

Cubic Trigonometric B-spline Galerkin Methods for the Regularized Long Wave Equation

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Abstract. A numerical solution of the Regularized Long Wave (RLW) equation is obtained using Galerkin finite element method, based on Crank Nicolson method for the time integration and cubic trigonometric B-spline functions for the space integration. After two different linearization techniques are applied, the proposed algorithms are tested on the problems of propagation of a solitary wave and interaction of two solitary waves.

1. Introduction

The RLW equation

$$u_t + u_x + \varepsilon uu_x - \mu u_{xxt} = 0, \quad (1)$$

was formulated by Peregrine as an alternative to KdV equation for studying soliton phenomenon with boundary conditions $u \rightarrow 0$ as $x \rightarrow \pm\infty$ [1]. In the equation, ε and μ are positive constants and u , x , t denote the amplitude, spatial and time coordinate, respectively. In this study the RLW equation will be considered with boundary condition over the space region as

$$u(a, t) = u(b, t) = u_x(a, t) = u_x(b, t) = 0, \quad t \in (0, T] \quad (2)$$

and the initial condition

$$u(x, 0) = f(x) \quad (3)$$

will be defined in the numerical experiments section.

Since the RLW equation has been solved analytically only for restricted set of boundary and initial conditions [2], the numerical solution of the RLW equation has been the subject of many papers over the last few years [3]-[7]. In this paper, the purpose of this study is to obtain numerical solution of the RLW equation based on the application of Crank Nicolson methods for the time discretization and cubic trigonometric B-spline Galerkin method for the space discretization using different two linearization techniques.

2. Cubic Trigonometric B-spline Galerkin Method

For computational work, the space-time plane is discretized by grid with the time step Δt and space step h . The exact solution of the unknown function at the grid points is denoted by

$$u(x_m, t_n) = u_m^n, \quad m = 0, 1, \dots, N; \quad n = 0, 1, 2, \dots$$



where $x_m = a + mh$, $t_n = n\Delta t$ and the notation U_m^n is used to represent the numerical value of u_m^n . For the space discretization, the space domain $[a, b]$ is discretized into partitions of N finite elements of equal length h by the knots

Using the Crank-Nicolson method for the time discretization of the RLW equation, we have

$$u^{n+1} + \frac{\Delta t}{2} (u_x)^{n+1} + \frac{\Delta t}{2} \varepsilon u^{n+1} (u_x)^{n+1} - \mu (u_{xx})^{n+1} = u^n - \mu (u_{xx})^n - \frac{\Delta t}{2} (u_x)^n - \frac{\Delta t}{2} \varepsilon u^n (u_x)^n. \quad (4)$$

Using the recurrence relation given in [8, 9], the cubic trigonometric B-spline functions are defined at these knots as

$$T_m(x) = \frac{1}{\theta} \begin{cases} g^3(x_{m-2}) & , [x_{m-2}, x_{m-1}) \\ -g^2(x_{m-2})g(x_m) - g(x_{m-2})g(x_{m+1})g(x_{m-1}) - g(x_{m+2})g^2(x_{m-1}) & , [x_{m-1}, x_m) \\ g(x_{m-2})g^2(x_{m+1}) + g(x_{m+2})g(x_{m-1})g(x_{m+1}) + g^2(x_{m+2})g(x_m) & , [x_m, x_{m+1}) \\ -g^3(x_{m+2}) & , [x_{m+1}, x_{m+2}) \\ 0 & , \text{otherwise} \end{cases} \quad (5)$$

where

$$\theta = \sin\left(\frac{h}{2}\right) \sin(h) \sin\left(\frac{3h}{2}\right), \quad g(x_m) = \sin\left(\frac{x - x_m}{2}\right).$$

Since from (5) each cubic trigonometric B-splines covers 4 intervals, each element $[x_m, x_{m+1}]$ is covered by four splines. Therefore over the element $[x_m, x_{m+1}]$, an approximation to the exact solution $u(x, t)$ in terms of cubic trigonometric B-splines can be written as

$$U(x, t) = \sum_{j=m-1}^{m+2} T_j(x) \delta_j(t) = T_{m-1} \delta_{m-1} + T_m \delta_m + T_{m+1} \delta_{m+1} + T_{m+2} \delta_{m+2}. \quad (6)$$

