Analytical solutions with the improved (G'/G)-expansion method for nonlinear evolution equations

Melike Kaplan, Ahmet Bekir, Arzu Akbulut

Eskisehir Osmangazi University, Art-Science Faculty, Mathematics-Computer Department, Eskisehir 26480, TURKEY

E-mail: mkaplan@ogu.edu.tr; abekir@ogu.edu.tr; ayakut1987@hotmail.com

Abstract. To seek the exact solutions of nonlinear partial differential equations (NPDEs), the improved $(G'/G)-$expansion method is proposed in the present work. With the aid of symbolic computation, this effective method is applied to construct exact solutions of the (1+1)-dimensional nonlinear dispersive modified Benjamin-Bona-Mahony equation and (3+1)-dimensional Kudryashov-Sinelshchikov equation. As a result, new types of exact solutions are obtained.

1. Introduction

Many important complex phenomena and dynamic processes in physics, mechanics, biology and chemistry can be described by NPDEs. Therefore, the investigation of the exact solutions for NPDEs have become more and more attractive in the study of soliton theory. Recently, a lot of direct methods have been proposed to construct exact solutions of these equations partly by virtue of the applicability of symbolic computation packages like Mathematica and Maple, which enables us to carry out the exact computation on computer [1-11].

The original $(G'/G)-$expansion method is a widely used method in soliton theory and mathematical physics. The keynote of this method is that the travelling wave solutions of NPDEs can be represented in terms of (G'/G) in which $G = G(\xi)$ satisfies the second order ordinary differential equation $G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0$, where λ and μ are constants [5]. Comparably, in the improved $(G'/G)-$expansion method the travelling wave solutions of NPDEs can be represented in terms of (G'/G) in which $G = G(\xi)$ satisfies the second order ordinary differential equation $GG'' = DG^2 + EGG' + F(G')^2$, where D, E and F are real parameters [13].

1.1. Algorithm of the improved $(G'/G)-$expansion method

We consider a general partial differential equation, say in the independent variables x and t is given by

$$P(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, ...) = 0, \tag{1}$$

where u is an unknown function, P is a polynomial in u and their various partial derivatives.

To find the exact solution of Eq.(1) , we introduce the following travelling wave transformation:

$$u(x,t) = u(\xi), \xi = x - ct. \tag{2}$$

Here c denotes the wave velocity.

Employing Eq.(2), we can rewrite Eq.(1) in a nonlinear ordinary differential equation (ODE) as follows

$$Q(u, u', u'', u''', ...) = 0. \tag{3}$$

Here the prime denotes the derivation with respect to ξ. Eq.(3) is then integrated as many times as possible and setting the integration cosntant to zero. In the improved $(G'/G)-$expansion method, the solution $u(\xi)$ is considered in the finite series form

$$u(\xi) = \sum_{i=0}^{n} a_i \left(\frac{G'}{G}\right)^i. \tag{4}$$

Here the positive integer n denotes the balancing number, which is determined by considering the homogeneous balance principle. Namely, it can be calculated by balancing the highest order derivative term and nonlinear term appears in Eq.(3). Here $G(\xi)$ satisfies the second order auxiliary ODE in the form:

$$GG'' = DG^2 + EGG' + F(G')^2, \tag{5}$$

where D, E and F are real parameters. Also note that Eq.(5) reduces into following Riccati equation as:

$$\frac{d}{d\xi}\left(\frac{G'}{G}\right) = D + E\left(\frac{G'}{G}\right) + (F-1)\left(\frac{G'}{G}\right)^2. \tag{6}$$

From the general solutions of the Eq.(6) , we have the following different cases:

Case 1: If $E \neq 0$ and $\Delta = E^2 + 4D - 4DF < 0$

$$\frac{G'(\xi)}{G(\xi)} = \frac{E}{2(1-F)} + \frac{E\sqrt{-\Delta}}{2(1-F)}\left(\frac{ic_1\cos\left(\frac{\sqrt{-\Delta}}{2}\xi\right) - c_2\sin\left(\frac{\sqrt{-\Delta}}{2}\xi\right)}{ic_1\sin\left(\frac{\sqrt{-\Delta}}{2}\xi\right) + c_2\cos\left(\frac{\sqrt{-\Delta}}{2}\xi\right)}\right) \tag{7}$$

