

Sheath formation in an oblique magnetic field – some comments on length scales and source terms

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Abstract. A one-dimensional fluid model is presented and used for the analysis of the potential formation in front of a negative, large, planar electrode immersed in magnetized plasma. The magnetic field lines can form an arbitrary angle with the electrode surface, with the exception that they must not be completely parallel to the surface. The model includes elastic ion collisions and 3 source terms that describe ionization mechanisms. If the exponential source term is used, the pre-sheath is longer than in the case of the constant source term. The zero source term results in qualitatively different solutions that may even exhibit nonphysical oscillations. The problem involves 4 characteristic length scales. Any of them can be used for the normalization of the space coordinate and this has no effect to the solutions.

1. Introduction

The boundary of magnetized plasma is of great interest to several fields ranging from material processing to magnetic confinement fusion. The scrape-off layer of a tokamak, for example has low density and strong magnetic fields. Further, the angle of incidence of the magnetic field lines onto the divertor spans nearly the whole range from perpendicular to parallel. In this paper we investigate the structure of the sheath and pre-sheath using a simple fluid description. Similar models [1]-[4] have been used to study the pre-sheath region only. In this work the sheath and the pre-sheath region are studied together. The presented model includes the effects of elastic collisions, finite ion temperature and three model of charged particle creation.

2. Model

Formation of a sheath in front of a negative planar electrode immersed in a magnetized plasma is studied by a one dimensional fluid [1,2] model. The ions are assumed to obey the continuity equation and equation of motion:

$$\frac{\partial n_i}{\partial t} + \nabla \cdot (n_i \mathbf{u}_i) = S_i, \quad (1)$$

$$m_i n_i \left(\frac{\partial \mathbf{u}_i}{\partial t} + (\mathbf{u}_i \cdot \nabla) \mathbf{u}_i \right) = n_i e_0 (\mathbf{E} + \mathbf{u}_i \times \mathbf{B}) - \nabla p_i + \mathbf{A}_i - m_i \mathbf{u}_i S_i. \quad (2)$$

Here m_i is the ion mass, t is time, \mathbf{u}_i is the ion fluid velocity, n_i is the ion density, e_0 is the elementary charge, \mathbf{E} is electric field, \mathbf{B} is magnetic field, p_i is the ion pressure, S_i is the source term and \mathbf{A}_i is the collision term. Let us first focus on the collision term \mathbf{A}_i . The ions can exchange momentum in elastic



collisions with other particle species. In our model those species could be electrons and neutral atoms of the same kind as the ions. The rate of change of momentum of the ions because of such collisions should therefore be written in the following way:

$$\mathbf{A}_i = -m_i n_i \nu_{in} (\mathbf{u}_i - \mathbf{u}_n) - m_i n_i \nu_{ie} (\mathbf{u}_i - \mathbf{u}_e).$$

Here ν_{in} is the frequency of ion neutral collisions, ν_{ie} is the frequency of ion electron collisions, \mathbf{u}_n is the neutral fluid velocity and \mathbf{u}_e is the electron fluid velocity. Elastic collisions with neutrals could be charge exchange collisions or also of some other type. Collisions with the electrons on the other hand are coulomb collisions. It is beyond the scope of this work to analyse in detail the collisions of ions with other particle species. No attempt is made to estimate the relative ion velocity with respect to other particle species. Instead the collision term \mathbf{A}_i is taken in the following simple form:

$$\mathbf{A}_i = -m_i n_0 \nu \mathbf{u}_i. \quad (3)$$

Here ν is the ion collision frequency and n_0 is the density of ions in the region where the plasma is not perturbed by the electrode, this means beyond the pre-sheath.

Three different source terms are considered. The first one is called *the zero source term*:

$$S_i = 0. \quad (4)$$

This means that no ions are produced or annihilated anywhere in the plasma. The second is the *constant source term*:

$$S_i = \frac{n_0}{\tau}. \quad (5)$$

Here τ is the so called ionization time. The source term S_i gives the difference between the number of created and the number of annihilated ions per unit volume and per unit time. The third source term that is considered in this work is the *exponential source term*:

$$S_i = \frac{n_0}{\tau} \exp\left(\frac{e_0 \Phi}{k T_e}\right). \quad (6)$$

Here Φ is the potential, which is space dependent, k is the Boltzmann constant and T_e is the electron temperature.

