Some Results About Concircular and Concurrent Vector Fields On Pseudo-Kaehler Manifolds

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Abstract. Kaehler manifolds which are used in physics have a lot of application fields. In this study we only state concircular and concurrent vector field that are defined on these manifolds. A vector field on a pseudo-Riemannian manifold N is called concircular, if it satisfies $\nabla_X v = \mu X$ for any vector X tangent to N, where ∇ is the Levi-Civita connection of N. Furthermore, a concircular vector field v is called a concurrent vector field if the function μ is non-constant. So, we provide some results on submanifolds of pseudo-Kaehler manifolds with respect to a concircular vector field or a concurrent vector field. Morever, we investigate this problem for another manifolds and proof some theorems.

1. Introduction

The notion of concircular vector field on a Riemannian manifold was introduced by A. Fialkow [7] by

$$\nabla_X v = \mu X \tag{1}$$

where X is tangent to N, ∇ denotes the Levi-Civita connection of N and μ is a non-trivial function on N. A concircular vector field satisfying (1) is called non-trivial if the function μ is non-constant. Furthermore, a concircular vector field v is called a concurrent vector field if the function μ is equal to one [2]. For simplicity, we call as a concurrent vector field v if the function μ in (1) is a non-zero constant.

Concircular vector fields have an important role in the theory of projective and conformal transformations. Additionally, such vector fields have interesting applications in general relativity. Particularly, a Lorentzian manifold is a generalized Robertson-Walker spacetime if and only if it admits a timelike concircular vector field. For some further results related to concircular vector fields, you can see [7, 3, 8].

A pseudo-Riemannian metric g on a complex manifold (M, J) is called pseudo-Hermitian if the metric g and the complex structure J on M are given by

$$g(JX, JY) = g(X, Y), X, Y \in T_pM, p \in M \tag{2}$$

A pseudo-Hermitian manifold is called a pseudo-Kaehler manifold if its complex structure J is parallel with respect to its Levi-Civita connection ∇. So it is clear that $\nabla J = 0$.

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Let N be an n-dimensional manifold with a 3-dimensional vector bundle Q consisting of three tensors J_1, J_2 and J_3 of type $(1,1)$ over N. Suppose, in any coordinate neighborhood U of N, there is a local basis $\{J_1, J_2, J_3\}$of Q such that

$$J_1^2 = J_2^2 = J_3^2 = -I$$

and

$$J_2J_3 = -J_3J_2 = J_1, J_3J_1 = -J_1J_3 = J_2, J_1J_2 = -J_2J_1 = J_3$$

Such a basis$\{J_1, J_2, J_3\}$is called a canonical local basis of the bundle Q in U. We say that the bundle Q has an almost quaternion structure in N and (N, Q) is called an almost quaternion manifold whose dimension is $n = 4m, (m \geq 1)$. In such a case, the given structure Q is called an integrable quaternionic structure. Suppose g is an indefinite metric on (N, Q) such that $g(\phi X, \phi Y) = g(X, Y), \forall X, Y \in TpN$, $p \in N$ and $\phi = J_1, J_2, J_3$, with$\{J_1, J_2, J_3\}$being a basis of Q at p. Then, (N, g, Q) is called an indefinite almost quaternion manifold.If the Levi-Civita connection ∇ of $\left(\tilde{M}, g, Q\right)$ satisfies

$$\begin{aligned} \nabla_X J_1 &= r(X) J_2 - q(X) J_3 \\ \nabla_X J_2 &= -r(X) J_1 - p(X) J_3 \\ \nabla_X J_3 &= q(X) J_1 - p(X) J_2 \end{aligned} \tag{3}$$

for any vector field X on $\tilde{M}$, then $\tilde{M}$ is called an indefinite quaternion Kaehler manifold, where p, q, r are certain local $1-$forms and $\{J_1, J_2, J_3\}$ is a local canonical basis of Q. Let M be a lightlike submanifold of an indefinite quaternion Kaehler manifold N. We say that M is a QR-lightlike submanifold if the following conditions are fulfilled:

$$\begin{aligned} &J_aRadTM \cap RadTM = \{0\}, \\ &S(TM) = \{J_aRadTM \oplus \widetilde{D}\} \perp D_0, \\ &J_a(D_0) = D_0, J_a(L_1 \perp L_2) = \widetilde{D} \subset S(TM), \end{aligned}$$

where $L_1 = ltr(TM)$ and L_2 is a vector sub-bundle of $S(TM^{\perp})$. Now, we consider

$$D = \{RadTM \oplus_{orth} J_aRadTM\} \oplus_{orth} D_0,$$

where D_0 is non-degenerate distribution. It is easy to check that D_0 is invariant with respect to each J_a.Thus we have

$$TM = D \oplus \widetilde{D}$$

[6].

2. Some results for concurent vector fields

Assume that N is a indefinite quaternionic Kaehler manifold and M is a pseudo-Kaehler submanifold of N which admits a concurrent vector field υ. We have the following:

Proposition 2.1. *Let N be a indefinite quaternionic Kaehler manifold and M be a submanifold with the indefinite metric of N. Then we have*
a) For any concurrent vector field v on M, $J_a\upsilon$ $(a = 1, 2, 3)$ is never a concurrent vector field on M.
b) The complex distribution $D = Sp\{v, J_1v, J_2v, J_3v\}$ of N is always integrable.

Proof. a) Let υ be a concurrent vector field on M. For $\bar{\nabla}$ Levi-Civita connection of N

$$\bar{\nabla}_X J_a \upsilon = J_a \bar{\nabla}_X \upsilon = c J_a X + h(X, \upsilon) J_a W_i$$

which implies that $J_a \upsilon$ is not a concurrent vector field for $\bar{\nabla}$ (since $\bar{\nabla}_X J\upsilon \neq cX$).

b) Due to the Gauss equation, for $W_i \in TM^{\perp}$

$$\begin{aligned} \bar{\nabla}_{J_a\upsilon}\upsilon &= \nabla_{J_a\upsilon}\upsilon + h(J_a\upsilon, \upsilon) W_i = cJ_a\upsilon + h(J_a\upsilon, \upsilon) W_i \\ \bar{\nabla}_{\upsilon} J_a \upsilon &= \nabla_{\upsilon} J_a \upsilon + h(\upsilon, J_a\upsilon) W_i \\ \bar{\nabla}_X J_a \upsilon &= J_a \nabla_X \upsilon = J_a c X = c J_a X \end{aligned}$$

then we find

$$\begin{aligned} \bar{\nabla}_{J_a\upsilon}\upsilon - \bar{\nabla}_{\upsilon} J_a \upsilon &= cJ_a\upsilon + h(J_a\upsilon, \upsilon) W_i - cJ_a\upsilon - h(\upsilon, J_a\upsilon) W_i = 0 \\ &\Rightarrow \quad [J_a\upsilon, \upsilon] = 0 \end{aligned} \tag{4}$$

which implies that for $\bar{\nabla}$ Levi-Civita connection of N, the distribution $\mathcal{D}$ is integrable. □

Corollary. *Let υ is a concurrent vector field on an indefinite quaternionic Kaehler manifold N. Then υ is a concurrent vector field on a submanifold of N if and only if $h(X, \upsilon) = 0$.*