Case 2: If $E \neq 0$ and $\Delta = E^2 + 4D - 4DF \geq 0$

$$\frac{G'(\xi)}{G(\xi)} = \frac{E}{2(1-F)} + \frac{E\sqrt{\Delta}}{2(1-F)}\left(\frac{c_1\exp\left(\frac{\sqrt{\Delta}}{2}\xi\right) + c_2\exp\left(\frac{-\sqrt{\Delta}}{2}\xi\right)}{c_1\exp\left(\frac{\sqrt{\Delta}}{2}\xi\right) - c_2\exp\left(\frac{-\sqrt{\Delta}}{2}\xi\right)}\right) \tag{8}$$

Case 3: If $E = 0$ and $\Delta = D(1-F) < 0$

$$\frac{G'(\xi)}{G(\xi)} = \frac{\sqrt{-\Delta}}{(1-F)}\left(\frac{ic_1\cosh\left(\sqrt{-\Delta}\xi\right) - c_2\sinh\left(\sqrt{-\Delta}\xi\right)}{-c_1\sinh\left(\sqrt{-\Delta}\xi\right) - c_2\cosh\left(\sqrt{-\Delta}\xi\right)}\right) \tag{9}$$

Case4: If $E = 0$ and $\Delta = D(1-F) \geq 0$

$$\frac{G'(\xi)}{G(\xi)} = \frac{\sqrt{\Delta}}{(1-F)}\left(\frac{c_1\cos\left(\sqrt{\Delta}\xi\right) + c_2\sin\left(\sqrt{\Delta}\xi\right)}{c_1\sin\left(\sqrt{\Delta}\xi\right) - c_2\cos\left(\sqrt{\Delta}\xi\right)}\right) \tag{10}$$

where $\xi = x - ct$ and D, E, F, c_1 and c_2 are arbitrary constants.

We substitute Eq.(4) into Eq.(3) along with Eq.(6) and collect the coefficients of $\left(\frac{G'(\xi)}{G(\xi)}\right)^i, (i = 0, 1, ...)$ then set each coefficient to zero to derive a set of algebraic equations for $a_i, (i = 0, 1, ..., n)$, D, E, F and c. We solve these set of algebraic equations with the aid of Maple packet program and substitute into Eq.(4) along with the general solutions of the Eq.(6) [13].

2. Implemantation of the improved $(G'/G)-$expansion method

In the current section, we apply our algoritm to the (1+1)-dimensional nonlinear dispersive modified Benjamin-Bona-Mahony equation and (3+1)-dimensional Kudryashov-Sinelshchikov equation.

2.1. (1+1)-dimensional nonlinear dispersive modified Benjamin-Bona-Mahony equation

The (1+1)-dimensional nonlinear dispersive modified Benjamin-Bona-Mahony equation was first derived to model an approximation for surface long waves in nonlinear dispersive media. This equation can also qualify the acoustic waves in inharmonic crystals, hydromagnetic waves in cold plasma and acoustic gravity waves in compressible fluids. It is given in the following form [12]:

$$u_t + u_x - \alpha u^2 u_x + u_{xxx} = 0, \tag{11}$$

where $u = u(\xi)$. Employing the travelling wave transformation (2), Eq.(11) reduced to an ODE and integrating the equation, we get

$$(1 - c)\, u - \frac{\alpha u^3}{3} + u'' = 0. \tag{12}$$

By balancing the highest order derivative terms and nonlinear terms in Eq.(12) , we get the balancing number $m = 1$. According to the improved $(G'/G)-$expansion method, the exact solution takes the form:

$$u(\xi) = a_0 + a_1 \frac{G'(\xi)}{G(\xi)}. \tag{13}$$

Then we substitute Eq.(13) into the Eq.(12) and collect the coefficients of $\left(\frac{G'(\xi)}{G(\xi)}\right)^i, (i = 0, 1, 2, 3)$ then set each coefficient to zero to derive a set of algebraic equations. By solving this system with the aid of symbolic computation, we get the following results

$$a_0 = \pm\tfrac{E}{2}\sqrt{\tfrac{6}{\alpha}}, \quad a_1 = \pm\sqrt{\tfrac{6}{\alpha}}\,(F - 1)\,, \quad c = 2DF - \tfrac{E^2}{2} + 1 - 2D, \quad D = D, \quad E = E, \quad F = F. \tag{14}$$

If we substitute these results into Eq.(13), we find the following cases for the exact solutions of (1+1) dimensional nonlinear dispersive modified Benjamin-Bona-Mahony equation.

Case 1: When $E \neq 0$ and $\Delta = E^2 + 4D - 4DF < 0$,

$$u_1(\xi) = \pm\frac{E}{2}\sqrt{\frac{6}{\alpha}} \pm \sqrt{\frac{6}{\alpha}}\,(F - 1)\left(\frac{E}{2\,(1 - F)} + \frac{E\sqrt{-\Delta}}{2\,(1 - F)}\left(\frac{ic_1 \cos\left(\frac{\sqrt{-\Delta}}{2}\xi\right) - c_2 \sin\left(\frac{\sqrt{-\Delta}}{2}\xi\right)}{ic_1 \sin\left(\frac{\sqrt{-\Delta}}{2}\xi\right) + c_2 \cos\left(\frac{\sqrt{-\Delta}}{2}\xi\right)}\right)\right) \tag{15}$$

Case 2: When $E \neq 0$ and $\Delta = E^2 + 4D - 4DF \geq 0$,

$$u_2(\xi) = \pm\frac{E}{2}\sqrt{\frac{6}{\alpha}} \pm \sqrt{\frac{6}{\alpha}}\,(C - 1)\left(\frac{E}{2\,(1 - F)} + \frac{E\sqrt{\Delta}}{2\,(1 - F)}\left(\frac{c_1 \exp\left(\frac{\sqrt{\Delta}}{2}\xi\right) + c_2 \exp\left(\frac{-\sqrt{\Delta}}{2}\xi\right)}{c_1 \exp\left(\frac{\sqrt{\Delta}}{2}\xi\right) - c_2 \exp\left(\frac{-\sqrt{\Delta}}{2}\xi\right)}\right)\right) \tag{16}$$

Case 3: When $E = 0$ and $\Delta = D(1-F) < 0$,

$$u_3(\xi) = \pm\sqrt{\frac{6}{\alpha}}(F-1)\left(\frac{\sqrt{-\Delta}}{(1-F)}\left(\frac{ic_1\cosh\left(\sqrt{-\Delta}\xi\right) - c_2\sinh\left(\sqrt{-\Delta}\xi\right)}{-c_1\sinh\left(\sqrt{-\Delta}\xi\right) - c_2\cosh\left(\sqrt{-\Delta}\xi\right)}\right)\right) \tag{17}$$

Case 4: When $E = 0$ and $\Delta = D(1-F) \geq 0$,

$$u_4(\xi) = \pm\sqrt{\frac{6}{\alpha}}(F-1)\left(\frac{\sqrt{\Delta}}{(1-F)}\left(\frac{c_1\cos\left(\sqrt{\Delta}\xi\right) + c_2\sin\left(\sqrt{\Delta}\xi\right)}{c_1\sin\left(\sqrt{\Delta}\xi\right) - c_2\cos\left(\sqrt{\Delta}\xi\right)}\right)\right) \tag{18}$$

Note that our solutions are different from the given ones in [12].