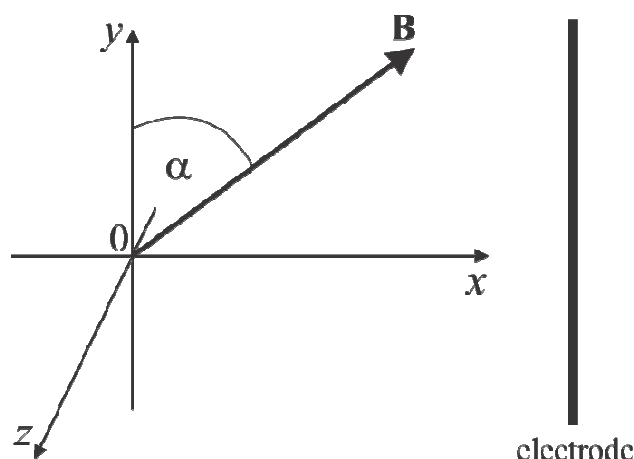


Figure 1. Schematic of the model. A large planar electrode is perpendicular to the x axis. The magnetic field lies in the xy plane. The magnetic field angle α is the angle between the y axis and the magnetic field vector \mathbf{B} .

An infinitely large planar electrode is considered. The electrode is perpendicular to the x axis. The magnetic field lies in the xy plane, as shown in figure 1. The components of the magnetic field are therefore given by $\mathbf{B} = B(\sin\alpha, \cos\alpha, 0)$. Our model is one dimensional. This means that in our model the electric field only has one component: $\mathbf{E} = (E_x, 0, 0)$, which is given by:

$$E_x = -\frac{d\Phi}{dx}.$$

The gradient operator is replaced by the derivative over x :

$$\nabla \rightarrow \frac{d}{dx}.$$

The following equation of state is assumed for the ions:

$$p_i(x) = \kappa n_i(x) k T_i.$$

In a one dimensional model the pressure depends only on x coordinate. Also the ion temperature T_i could in principle be a function of x . But in our model we assume that the ion temperature is constant everywhere and consequently the polytropic coefficient κ is equal to 1, $\kappa = 1$. But in general κ can also be a function of x [5]. So the pressure gradient in equation (2) is replaced by

$$\nabla p_i \rightarrow k T_i \frac{dn_i}{dx}.$$

The electrons are assumed to be Boltzmann distributed with a given temperature T_e :

$$n_e(x) = n_0 \exp\left(\frac{e_0 \Phi(x)}{k T_e}\right).$$

Now also the idea behind the exponential source term becomes clear. The assumption behind the exponential source term is that the main ionization mechanism are ionizing collisions of electrons with neutral atoms.

In the steady state the time derivatives are zero, so the equations (1) and (2) become ordinary differential equations. In the coordinate system defined in figure 1 and using the one dimensional model, the equations (1) and (2) are written in the following way:

$$\frac{dn}{dX} V_x + n \frac{dV_x}{dX} = \frac{d}{dX} (n V_x) = A_1 \begin{Bmatrix} 0 \\ 1 \\ \exp(\Psi) \end{Bmatrix}, \quad (7)$$

$$n V_x \frac{dV_x}{dX} + n \frac{d\Psi}{dX} + \Theta \frac{dn}{dX} + A_2 \cos \alpha n V_z + A_1 V_x \begin{Bmatrix} Z \\ 1 + Z \\ \exp(\Psi) + Z \end{Bmatrix} = 0, \quad (8)$$

$$n V_x \frac{dV_y}{dX} - A_2 \sin \alpha n V_z + A_1 V_y \begin{Bmatrix} Z \\ 1 + Z \\ \exp(\Psi) + Z \end{Bmatrix} = 0, \quad (9)$$

$$n V_x \frac{dV_z}{dX} - A_2 \cos \alpha n V_x + A_2 \sin \alpha n V_y + A_1 V_z \begin{Bmatrix} Z \\ 1 + Z \\ \exp(\Psi) + Z \end{Bmatrix} = 0. \quad (10)$$

The system is completed with the Poisson equation in one dimension, which determines the potential profile:

$$A_3 \frac{d^2 \Psi}{dX^2} = \exp \Psi - n. \quad (11)$$

The following variables have been introduced:

$$\begin{aligned} \lambda_D &= \sqrt{\frac{\varepsilon_0 k T_e}{n_0 e_0^2}}, \quad c_0 = \sqrt{\frac{k T_e}{m_i}}, \quad \omega_c = \frac{e_0 B}{m_i} = \frac{2\pi}{T_{\text{cyc}}}, \quad K = \omega_c \tau, \quad L = c_0 \tau, \quad \mu = \frac{m_e}{m_i}, \\ L_c &= \frac{c_0}{\nu}, \quad r_{Li} = \frac{c_0}{\omega_c}, \quad n = \frac{n_i}{n_0}, \quad \Psi = \frac{e_0 \Phi}{k T_e}, \quad \Theta = \frac{T_i}{T_e}, \quad \varepsilon = \frac{\lambda_D}{L}, \quad Z = \nu \tau = \frac{\tau}{T_{\text{coll}}}, \\ V_x &= \frac{u_x}{c_0}, \quad V_y = \frac{u_y}{c_0}, \quad V_z = \frac{u_z}{c_0}, \quad X = \frac{x}{\lambda_D} \quad \text{or} \quad X = \frac{x}{L} \quad \text{or} \quad X = \frac{x}{r_{Li}} \quad \text{or} \quad X = \frac{x}{L_c}. \end{aligned} \quad (12)$$