Proof. Let N be an indefinite quaternionic Kaehler manifold and M be a submanifold of N. For $\bar{\nabla}$ Levi-Civita connection of N we can write $\bar{\nabla}_X \upsilon = cX$, then

$$\bar{\nabla}_X \upsilon = \nabla_X \upsilon + h(X, \upsilon) W_i = cX$$

and we find

$$h(X, \upsilon) = 0$$

Obviously, the opposite of the theorem is true. □

Now we define the Ricci tensor of N. Let $e_1, ..., e_{2n}$ be an orthonormal frame on M, then the Ricci tensor Ric of M is defined by

$$Ric(X, Y) = \sum_{i=1}^{2n} \varepsilon_i g(R(X, e_i) Y, e_i) + \sum_{j=2n}^{2m} \varepsilon_j g(R(X, e_j) Y, e_j) \tag{5}$$

where $g(e_i, e_j) = \varepsilon_i \delta_{ij}$. Let R denote the Riemannian curvature tensor of M which is given by

$$R(X, Y) Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \tag{6}$$

for vector fields X, Y, Z tangent to M. Furthermore, it is well-known that the Riemannian curvature tensor R satisfies

$$\begin{aligned} &i) R(X, Y) = -R(Y, X) \\ &ii) R(X, Y) JZ = J(R(X, Y) Z) \\ &iii) R(JX, JY) = R(X, Y) Z \\ &iv) g(R(X, Y) Z, W) = g(R(Z, W) X, Y) \end{aligned} \tag{7}$$

Since υ is a concurrent vector field,

$$\begin{aligned} \bar{R}(X, \upsilon)\upsilon &= \bar{\nabla}_X \bar{\nabla}_{\upsilon}\upsilon - \bar{\nabla}_{\upsilon}\bar{\nabla}_X \upsilon - \bar{\nabla}_{[X,\upsilon]}\upsilon \\ &= \bar{\nabla}_X (c\upsilon + h(\upsilon, \upsilon) W_i) - \bar{\nabla}_{\upsilon}(cX + h(X, \upsilon) W_i) \\ &\quad -c[X, \upsilon] - h([X, \upsilon], \upsilon) W_i \\ &= X(h(\upsilon, \upsilon)) W_i + h(\upsilon, \upsilon) \nabla^{\perp}_X W_i - h(\upsilon, \upsilon) A_{W_i} X - \upsilon(h(X, \upsilon)) W_i \\ &\quad -h(X, \upsilon) \nabla^{\perp}_{\upsilon} W_i + h(X, \upsilon) A_{W_i}\upsilon - h([X, \upsilon], \upsilon) W_i \end{aligned} \tag{8}$$

So, we can write

$$Ric(\upsilon,\upsilon)=\sum_{j=2n}^{2m}\varepsilon_j g\left(R\left(\upsilon,e_j\right)\upsilon,e_j\right)=-\sum_{j=2n}^{2m}\varepsilon_j g\left(R\left(e_j,\upsilon\right)\upsilon,e_j\right) \tag{9}$$

Multiplying with e_j we find

$$g\left(R\left(e_j,\upsilon\right)\upsilon,e_j\right)=g\left(-h\left(\upsilon,\upsilon\right)A_{W_i}X+h\left(X,\upsilon\right)A_{W_i}\upsilon,e_j\right) \tag{10}$$

From this,

$$Ric(\upsilon,\upsilon)=-\sum_{j=2n}^{2m}\varepsilon_j g\left(-h\left(\upsilon,\upsilon\right)A_{W_i}X+h\left(X,\upsilon\right)A_{W_i}\upsilon,e_j\right) \tag{11}$$

Consequently, we have the following:

Proposition 2.2. *Let N be a indefinite quaternionic Kaehler manifold and M be a submanifold with the indefinite metric of N which admits a concurrent vector field (e.i. M is Ricci-flat). If*

$$-h\left(\upsilon,\upsilon\right)\mathcal{A}_{W_i}X+h\left(X,\upsilon\right)\mathcal{A}_{W_i}\upsilon=0 \tag{12}$$

then N is Ricci-flat ($Ric=0$).