2.2. (3+1)-dimensional Kudryashov-Sinelshchikov equation
We handle with the (3+1)-dimensional Kudryashov-Sinelshchikov equation, which has been examined to model the physical characteristics of nonlinear waves in a bubbly liquid. It is given in the form

$$(u_t + uu_x + u_{xxx} - \gamma u_{xx})_x + \frac{1}{2}(u_{yy} + u_{zz}) = 0, \tag{19}$$

where $u = u(x, y, z, t)$ represents the density of the bubbly liquid, the scalar quantity γ depends on the kinematic viscosity of the bubbly liquid and the independent variables x, y and z are the scaled space coordinates, t is the scaled time coordinate [14].

By using the travelling wave transformation

$$\xi = x + y + z - ct, u(x, y, z, t) = u(\xi), \tag{20}$$

Eq.(19) can be reduced to a nonlinear ODE and integrating the equation twice with respect to ξ, and taking the integration constants as zero, we get

$$(1-c)u + \frac{u^2}{2} + u'' - \gamma u' = 0. \tag{21}$$

Balancing the highest order derivative term with the nonlinear term appearing in Eq.(21) , we find the balancing number as $n = 2$. By means of the improved $(G'/G)-$expansion method, the solutions takes the form as follows

$$u(\xi) = a_0 + a_1\frac{G'(\xi)}{G(\xi)} + a_2\left(\frac{G'(\xi)}{G(\xi)}\right)^2. \tag{22}$$

Then we substitute Eq.(22) into the Eq.(21) and collect the coefficients of $\left(\frac{G'(\xi)}{G(\xi)}\right)^i, (i = 0, 1, 2, 3, 4)$, then set each coefficient to zero to derive a set of algebraic equations. If we solve the set of algebraic equations above, we get different cases for the solutions of D, E, F, a_0, a_1, a_2 and c.

(**I**)

$$D = D, E = E, \qquad c = \tfrac{6}{25}\gamma^2 + 1, \qquad F = \tfrac{-\gamma^2+100D+25E^2}{100D},$$

$$a_0 = -3E^2 + \tfrac{6}{5}\gamma E + \tfrac{9}{25}\gamma^2, \quad a_1 = -\tfrac{3(\gamma^3-25E^2\gamma-5\gamma^2E+125E^3)}{125D}, \quad a_2 = -\tfrac{3(\gamma^4-50\gamma^2E^2+625E^4)}{2500D^2}. \tag{23}$$

Substituting these results into Eq.(22) we get the following cases for the exact solutions of (3+1)-dimensional Kudryashov-Sinelshchikov equation.

Case 1: When $E \neq 0$ and $\Delta = E^2 + 4D - 4DF < 0$,

$$\begin{aligned} u_1(\xi) &= -3E^2 + \tfrac{6}{5}\gamma E + \tfrac{9}{25}\gamma^2 \\ &- \tfrac{3(\gamma^3 - 25E^2\gamma - 5\gamma^2 E + 125E^3)}{125D}\left(\tfrac{E}{2(1-F)} + \tfrac{E\sqrt{-\Delta}}{2(1-F)}\left(\tfrac{ic_1\cos\left(\frac{\sqrt{-\Delta}}{2}\xi\right) - c_2\sin\left(\frac{\sqrt{-\Delta}}{2}\xi\right)}{ic_1\sin\left(\frac{\sqrt{-\Delta}}{2}\xi\right) + c_2\cos\left(\frac{\sqrt{-\Delta}}{2}\xi\right)}\right)\right) \\ &- \tfrac{3(\gamma^4 - 50\gamma^2E^2 + 625E^4)}{2500D^2}\left(\tfrac{E}{2(1-F)} + \tfrac{E\sqrt{-\Delta}}{2(1-F)}\left(\tfrac{ic_1\cos\left(\frac{\sqrt{-\Delta}}{2}\xi\right) - c_2\sin\left(\frac{\sqrt{-\Delta}}{2}\xi\right)}{ic_1\sin\left(\frac{\sqrt{-\Delta}}{2}\xi\right) + c_2\cos\left(\frac{\sqrt{-\Delta}}{2}\xi\right)}\right)\right)^2 \end{aligned} \tag{24}$$