Here ε_0 is the permittivity of the free space and T_{cyc} is the cyclotron period. The potential Φ is normalized to the electron temperature divided by elementary charge kT_e/e_0 . The components of the ion velocity are normalized to the so called normalizing velocity c_0 , defined in (12). The problem is characterized by 4 characteristic length scales: the ionization length L , the Debye length λ_D , the mean free path or collision length L_c and the ion Larmor radius r_{Li} . The ionization length L is the distance that an ion moving with the normalizing velocity c_0 passes in one ionization time τ . The mean free path L_c is the distance that an ion moving with the normalizing velocity c_0 passes in the so called collision time $T_{\text{coll}} = 1/\nu$. An ion that enters the magnetic field B with normalizing velocity c_0 perpendicularly to the magnetic field line gyrates with the Larmor radius r_{Li} . Obviously the space coordinate x can be normalized to any of these lengths. From (12) it can be seen easily that:

$$K = \frac{L}{r_{Li}} = \frac{2\pi\tau}{T_{\text{cyc}}}, \quad Z = \frac{L}{L_c}, \quad \frac{K}{Z} = \frac{L_c}{r_{Li}} = \frac{2\pi T_{\text{coll}}}{T_{\text{cyc}}}, \quad K\varepsilon = \frac{\lambda_D}{r_{Li}}, \quad Z\varepsilon = \frac{\lambda_D}{L_c}. \quad (13)$$

A remark should be given about the ion sound velocity c_s . The ion sound velocity c_s is defined as:

$$c_s = \sqrt{\frac{kT_e^* + \kappa kT_i}{m_i}}. \quad (14)$$

Here T_e^* is the so called screening temperature [5]. In our case it is equal to T_e because the electrons are the only negatively biased particle species. Using (12) and taking into account that $\kappa = 1$ the ion sound velocity (14) is written as:

$$V_s = \frac{c_s}{c_0} = \sqrt{1 + \Theta}.$$

The coefficients A_1 , A_2 and A_3 depend on the normalization of the space coordinate x and they are given in table 1.

Table 1. The coefficients A_1 , A_2 and A_3 for various normalizations of the space coordinate x .

| Symbol | $X=x/L$ | $X=x/\lambda_D$ | $X=x/L_c$ | $X=x/r_{Li}$ |
|--------|-----------------|-----------------|---------------------|---------------------|
| A_1 | 1 | ε | $1/Z$ | $1/K$ |
| A_2 | K | $K\varepsilon$ | K/Z | 1 |
| A_3 | ε^2 | 1 | $Z^2 \varepsilon^2$ | $K^2 \varepsilon^2$ |

The system (7) - (11) represents 12 (3 source terms \times 4 normalizations of x) systems of 5 equations for 5 unknown functions of X , which are $n(X)$, $\Psi(X)$, $V_x(X)$, $V_y(X)$ and $V_z(X)$. Once these functions are

found, one may find also the parallel velocity V_{par} , the perpendicular velocity V_{perp} and the angle of incidence β , which are given by:

$$V_{par} = \frac{\mathbf{V} \cdot \mathbf{B}}{B} = V_x \sin \alpha + V_y \cos \alpha,$$

$$V_{perp} = \frac{|\mathbf{V} \times \mathbf{B}|}{B} = \sqrt{V_z^2 + (V_x \cos \alpha - V_y \sin \alpha)^2},$$

$$\beta = \arctan \left(\frac{V_x}{V_y} \right).$$

Here parallel and perpendicular refers to the magnetic field.

3. Results

We now present some results of the model presented in the previous section. In order to find the unknown functions $n(X)$, $\Psi(X)$, $V_x(X)$, $V_y(X)$ and $V_z(X)$ from the system (7) - (11) first the normalization of x and the source term must be selected and then the parameters K , Z , ε , α and Θ must also be selected. In the next step boundary conditions must be selected. From figure 1 it can be seen that the numerical integration of the equations starts at $X = 0$ and proceeds in the positive X direction towards the electrode, so strictly mathematically speaking, one is dealing with initial value problem. The “initial” conditions (since X is the space coordinate these are in fact *boundary conditions*) that are usually selected are the following:

$$n(0) = 1, \Psi(0) = 0, \frac{d\Psi}{dX}(0) = 0, V_x(0) = V_0, V_y(0) = 0, V_z(0) = 0. \quad (15)$$

It is assumed that at the starting point $X = 0$ the plasma is not perturbed, so the plasma density there is n_0 , which gives the first boundary condition. In the unperturbed plasma also electric field must be zero and this gives the third boundary condition. The second boundary condition simply tells that the plasma potential at $X = 0$ is selected as the zero of the potential. In our model the ions are born at rest so all three velocity components should be zero. But in this case only the zero solution of the system can be found. So a small starting velocity [1,3,4] in the positive X direction must be selected. A typical value is $V_0 = 10^{-8}$. The system of equations is integrated using the fourth order Runge-Kutta method with variable step size. The size of the step is decreased only a limited number of times (called iterations), which is specified beforehand. If after the prescribed number of iterations the desired precision has not yet been reached, the system is declared to be stiff, since in the point, where this occurs there is very probably a singularity.