Using (5) and the trial solution (6) the values of $U_m = U(x_m, t)$ and their first two space derivatives at knots can be written in terms of the parameters δ_m as

$$U_m = \sin^2\left(\frac{h}{2}\right) \csc(h) \csc\left(\frac{3h}{2}\right) (\delta_{m-1} + \delta_{m+1}) + \left(\frac{2}{1 + 2\cos(h)}\right) \delta_m, \quad (7)$$

$$U'_m = \left(\frac{3}{4} \csc\left(\frac{3h}{2}\right)\right) (-\delta_{m-1} + \delta_{m+1}), \quad (8)$$

$$U''_m = \left(\frac{3(3\cos^2(\frac{h}{2}) - 1)}{4(\sin(h)\sin(\frac{3h}{2}))}\right) (\delta_{m-1} + \delta_{m+1}) - \left(\frac{3\cot^2(\frac{h}{2})}{(2 + 4\cos(h))}\right) \delta_m. \quad (9)$$

2.1. Linearization Technique 1 (CN1)

Applying Galerkin method to Eq. (4) with weight function $W(x)$ and identifying the weight function $W(x)$ with the cubic trigonometric B-spline T_m and using the expression (6), a fully discrete approximation for the RLW equation is obtained over the element $[x_m, x_{m+1}]$ as

$$\begin{aligned} & \sum_{j=m-1}^{m+2} \left\{ \left(\int_{x_m}^{x_{m+1}} T_i T_j dx \right) \delta_j^{n+1} - \mu \left(\int_{x_m}^{x_{m+1}} T_i T_j'' dx \right) \delta_j^{n+1} + \frac{\Delta t}{2} \left(\int_{x_m}^{x_{m+1}} T_i T_j' dx \right) \delta_j^{n+1} \right. \\ & + \frac{\Delta t}{2} \varepsilon \sum_{k=m-1}^{m+2} \left(\int_{x_m}^{x_{m+1}} T_i T_k (\delta_k^{n+1}) T_j' dx \right) \delta_j^{n+1} \Big\} - \sum_{j=m-1}^{m+2} \left\{ \left(\int_{x_m}^{x_{m+1}} T_i T_j dx \right) \delta_j^n \right. \\ & \left. - \mu \left(\int_{x_m}^{x_{m+1}} T_i T_j'' dx \right) \delta_j^n - \frac{\Delta t}{2} \left(\int_{x_m}^{x_{m+1}} T_i T_j' dx \right) \delta_j^n - \frac{\Delta t}{2} \varepsilon \sum_{k=m-1}^{m+2} \left(\int_{x_m}^{x_{m+1}} T_i T_k (\delta_k^n) T_j' dx \right) \delta_j^n \right\} \end{aligned} \quad (10)$$

where i, j and k take only the values $m-1, m, m+1, m+2$ for this typical element $[x_m, x_{m+1}]$. (10) can be written in the matrices form as

$$\left[\mathbf{A}^e - \mu \mathbf{D}^e + \frac{\Delta t}{2} \mathbf{B}^e + \frac{\Delta t}{2} \varepsilon \mathbf{C}^e \left((\delta^e)^{n+1} \right) \right] (\delta^e)^{n+1} - \left[\mathbf{A}^e - \mu \mathbf{D}^e - \frac{\Delta t}{2} \mathbf{B}^e - \frac{\Delta t}{2} \varepsilon \mathbf{C}^e \left((\delta^e)^n \right) \right] (\delta^e)^n \quad (11)$$

where the element matrices and element parameters are

$$\begin{aligned} A_{ij}^e &= \int_{x_m}^{x_{m+1}} T_i T_j dx, \quad B_{ij}^e = \int_{x_m}^{x_{m+1}} T_i T_j' dx, \quad C_{ij}^e \left((\delta^e)^{n+1} \right) = \int_{x_m}^{x_{m+1}} T_i T_k (\delta_k^{n+1}) T_j' dx, \\ D_{ij}^e &= \int_{x_m}^{x_{m+1}} T_i T_j'' dx, \quad (\delta^e)^{n+1} = (\delta_{m-1}^{n+1}, \delta_m^{n+1}, \delta_{m+1}^{n+1}, \delta_{m+2}^{n+1})^T. \end{aligned}$$

The 4×4 element matrices \mathbf{A}^e , \mathbf{B}^e and \mathbf{D}^e are independent of the parameters δ^e and $4 \times 4 \times 4$ element matrices \mathbf{C}^e depends on the parameters δ^e .