Case 2: When $E \neq 0$ and $\Delta = E^2 + 4D - 4DF \geq 0$,

$$\begin{aligned} u_2(\xi) &= -3E^2 + \tfrac{6}{5}\gamma E + \tfrac{9}{25}\gamma^2 \\ &- \tfrac{3(\gamma^3 - 25E^2\gamma - 5\gamma^2 E + 125E^3)}{125D}\left(\tfrac{E}{2(1-F)} + \tfrac{E\sqrt{\Delta}}{2(1-F)}\left(\tfrac{c_1\exp\left(\frac{\sqrt{\Delta}}{2}\xi\right) + c_2\exp\left(\frac{-\sqrt{\Delta}}{2}\xi\right)}{c_1\exp\left(\frac{\sqrt{\Delta}}{2}\xi\right) - c_2\exp\left(\frac{-\sqrt{\Delta}}{2}\xi\right)}\right)\right) \\ &- \tfrac{3(\gamma^4 - 50\gamma^2E^2 + 625E^4)}{2500D^2}\left(\tfrac{E}{2(1-F)} + \tfrac{E\sqrt{\Delta}}{2(1-F)}\left(\tfrac{c_1\exp\left(\frac{\sqrt{\Delta}}{2}\xi\right) + c_2\exp\left(\frac{-\sqrt{\Delta}}{2}\xi\right)}{c_1\exp\left(\frac{\sqrt{\Delta}}{2}\xi\right) - c_2\exp\left(\frac{-\sqrt{\Delta}}{2}\xi\right)}\right)\right)^2 \end{aligned} \tag{25}$$

Case 3: When $E = 0$ and $\Delta = D(1-F) < 0$,

$$\begin{aligned} u_3(\xi) &= \tfrac{9}{25}\gamma^2 - \tfrac{3\gamma^3}{125D}\left(\tfrac{\sqrt{-\Delta}}{(1-F)}\left(\tfrac{ic_1\cosh(\sqrt{-\Delta}\xi) - c_2\sinh(\sqrt{-\Delta}\xi)}{-c_1\sinh(\sqrt{-\Delta}\xi) - c_2\cosh(\sqrt{-\Delta}\xi)}\right)\right) \\ &- \tfrac{3\gamma^4}{2500D^2}\left(\tfrac{\sqrt{-\Delta}}{(1-F)}\left(\tfrac{ic_1\cosh(\sqrt{-\Delta}\xi) - c_2\sinh(\sqrt{-\Delta}\xi)}{-c_1\sinh(\sqrt{-\Delta}\xi) - c_2\cosh(\sqrt{-\Delta}\xi)}\right)\right)^2 \end{aligned} \tag{26}$$

Case 4: When $E = 0$ and $\Delta = D(1-F) \geq 0$,

$$u_4(\xi) = \frac{9}{25}\gamma^2 - \frac{3\gamma^4}{2500D^2}\left(\frac{\sqrt{\Delta}}{(1-F)}\left(\frac{c_1\cos\left(\sqrt{\Delta}\xi\right) + c_2\sin\left(\sqrt{\Delta}\xi\right)}{c_1\sin\left(\sqrt{\Delta}\xi\right) - c_2\cos\left(\sqrt{\Delta}\xi\right)}\right)\right)^2 \tag{27}$$

(**II***)*

$$\begin{gathered} D = D, E = E, \qquad c = -\tfrac{6}{25}\gamma^2 + 1, \qquad F = \tfrac{-\gamma^2 + 100D + 25E^2}{100D}, \\ a_0 = -3E^2 + \tfrac{6}{5}\gamma E - \tfrac{3}{25}\gamma^2, \quad a_1 = -\tfrac{3(\gamma^3 - 25E^2\gamma - 5\gamma^2 E + 125E^3)}{125D}, \quad a_2 = -\tfrac{3(\gamma^4 - 50\gamma^2E^2 + 625E^4)}{2500D^2}. \end{gathered} \tag{28}$$

Substituting these results into Eq.(22) we get the following cases for the exact solutions of (3+1)-dimensional Kudryashov-Sinelshchikov equation.