Theorem 2.3. *Let N be an indefinite quatenionic Kaehler manifold and M be a QR-lightlike submanifold of manifold N. Then, there are the followings.*
a) *For any concurent vector field υ on M, $J_a\upsilon$ is never a concurent vector field on M.*
b) *The complex distribution $\mathcal{D}$ of N is always integrable.*

Proof. a) Let υ be a concurent vector field on M. For $\bar{\nabla}$ Levi-Civita connection of N

$$\begin{aligned}\bar{\nabla}_X J_a\upsilon &= J_a\bar{\nabla}_X\upsilon=J_a\left(\nabla_X\upsilon+h^l\left(X,\upsilon\right)N_j+h^s\left(X,\upsilon\right)W_i\right)\\ &= J_a\left(cX+h^l\left(X,\upsilon\right)N_j+h^s\left(X,\upsilon\right)W_i\right)=cJ_aX+h^l\left(X,\upsilon\right)J_aN_j+h^s\left(X,\upsilon\right)J_aW_i\end{aligned}$$

where $N_j\in ltr\left(TM\right),W_i\in S\left(TM\right)^{\perp}$. Since $\bar{\nabla}_X J_a\upsilon\neq cX$, $J_a\upsilon$ is not a concurent vector field for $\bar{\nabla}$.

b)

$$\begin{aligned}\bar{\nabla}_{J_a\upsilon}\upsilon &= \nabla_{J_a\upsilon}\upsilon+h^l\left(X,\upsilon\right)N_j+h^s\left(X,\upsilon\right)W_i\\ \bar{\nabla}_{\upsilon}J_a\upsilon &= \nabla_{\upsilon}J_a\upsilon+h^l\left(X,\upsilon\right)N_j+h^s\left(X,\upsilon\right)W_i\end{aligned}$$

then we find

$$\bar{\nabla}_{J_a\upsilon}\upsilon-\bar{\nabla}_{\upsilon}J_a\upsilon=cJ_a\upsilon-\nabla_{\upsilon}J_a\upsilon=cJ_a\upsilon-J_a\nabla_{\upsilon}\upsilon=cJ_a\upsilon-J_ac\upsilon=0\Rightarrow\left[J_a\upsilon,\upsilon\right]=0$$

which implies that for $\bar{\nabla}$ Levi-Civita connection of N, the distribution $\mathcal{D}$ is integrable. □

Corollary. *Let υ is a concurrent vector field on a indefinite quaternionic Kaehler manifold N. Then υ is a concurrent vector field on a QR-lightlike submanifold of N if and only if $h^l\left(X,\upsilon\right)=0$ and $h^s\left(X,\upsilon\right)=0$*

Proof. Let N be a indefinite quaternionic Kaehler manifold and M be a submanifold of N. For $\bar{\nabla}$ Levi-Civita connection of N we can write $\bar{\nabla}_X\upsilon=cX$, then

$$\bar{\nabla}_X\upsilon=\nabla_X\upsilon+h^l\left(X,\upsilon\right)N_j+h^s\left(X,\upsilon\right)W_i=cX$$

and we find

$$h^l\left(X,\upsilon\right)=0\text{ and }h^s\left(X,\upsilon\right)=0$$

Obviously, the opposite of the theorem is true. □

3. QR-lightlike hypersurfaces with concircular vector fields

Theorem 3.1. *Let M be a QR-lightlike hypersurface of an indefinite Kaehler manifold N. Then we have, $h(X,v)=0, X\in\Gamma(D_0)$ such that v is a concircular vector field.*

Proof. For v$\in\Gamma(TM)$ and $\xi\in\Gamma(RadTM)$,

$$\begin{aligned}\widetilde{g}\left(\tilde{\nabla}_X v, J_a\xi\right) &= \widetilde{g}\left(\nabla_X v + h(X,v)N, J_a\xi\right) \\ &= \widetilde{g}\left(\nabla_X v, J_a\xi\right) + \widetilde{g}\left(h(X,v)N, J_a\xi\right)\end{aligned} \tag{13}$$

Due to $N\in\Gamma(ltrTM)$ and $J_a\xi\in\Gamma(S(TM))$, $\widetilde{g}(N,J_a\xi)=0$. So,

$$\widetilde{g}\left(\tilde{\nabla}_X v, J_a\xi\right) = g(\mu X, J_a\xi) \tag{14}$$