Combining contributions from all elements lead to the nonlinear matrix equation

$$\left[\mathbf{A} - \mu \mathbf{D} + \frac{\Delta t}{2} \mathbf{B} + \frac{\Delta t}{2} \varepsilon \mathbf{C} (\delta^{n+1}) \right] \delta^{n+1} = \left[\mathbf{A} - \mu \mathbf{D} - \frac{\Delta t}{2} \mathbf{B} - \frac{\Delta t}{2} \varepsilon \mathbf{C} (\delta^n) \right] \delta^n. \quad (12)$$

After the first and last equations are deleted in the system (12), imposition of the boundary conditions $U(a, x) = U(b, x) = 0$ at the both ends of the region yields to eliminate δ_{-1}^{n+1} and δ_{N+1}^{n+1} from the above system. Therefore the solution of the seven band matrix equations with the dimensions $(N+1) \times (N+1)$ is obtained by way of Thomas algorithms. After initial vector $\mathbf{d}^0 = (\delta_{-1}^0, \dots, \delta_{N-1}^0, \delta_N^0)$ are found with the help of the boundary and initial conditions, \mathbf{d}^{n+1} , $(n = 0, 1, \dots)$ unknown vectors can be found repeatedly by solving the recurrence relation (12) using previous \mathbf{d}^{n-1} unknown vector. Note that since the system (12) is an implicit system, we have used an inner iteration algorithm for all time steps to increase the accuracy of the system.

- Step 1: Set $error = 1$ and $\delta_m^* = \delta_m^{n+1}$ in $\mathbf{C} (\delta^{n+1})$ and taking $\delta_m^* = \delta_m^n$ then find U_m^* ,
- Step 2: While $error \geq 10^{-10}$ do Steps 3-4,
- Step 3: Find U_m^{n+1} ,
- Step 4: Find $\max_m |U_m^{n+1} - U_m^*|$ and set $\delta_m^* = \delta_m^{n+1}$,
- Step 5: Stop and go to next time step.

2.2. Linearization Technique 2(CN2)

The nonlinear term in (4) is approximated by the following formula based on Taylor series:

$$(UU_x)^{n+1} \approx U^{n+1}U_x^n + U^nU_x^{n+1} - U^nU_x^n.$$

Using this approximation and applying Galerkin method by using the expression (6) in (4), a fully discrete approximation is obtained over the element $[x_m, x_{m+1}]$ as

$$\begin{aligned} & \sum_{j=m-1}^{m+2} \left\{ \left(\int_{x_m}^{x_{m+1}} T_i T_j dx \right) \delta_j^{n+1} - \mu \left(\int_{x_m}^{x_{m+1}} T_i T_j'' dx \right) \delta_j^{n+1} + \frac{\Delta t}{2} \left(\int_{x_m}^{x_{m+1}} T_i T_j' dx \right) \delta_j^{n+1} + \right. \\ & + \frac{\Delta t}{2} \varepsilon \sum_{k=m-1}^{m+2} \left[\left(\int_{x_m}^{x_{m+1}} T_i T_k' (\delta_k^n) T_j dx \right) \delta_j^{n+1} + \left(\int_{x_m}^{x_{m+1}} T_i T_k (\delta_k^n) T_j' dx \right) \delta_j^{n+1} \right] \Big\} - \\ & \sum_{j=m-1}^{m+2} \left\{ \left(\int_{x_m}^{x_{m+1}} T_i T_j dx \right) \delta_j^n - \mu \left(\int_{x_m}^{x_{m+1}} T_i T_j'' dx \right) \delta_j^n - \frac{\Delta t}{2} \left(\int_{x_m}^{x_{m+1}} T_i T_j' dx \right) \delta_j^n \right\} \end{aligned} \quad (13)$$

where i, j and k take only the values $m-1, m, m+1, m+2$ for this typical element $[x_m, x_{m+1}]$. (13) can be written in the matrices form as

$$\left[\mathbf{A}^e - \mu \mathbf{D}^e + \frac{\Delta t}{2} \mathbf{B}^e + \frac{\Delta t}{2} \varepsilon (\mathbf{E}^e ((\delta^e)^n) + \mathbf{C}^e ((\delta^e)^n)) \right] (\delta^e)^{n+1} - \left[\mathbf{A}^e - \mu \mathbf{D}^e - \frac{\Delta t}{2} \mathbf{B}^e \right] (\delta^e)^n \quad (14)$$

where the element matrices and element parameters are

$$\begin{aligned} A_{ij}^e &= \int_{x_m}^{x_{m+1}} T_i T_j dx, \quad B_{ij}^e = \int_{x_m}^{x_{m+1}} T_i T_j' dx, \quad C_{ij}^e(\delta^n) = \int_{x_m}^{x_{m+1}} T_i T_k(\delta_k^n) T_j' dx, \\ D_{ij}^e &= \int_{x_m}^{x_{m+1}} T_i T_j'' dx, \quad E_{ij}^e(\delta^n) = \int_{x_m}^{x_{m+1}} T_i T_k'(\delta_k^n) T_j dx, \quad (\delta^e)^n = (\delta_{m-1}^n, \delta_m^n, \delta_{m+1}^n, \delta_{m+2}^n)^T. \end{aligned}$$