In figure 2 an example of the solution of the system (7) - (11) is shown. The following parameters are selected: $K = 50$, $Z = 2$, $\alpha = 20^\circ$, $\Theta = 0$ and 2 values of ε are selected, $\varepsilon = 10^{-3}$ and $\varepsilon = 10^{-5}$. The boundary conditions are given (15) with $V_0 = 10^{-8}$ and the exponential source term (6) is selected. The space coordinate is normalized to the Larmor radius r_{Li} . Solutions obtained with $\varepsilon = 10^{-3}$ are plotted with solid line while the solutions obtained with $\varepsilon = 10^{-5}$ are plotted with a dotted line. In the top left figure the velocity component $V_x(X)$ is shown. The ion sound velocity V_s is also labelled. The point where V_x reaches V_s is identified as the sheath edge. In the top middle graph the potential profile $\Psi(X)$ is presented. In the top right plot the electric field profile is displayed. In the bottom left figure the ion $n(X)$ and electron density $\exp(\Psi(X))$ profiles are shown. On the insert a part of the figure is shown on an expanded scale. In the bottom middle plot the space charge density profile is shown. This is simply the difference between the ion and electron density $n(X) - \exp(\Psi(X))$. Finally in the bottom right figure the angle of incidence $\beta(X)$ is plotted. Note that according to the boundary conditions, $\beta(0)$ should be 90 degrees. The starting point of the integration is eliminated from the plot. The ion flow follows the

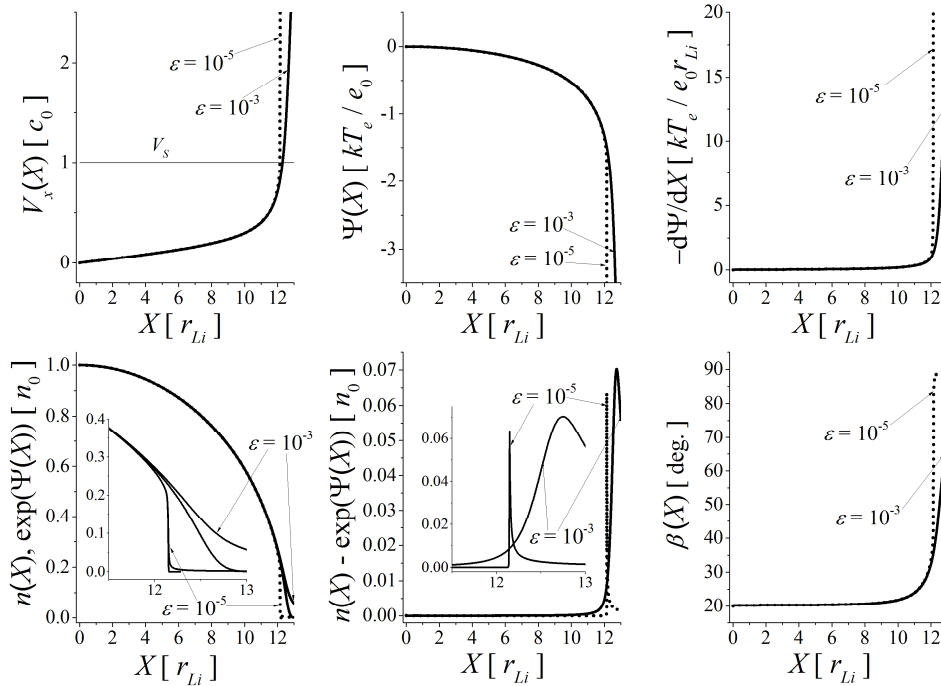


Figure 2. An example of the solution of the system (7) - (11). The parameters are: $K = 50$, $Z = 2$, $\alpha = 20^\circ$, $\Theta = 0$ and 2 values of ε are selected, $\varepsilon = 10^{-3}$ (solid) and $\varepsilon = 10^{-5}$ (dotted). The boundary conditions are given by (15) with $V_0 = 10^{-8}$ and the exponential source term (6) is selected. The space coordinate is normalized to the Larmor radius r_{Li} .