By using this equation, we find

$$\widetilde{g}\left(\tilde{\nabla}_X v, J_a\xi\right) = 0 \tag{15}$$

From here, we get

$$\begin{aligned}\widetilde{g}\left(\tilde{\nabla}_X v, J_aJ_b\xi\right) &= \widetilde{g}\left(\nabla_X v + h(X,v)N, J_aJ_b\xi\right) \\ &= \widetilde{g}\left(\mu X, J_aJ_b\xi\right) + h(X,v)\,\widetilde{g}\left(N, J_aJ_b\xi\right)\end{aligned} \tag{16}$$

This equation must equal to zero. Then, let us find the needed conditions for this. For $a=b$,

$$\widetilde{g}\left(\tilde{\nabla}_X v, J_aJ_b\xi\right) = \widetilde{g}(\mu X, -\xi) + h(X,v)\,\widetilde{g}(N,-\xi) = h(X,v) \tag{17}$$

So, $h(X,v)=0$ is. For $a\neq b$,

$$\widetilde{g}\left(\tilde{\nabla}_X v, J_aJ_b\xi\right) = \widetilde{g}(\mu X, J_c\xi) + h(X,v)\,\widetilde{g}(N,J_c\xi) = 0 \tag{18}$$

where $J_c = J_aJ_b$. □

We say that M is a mixed geodesic QR-lightlike surface if $h(X,Z)=0$ for any $X\in\Gamma(D), Z\in\Gamma\left(\widetilde{D}\right)$.

Theorem 3.2. *Let M be a QR-lightlike hypersurface of an indefinite Kaehler manifold N.*
a) M is a mixed geodesic if $X(J_a)=0$ while Z is a concircular vector field.
b) M is a mixed geodesic if $\widetilde{g}([J_aN,X],\xi)+\widetilde{g}(A_NX,J_a\xi)=X(J_a)$ while X is a concircular vector field.

Proof. a) Let Z be a concircular vector field and $Z=J_aN$.

$$\begin{aligned}h(X,J_aN)N &= \tilde{\nabla}_X J_aN - \nabla_X J_aN \\ &= X(J_aN) + J_a\left(\tilde{\nabla}_X N\right) - \mu X \\ &= X(J_aN) + J_a\left(\nabla^{\perp}_X N\right) + J_a(A_NX) - \mu X\end{aligned}$$

So, we find that

$$\widetilde{g}(h(X,J_aN)N,\xi) = X(J_a)$$

b) Let X be a concircular vector field.

$$\begin{aligned} h\left(X, J_aN\right)N &= \tilde{\nabla}_{J_aN}X - \nabla_{J_aN}X \\ &= [J_aN, X] - \tilde{\nabla}_X J_aN - \nabla_{J_aN}X \end{aligned}$$

So, we find that

$$\begin{aligned} \widetilde{g}\left(h\left(X, J_aN\right)N, \xi\right) &= \widetilde{g}\left([J_aN, X], \xi\right) - \widetilde{g}\left(\tilde{\nabla}_X\left(J_aN\right), \xi\right) - \widetilde{g}\left(\nabla_{J_aN}X, \xi\right) \\ &= \widetilde{g}\left([J_aN, X], \xi\right) - \widetilde{g}\left(X\left(J_a\right)N, \xi\right) - \widetilde{g}\left(J_a\left(\tilde{\nabla}_X N\right), \xi\right) - \mu\widetilde{g}\left(J_aN, \xi\right) \\ &= \widetilde{g}\left([J_aN, X], \xi\right) - X\left(J_a\right) - \widetilde{g}\left(\tilde{\nabla}_X N, J_a\xi\right) - \mu\widetilde{g}\left(J_aN, \xi\right) \\ &= \widetilde{g}\left([J_aN, X], \xi\right) - X\left(J_a\right) - \widetilde{g}\left(A_N X, J_a\xi\right) \end{aligned}$$

□

4. References

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