Assembling together contributions from all elements leads to the matrix equation

$$\left[\mathbf{A} - \mu \mathbf{D} + \frac{\Delta t}{2} \mathbf{B} + \frac{\Delta t}{2} \varepsilon (\mathbf{E}(\delta^n) + \mathbf{C}(\delta^n)) \right] \delta^{n+1} = \left[\mathbf{A} - \mu \mathbf{D} - \frac{\Delta t}{2} \mathbf{B} \right] \delta^n. \quad (15)$$

Eliminating δ_{-1}^{n+1} and δ_{N+1}^{n+1} from (15) just like in previous subsection, the solution of the seven band final matrix equations with the dimensions $(N+1) \times (N+1)$ is obtained by way of Thomas algorithms.

3. Numerical Experiments

Since an accurate numerical scheme must keep the conservation properties of the RLW equation, we will monitor the three invariants of numerical solution for the equation corresponding to conservation of mass, momentum and energy given by the following integrals [10]:

$$I_1 = \int_{-\infty}^{\infty} u dx \approx \int_a^b U dx, \quad I_2 = \int_{-\infty}^{\infty} (u^2 + \mu(u_x)^2) dx, \quad I_3 = \int_{-\infty}^{\infty} (u^3 + 3u^2) dx. \quad (16)$$

Integrals for the conservation invariants are computed approximately with the trapezoidal rule for the space interval at all time steps. For the first test problem, accuracy of the proposed algorithms is worked out by measuring error norm L_∞

$$L_\infty = \max_m |u_m - U_m|, \quad (17)$$

and the order of convergence is computed with fixed space step by the formula

$$\text{order} = \frac{\log \left| \frac{u - U_{\Delta t_m}}{u - U_{\Delta t_{m+1}}} \right|}{\log \left| \frac{\Delta t_m}{\Delta t_{m+1}} \right|}, \quad (18)$$

where u is the exact solution and $U_{\Delta t_m}$ is the numerical solution with time step Δt_m .

3.1. Motion of Single Solitary Wave for the RLW Equation

The solitary wave theoretical solution of the RLW equation is

$$u(x, t) = 3c \operatorname{sech}^2(k[x - \tilde{x}_0 - vt]), \quad (19)$$

where $v = 1 + \varepsilon c$ is the wave velocity, $3c$ is amplitude of the solitary wave, \tilde{x}_0 is peak position of the initially centered wave and $k = \sqrt{\frac{\varepsilon c}{4\mu v}}$. This solution corresponds to a solitary wave of magnitude $3c$, initially centered on the position \tilde{x}_0 propagating towards the right across the interval $[a, b]$ over the up to the time T without change of shape at a steady velocity v .

After the program is run up to time $t = 20$ with fixed space and various time step, error norm L_∞ , invariants, Cpu time and Order of convergence for the proposed algorithms are presented in Table 1.

Table 1: Error norms L_∞ and invariants for a single solitary wave at time $t = 20$ with $c = 0.1$, $-80 \leq x \leq 100$, $h = 0.01$ and various time step.

CN1	L_∞	I_1	I_2	I_3	Cpu	Order
$\Delta t = 0.1$	8.79×10^{-5}	3.9799497	0.8104625	2.5790074	249.63	2.00
$\Delta t = 0.05$	2.20×10^{-5}	3.9799497	0.8104624	2.5790074	359.13	2.00
$\Delta t = 0.02$	3.52×10^{-6}	3.9799497	0.8104624	2.5790074	798.56	2.00
$\Delta t = 0.01$	8.80×10^{-7}	3.9799497	0.8104624	2.5790074	1603.80	
CN2	L_∞	I_1	I_2	I_3	Cpu	Order
$\Delta t = 0.1$	7.80×10^{-5}	3.9799497	0.8104625	2.5790074	31.40	2.00
$\Delta t = 0.05$	1.95×10^{-5}	3.9799497	0.8104624	2.5790074	62.55	2.00
$\Delta t = 0.02$	3.12×10^{-6}	3.9799497	0.8104624	2.5790074	156.68	2.00
$\Delta t = 0.01$	7.81×10^{-7}	3.9799497	0.8104624	2.5790074	313.43	

Absolute error (analytical–numerical) distributions for all of the proposed methods are drawn at time $t = 20$ in Figure 1.