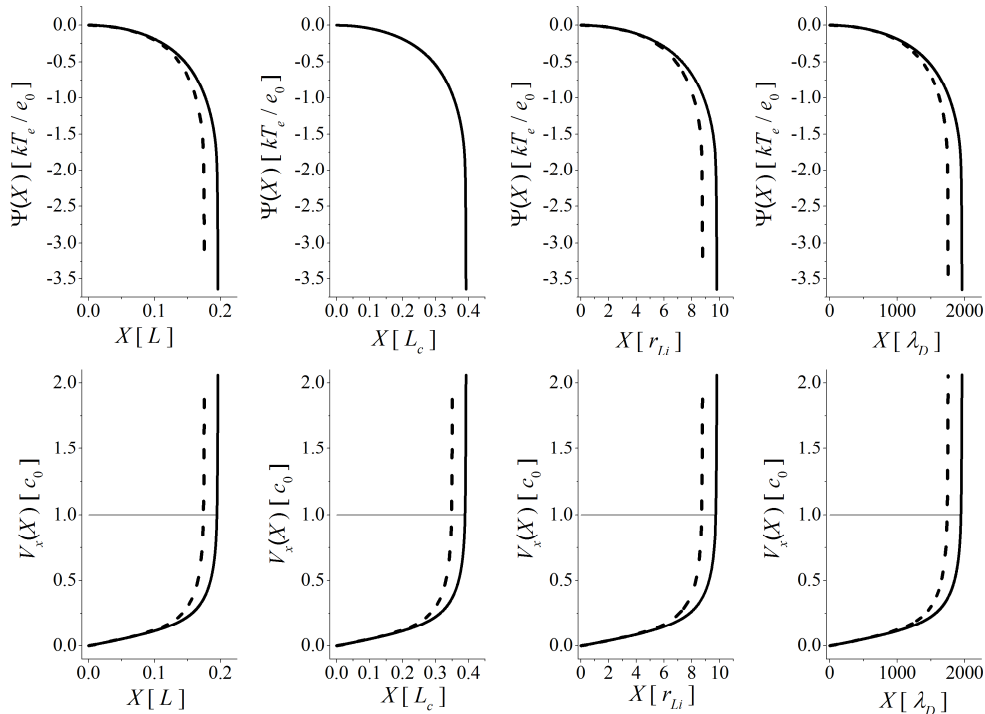


Figure 3. Potential $\Psi(X)$ (top) and velocity profiles $V_x(X)$ (bottom) found from the system (7) - (11) for the parameters $K = 50$, $Z = 2$, $\alpha = 20^\circ$, $\Theta = 0$ and $\varepsilon = 10^{-4}$. The boundary conditions are given by (15) with $V_0 = 10^{-8}$. The exponential (solid) and the constant (dashed) source term are used. The space coordinate is normalized to all 4 characteristic lengths.

magnetic field lines almost to the sheath edge and then within approximately 1 Larmor radius it is diverted by the electric field into direction which is almost perpendicular to the electrode. The selection of ε has strong impact to the solutions especially close to the sheath edge.

In figure 3 properties of the solutions of the system (7) - (11) are illustrated further. The following parameters are selected: $K = 50$, $Z = 2$, $\alpha = 20^\circ$, $\Theta = 0$ and $\varepsilon = 10^{-4}$. The boundary conditions are given by (15) with $V_0 = 10^{-8}$. The constant and the exponential source term are selected. The solutions obtained with the exponential source term are shown with solid line, while the solutions that correspond to the constant source term are dashed. The space coordinate is normalized to all 4 characteristic lengths. The potential profiles $\Psi(X)$ are shown in the top graphs, while the velocity profiles $V_x(X)$ are displayed in the bottom plots for all space coordinate normalizations. From the left to the right the normalizing lengths go as follows: the ionization length L , the mean free path L_c , the Larmor radius r_{Li} and the Debye length λ_D . It can be seen clearly that the normalization of the space coordinate has no effect to the solutions, except that the horizontal scale is expanded or stretched, as prescribed by the values of K , Z and ε . Note that one ionization length correspond to 2 mean free paths and to 50 Larmor radii and to 10000 Debye lengths, exactly as determined by $K = 50$, $Z = 2$ and $\varepsilon = 10^{-4}$.

In figure 4 very similar solutions of the system (7) - (11) are shown as in figure 3. Even the same parameters are selected: $K = 50$, $Z = 2$, $\alpha = 20^\circ$, $\Theta = 0$ and $\varepsilon = 10^{-4}$. Only this time the zero source term is used. This time V_0 must be increased for 7 orders of magnitude in order to obtain the solutions with similar pre-sheath lengths as with the exponential and constant source terms.