Case 1: When $E \neq 0$ and $\Delta = E^2 + 4D - 4DF < 0$,

$$\begin{aligned} u_5(\xi) &= -3E^2 + \tfrac{6}{5}\gamma E - \tfrac{3}{25}\gamma^2 \\ &- \tfrac{3(\gamma^3 - 25E^2\gamma - 5\gamma^2 E + 125E^3)}{125D}\left(\tfrac{E}{2(1-F)} + \tfrac{E\sqrt{-\Delta}}{2(1-F)}\left(\tfrac{ic_1\cos\left(\frac{\sqrt{-\Delta}}{2}\xi\right) - c_2\sin\left(\frac{\sqrt{-\Delta}}{2}\xi\right)}{ic_1\sin\left(\frac{\sqrt{-\Delta}}{2}\xi\right) + c_2\cos\left(\frac{\sqrt{-\Delta}}{2}\xi\right)}\right)\right) \\ &- \tfrac{3(\gamma^4 - 50\gamma^2E^2 + 625E^4)}{2500D^2}\left(\tfrac{E}{2(1-F)} + \tfrac{E\sqrt{-\Delta}}{2(1-F)}\left(\tfrac{ic_1\cos\left(\frac{\sqrt{-\Delta}}{2}\xi\right) - c_2\sin\left(\frac{\sqrt{-\Delta}}{2}\xi\right)}{ic_1\sin\left(\frac{\sqrt{-\Delta}}{2}\xi\right) + c_2\cos\left(\frac{\sqrt{-\Delta}}{2}\xi\right)}\right)\right)^2 \end{aligned} \tag{29}$$

Case 2: When $E \neq 0$ and $\Delta = E^2 + 4D - 4DF \geq 0$,

$$\begin{aligned} u_6(\xi) &= -3E^2 + \tfrac{6}{5}\gamma E - \tfrac{3}{25}\gamma^2 \\ &-\tfrac{3(\gamma^3-25E^2\gamma-5\gamma^2E+125E^3)}{125D}\left(\tfrac{E}{2(1-F)} + \tfrac{E\sqrt{\Delta}}{2(1-F)}\left(\tfrac{c_1\exp\left(\frac{\sqrt{\Delta}}{2}\xi\right)+c_2\exp\left(\frac{-\sqrt{\Delta}}{2}\xi\right)}{c_1\exp\left(\frac{\sqrt{\Delta}}{2}\xi\right)-c_2\exp\left(\frac{-\sqrt{\Delta}}{2}\xi\right)}\right)\right) \\ &-\tfrac{3(\gamma^4-50\gamma^2E^2+625E^4)}{2500D^2}\left(\tfrac{E}{2(1-F)} + \tfrac{E\sqrt{\Delta}}{2(1-F)}\left(\tfrac{c_1\exp\left(\frac{\sqrt{\Delta}}{2}\xi\right)+c_2\exp\left(\frac{-\sqrt{\Delta}}{2}\xi\right)}{c_1\exp\left(\frac{\sqrt{\Delta}}{2}\xi\right)-c_2\exp\left(\frac{-\sqrt{\Delta}}{2}\xi\right)}\right)\right)^2 \end{aligned} \tag{30}$$

Case 3: When $E = 0$ and $\Delta = D(1-F) < 0$,

$$\begin{aligned} u_7(\xi) &= -\tfrac{3}{25}\gamma^2 - \tfrac{3\gamma^3}{125A}\left(\tfrac{\sqrt{-\Delta}}{(1-F)}\left(\tfrac{ic_1\cosh\left(\sqrt{-\Delta}\xi\right)-c_2\sinh\left(\sqrt{-\Delta}\xi\right)}{-c_1\sinh\left(\sqrt{-\Delta}\xi\right)-c_2\cosh\left(\sqrt{-\Delta}\xi\right)}\right)\right) \\ &-\tfrac{3\gamma^4}{2500D^2}\left(\tfrac{\sqrt{-\Delta}}{(1-F)}\left(\tfrac{ic_1\cosh\left(\sqrt{-\Delta}\xi\right)-c_2\sinh\left(\sqrt{-\Delta}\xi\right)}{-c_1\sinh\left(\sqrt{-\Delta}\xi\right)-c_2\cosh\left(\sqrt{-\Delta}\xi\right)}\right)\right)^2 \end{aligned} \tag{31}$$