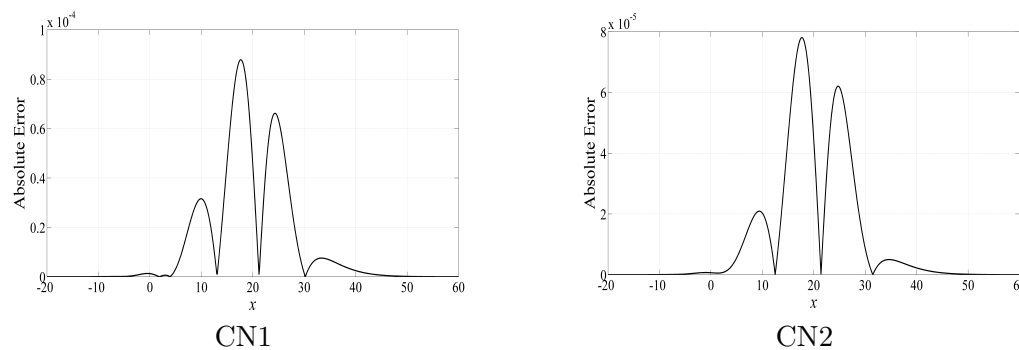


Figure 1: Absolute errors with $h = \Delta t = 0.1$

3.2. Interaction of Two Solitary Waves for RLW Equation

We consider interaction of two solitary waves using the following initial condition

$$u(x, 0) = 3c_1 \operatorname{sech}^2(k_1[x - \tilde{x}_1]) + 3c_2 \operatorname{sech}^2(k_2[x - \tilde{x}_2]), \quad (20)$$

where $k_i = \sqrt{\frac{\varepsilon c_i}{4\mu(1 + \varepsilon c_i)}}$, $i = 1, 2$. To ensure an interaction of two solitary waves, all of the computations are done for the parameters $\varepsilon = \mu = 1$, $c_1 = 0.2$, $c_2 = 0.1$, $\tilde{x}_1 = -177$ and $\tilde{x}_2 = -147$ over the region $-200 \leq x \leq 600$. These parameters provide two well separated

solitary waves of magnitudes 0.6 and 0.3 and peak positions of them are located at $x = -177$ and -147 .

The program is run until $t = 400$ with $h = 0.12$, $\Delta t = 0.1$ and numerical solutions of $U(x, t)$ at several times are drawn for visual views of the solutions. According to the figure, the nonlinear interaction takes place about time 200. Then, two solitary waves regain their original shapes after the interaction.

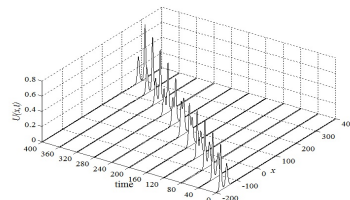


Figure 2: Interaction of two solitary waves.

Table 2 displays a comparison of the values of the invariants obtained by the proposed methods CN1, CN2 at some selected time t .

Table 2: Invariants for interaction of two solitary waves.

	CN1			CN2		
$t = 0$	9.8582	3.2446	10.7783	9.8582	3.2446	10.7783
$t = 80$	9.8582	3.2446	10.7783	9.8582	3.2446	10.7784
$t = 160$	9.8581	3.2448	10.7785	9.8581	3.2448	10.7787
$t = 240$	9.8580	3.2448	10.7785	9.8580	3.2449	10.7787
$t = 320$	9.8579	3.2446	10.7781	9.8580	3.2446	10.7784
$t = 400$	9.8577	3.2445	10.7779	9.8579	3.2446	10.7783

4. Conclusion

In this study, two numerical algorithms for the numerical solution of the RLW equation have been presented using Galerkin method based on cubic trigonometric B-splines as weight and trial functions for space discretization and Crank Nicolson method for time discretization. To compare all the proposed methods, CN2 gives accurate, reliable results and less Cpu time for the RLW equation. Also all of the methods have an advantage due to their small matrix operations. Therefore, the obtained results show that proposed algorithms, exhibit high accuracy and efficiency in both conservation of the invariants and error norm for the numerical solution of the RLW equation.

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6. References

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