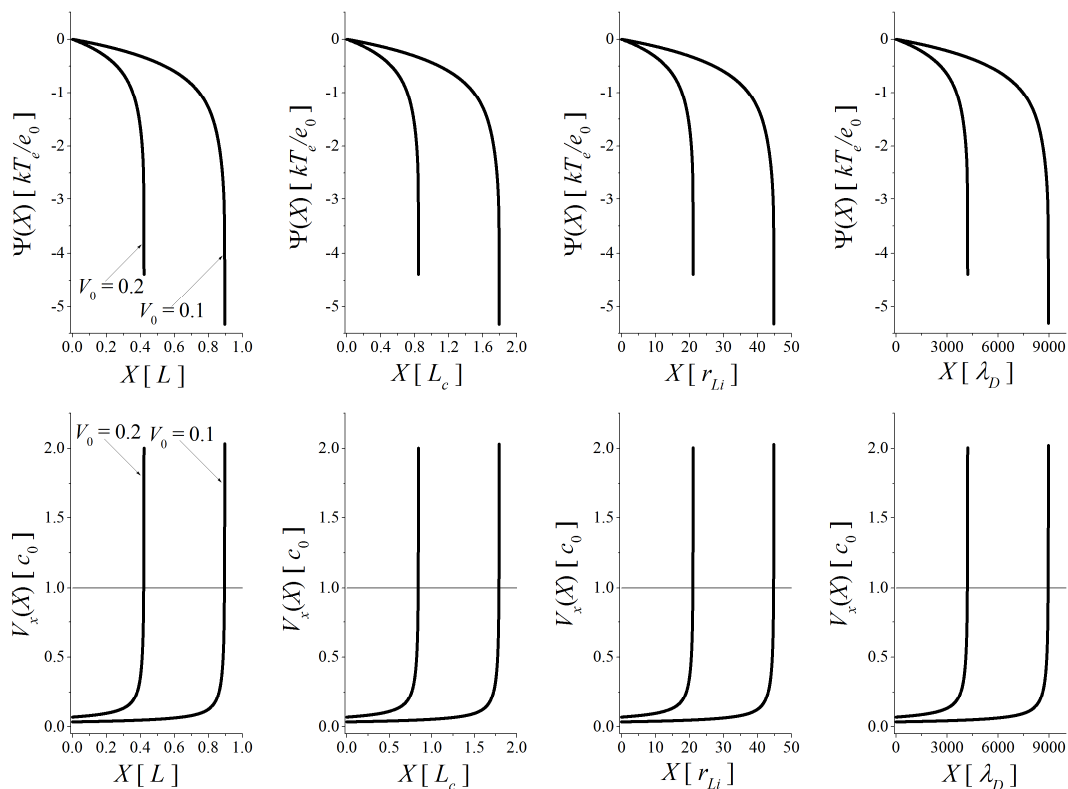


Figure 4. Potential $\Psi(X)$ (top) and velocity profiles $V_x(X)$ (bottom) found from the system (7) - (11) for the parameters $K = 50$, $Z = 2$, $\alpha = 20^\circ$, $\Theta = 0$ and $\varepsilon = 10^{-4}$. The boundary conditions are given by (15) with $V_0 = 0.1$ and $V_0 = 0.2$. The zero source term is used. The space coordinate is normalized to all 4 characteristic lengths.

When the zero source term is selected the solutions of the system (7) - (11) are much different than in the case when the other two source terms are used. One interesting property is illustrated in figure 5.

When the collision rate Z is low, the solutions of the system (7) - (11) exhibit coherent oscillations for certain boundary conditions and parameters. The following parameters are selected: $Z = 0.001$, $\varepsilon = 10^{-5}$, $K = 50$, $\alpha = 20^\circ$ and $\Theta = 0$.