Case 4: When $E = 0$ and $\Delta = A(1-F) \geq 0$,

$$u_8(\xi) = -\frac{3}{25}\gamma^2 - \frac{3\gamma^3}{125A}\left(\frac{\sqrt{\Delta}}{(1-F)}\left(\frac{c_1\cos\left(\sqrt{\Delta}\xi\right)+c_2\sin\left(\sqrt{\Delta}\xi\right)}{c_1\sin\left(\sqrt{\Delta}\xi\right)-c_2\cos\left(\sqrt{\Delta}\xi\right)}\right)\right) - \frac{3\gamma^4}{2500D^2}\left(\frac{\sqrt{\Delta}}{(1-F)}\left(\frac{c_1\cos\left(\sqrt{\Delta}\xi\right)+c_2\sin\left(\sqrt{\Delta}\xi\right)}{c_1\sin\left(\sqrt{\Delta}\xi\right)-c_2\cos\left(\sqrt{\Delta}\xi\right)}\right)\right)^2 \tag{32}$$

Note that, our solutions are different from the given ones in [14].

3. Conclusion

In this paper, the improved $(G'/G)-$expansion method has been successfully applied to get analytical solutions two nonlinear evolution equation. In the original $(G'/G)-$expansion method, the auxiliary equation $G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0$, has three different general solutions. But in the improved $(G'/G)-$expansion method, the auxiliary differential equations has $GG'' = DG^2 + EGG' + F(G')^2$ four different general solutions. By this way, the improved $(G'/G)-$expansion method can give more different solutions comparably the original $(G'/G)-$expansion method and it is suggested to get new and more general type analytical exact solutions. Accordingly, we use the improved $(G'/G)-$expansion method in this work. The adopted method also is a direct and powerful technique in obtaining the exact solutions of NPDEs. To our bestknowledge, the exact solutions obtained in this work will be significant to reveal the pertinent features of the physical phenomena.

4. References

[1] Ablowitz M J and Clarkson P A 1990, *Solitons, Nonlinear Evolution Equations and Inverse Scattering Transform* (Cambridge:Cambridge University Press)
[2] Wazwaz A M 2005 *Appl. Math. Comput.* **167** 1196
[3] Misirli E.and Gurefe Y 2011 *Mathematical and Computational Applications* **16** 258
[4] Tascan F and Bekir A 2009 *Appl. Math. Comput.* **207** 279
[5] Naher H, Abdullah F A and Akbar M A 2011 *Math. Prob. Eng.* **2011** 218216
[6] Kaplan M, Akbulut A and Bekir A 2015 *Z. Naturforsch. A* **70** 969
[7] Bekir A, Kaplan M and Guner O 2014 *AIP Conf. Proc.* **1611** 30
[8] Mirzazadeh M and Eslami M 2012 *Nonlinear Anal. Model. Control* **4** 481
[9] Zayed E M E 2010 *J Appl Math Informatics* **28** 383
[10] Pandir Y 2014 *Pramana-J Phys.* **82** 949
[11] Kaplan M, Bekir A and Ozer M N 2015 *Open Phys.* **13** 383
[12] Khan K,Akbar M A and Islam S M R 2014 *SpringerPlus* **3** 724
[13] Sahoo S and Saha Ray S 2016 *Physica A* **448** 265
[14] Yang H, Liu W, Yang B and He B 2015 *Commun Nonlinear Sci Numer Simulat* **27** 271