture computed using the isothermal approach [5, 10] (the dashed curve) and the non-isothermal approach of this paper (the solid curve). The difference is quite visible. The helium fraction can be important in the heat-blanketing envelopes, in particular, for diffusive nuclear burning. Note that other quantities can be significantly less affected by the temperature term in (1). For example, the relation between internal and surface temperatures of the star, the pressure and total density profiles differ only slightly when computed via the isothermal and non-isothermal approaches.

Fig. 2 shows a transition from a pure hydrogen to a pure helium envelope as well as a weak dependence of internal stellar temperature on the density ρ_b at the bottom of the heat-blanketing envelope. The effective density $\rho_{\rm H}^*$ of the transition is determined by the total amount of hydrogen in the envelope. It is introduced by artificial replacement of a smooth transition by a step-like one. More details will be published elsewhere.

4. Conclusions

We have derived general expressions for the diffusive flux in multicomponent non-isothermal Coulomb systems with arbitrary Coulomb coupling. In the limit of weakly coupled plasma these expressions turn into the classical expressions for diffusion in ideal gas mixtures. Our new expressions are valid not only for Coulomb systems, but for any gaseous or liquid system, provided that one knows chemical potentials of its constituents and corresponding diffusion coefficients (although diffusion is generally available in solids, it is greatly suppressed there compared to gases and liquids).

The expressions for the diffusive flux combined with the diffusion coefficients (see, e.g., [4]) allow one to calculate not only diffusive equilibrium configurations of heat-blanketing envelopes of neutron stars, but also the equilibration of these configurations with time provided that the initial configurations are out of equilibrium.

Acknowledgments

The authors are grateful to A.Y. Potekhin for many useful advises. The work of MB was partly supported by the Dynasty Foundation, and the work of DY by the Russian Foundation for Basic Research (grants 14-02-00868-a and 13-02-12017-ofi-M).

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