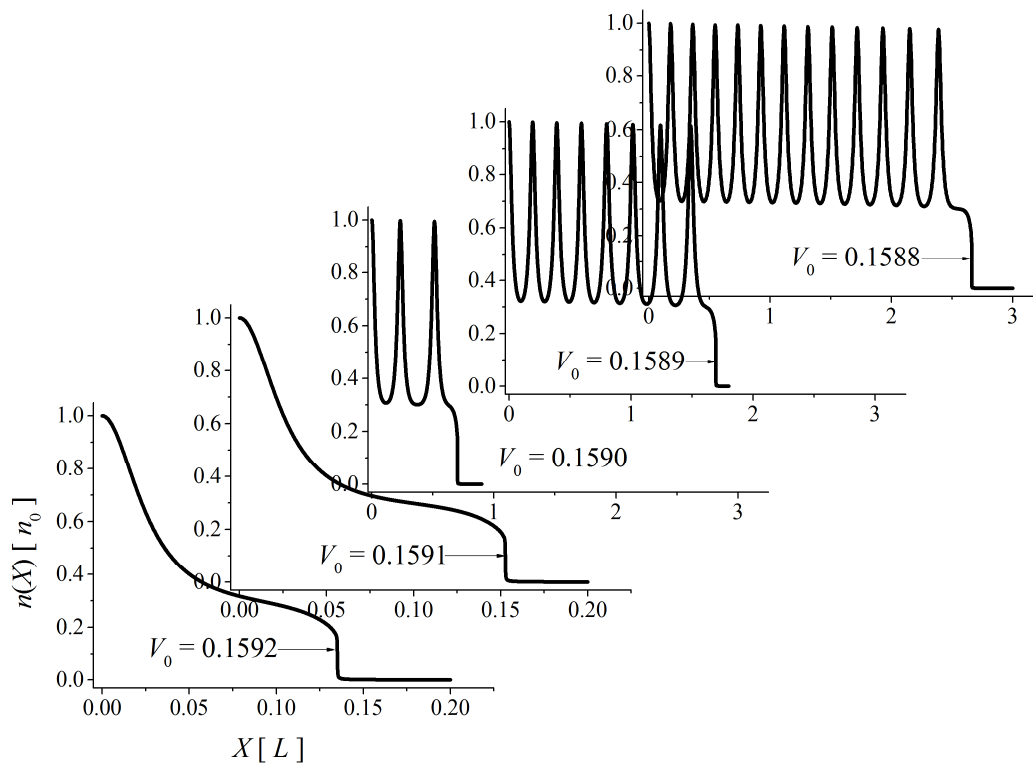


Figure 5. Oscillating solutions (ion density profiles $n(X)$) of the system (7) - (11) for the parameters: $Z = 0.001$, $\varepsilon = 10^{-5}$, $K = 50$, $\alpha = 20^\circ$ and $\Theta = 0$. The zero source term is used. The starting velocities are labelled in the figure.

The zero source term is used. In figure 5 the ion density profiles $n(X)$ are displayed. Whether these oscillations have some physical explanation, or are they a purely numerical effect is still an open question.

So far the ion temperature Θ has always been set to zero. If it is increased above zero the system (7) - (11) becomes singular. This stiffness of the system (7) - (11) can only be removed by increasing the velocity V_0 considerably. In figure 6 the minimum V_0 where the solution of the system (7) - (11) can be found is plotted versus the ion temperature Θ . The parameters are: $Z = 2$, $\varepsilon = 10^{-4}$, $K = 50$ and $\alpha = 20^\circ$. The exponential source is used. The results are very similar also for the constant and even for the zero source term. In the left plot both axis have linear scale. In the right plot the same results are shown, only both axis are in the logarithmic scale. The minimum V_0 is always slightly above the square root of the selected ion temperature. From (12) it can be seen easily that $\sqrt{\Theta}$ corresponds to the ion thermal velocity $\sqrt{kT_i/m_i}$.

Some examples of the ion density profiles $n(X)$ obtained with non-zero ion temperatures are shown in figure 7. The exponential source term is used and the following parameters are selected: $Z = 2$, $\varepsilon = 10^{-4}$, $K = 50$ and $\alpha = 20^\circ$. The ion temperatures 10^{-4} , 10^{-3} , 10^{-2} , 0.1 and 1 are selected. At each temperature the minimum velocity V_0 is used to solve the system (7) - (11). Because the increased ion temperature requires larger velocity V_0 the pre-sheath length decreases.

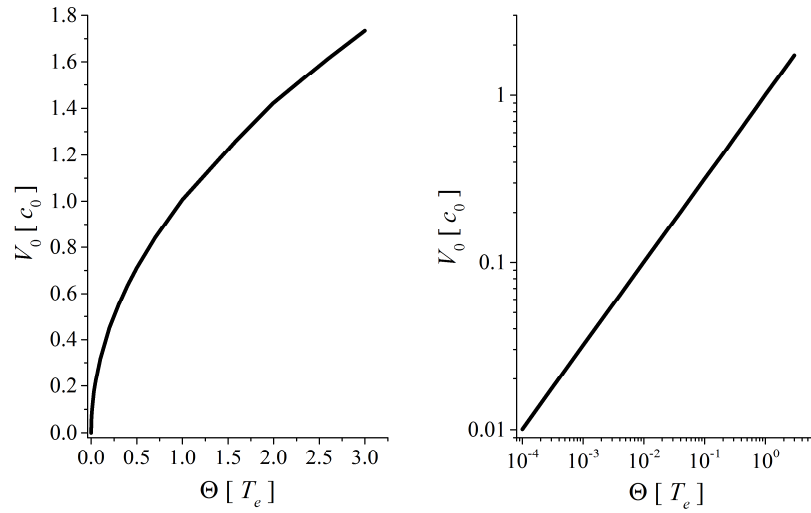


Figure 6. The minimum velocity V_0 for which the system (7) - (11) can be solved versus the ion temperature Θ . In the left graph the linear scale of the axis is used and in the right plot the logarithmic scale is applied.

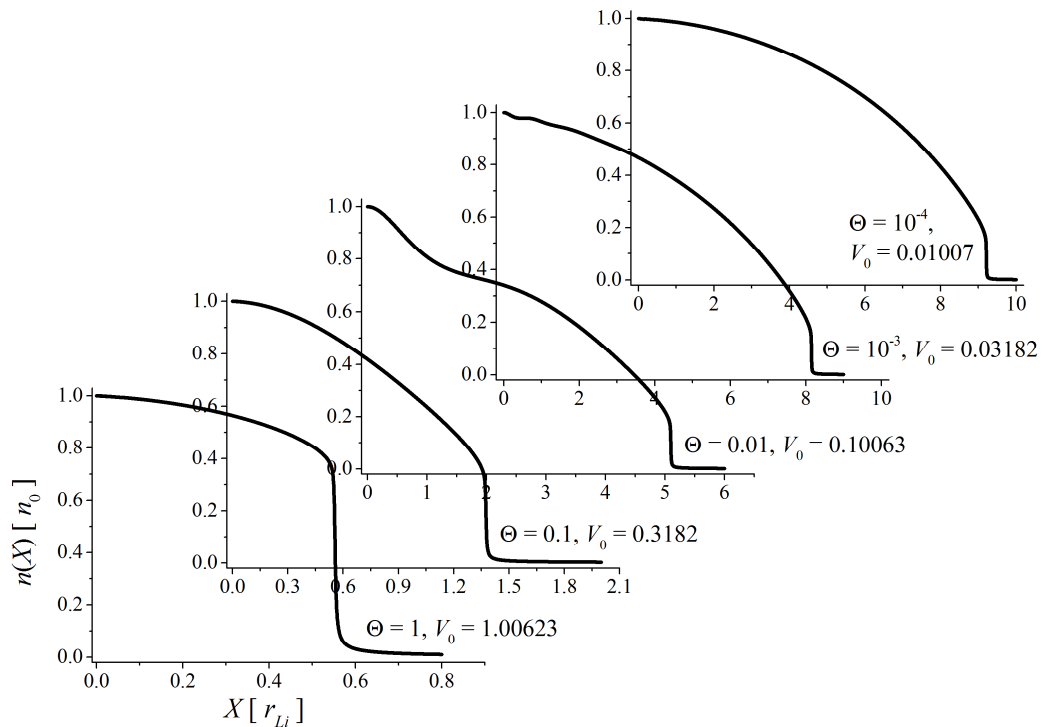


Figure 7. The ion density profiles $n(X)$ obtained from the system (7) - (11) for the parameters $Z = 2$, $\varepsilon = 10^{-4}$, $K = 50$ and $\alpha = 20^\circ$. Several ion temperatures Θ are selected and the minimum necessary V_0 is used at each Θ .

4. Conclusions

A one-dimensional fluid has been presented and used for the analysis of the potential formation in front of a negative planar electrode immersed in magnetized plasma. The magnetic field lines form an

arbitrary angle with the electrode surface. The only exception is that they must not be completely parallel to the electrode surface. Three different source terms have been considered. For the constant source term the pre-sheath region is shorter than for the exponential source term if the other parameters are the same. If the zero source term is selected, the behaviour of the solutions is different. The starting ion velocity must be increased for several orders of magnitude in order to obtain solutions with comparable pre-sheath lengths as for the constant and the exponential source terms. When the zero source term is selected and the collision rate is decreased, the solutions exhibit strong oscillations. The nature of these oscillations remains to be investigated.

Four characteristic length scales appear in the problem – the ionization length L , the collision length or mean free path L_c , the ion Larmor radius r_{Li} and the Debye length λ_D . The space coordinate x can be normalized to any of them. In all four cases the solutions are exactly the same only the horizontal axis is expanded or stretched. The selection of the coordinate scaling is purely a matter of taste and has no physical or mathematical consequences. If one is interested in the pre-sheath region the ionization length L is appropriate choice for scaling. If one is more interested in the sheath region, the Debye length λ_D is more suitable. If the collision effects are studied, the mean free path L_c is most probably the best choice.

If the zero source term (4) is selected, the ionization time τ loses its physical meaning because it becomes infinitely large. But formally it is still possible to derive dimensionless equations (7) - (10) from (1) and (2) by multiplication with τ . In this case τ should be interpreted as a formally defined time constant, which is equal to $K/2\pi$ cyclotron periods T_{cyc} and to Z collision times T_{coll} . If the zero source term is selected, also the ionization length L loses its physical meaning. But one can still introduce a formal length scale L , which is equal to K Larmor radii r_{Li} and to Z mean free paths L_c and to $1/\varepsilon$ Debye lengths λ_D . The Debye length λ_D , the Larmor radius r_{Li} and the mean free path L_c are still well defined relatively to each other.

If the ion temperature is increased above zero, the system of equations becomes stiff. This stiffness is removed by increasing the starting ion velocity V_0 . The minimum V_0 , where the stiffness is removed, is slightly above the square root of the ion temperature. From (12) it can be seen that this corresponds to the ion thermal velocity.

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