# A polynomial type Jost solution and spectrum of the self-adjoint quantum Dirac system

**N Coskun and N Yokus**

Department of Mathematics, Karamanoglu Mehmetbey University, Karaman, 70100, Turkey

E-mail: cannimet@kmu.edu.tr

**Abstract**. This paper is devoted to investigation of a polynomial type Jost solution for the self-adjoint quantum Dirac system. After examination of analytical features and asymptotic behaviour of the Jost solution, continuous and discrete spectrum and some properties of the eigenvalues of the operator $L$ generated by the quantum difference system have been discussed.

## 1. Introduction

Spectral analysis of differential and difference operators is of great importance for the solutions of certain problems in many areas including engineering, economics, quantum mechanics and mathematical physics [1-5]. In this context, Dirac system of differential and discrete operators have been studied in [6-8].

In the last years, an important effort has been devoted to quantum calculus [9]. As a consequence of developments in quantum theory, quantum difference equations has been subject matter of various studies [10-12]. In particular, spectral analysis of quantum difference equations has been studied in [13-15]. However, the Dirac system of quantum difference equations including a polynomial type Jost solution has not been examined in the known literature yet.

In this paper, we assume $q>1$ and use the notation

$$q^{\mathbb{N}_0} := \left\{ q^n : n \in \mathbb{N}_0 \right\}$$

where $\mathbb{N}_0$ indicates the set of nonnegative integers. The q-derivative of a function $f : q^{\mathbb{N}} \to \mathbb{C}$ is defined by

$$f^{\Delta}(t) := \frac{f(qt)-f(t)}{\mu(t)}, \ \forall t \in q^{\mathbb{N}},$$

where $\mu(t)=(q-1)t$ is the graininess function [10]. Hereafter, we will denote the Hilbert space $l_2(q^{\mathbb{N}},\mathbb{C}^2)$ including all sequences $y=\{y(t)\}=\begin{Bmatrix} y^{(1)}(t) \\ y^{(2)}(t) \end{Bmatrix}$ with the inner product,

$$\langle y, f \rangle_{(q^{\mathbb{N}},\mathbb{C}^2)} := \sum_{t \in q^{\mathbb{N}}} y^{(1)}(t)\overline{f^{(1)}(t)} + y^{(2)}(t)\overline{f^{(2)}(t)}, \quad y, f : q^{\mathbb{N}} \to \mathbb{C}$$

and the norm

$$\|y(t)\|_{(q^{\mathbb{N}},\mathbb{C}^2)}=\left(\sum_{t\in q^{\mathbb{N}}}\left|y^{(1)}(t)\right|^2+\left|y^{(2)}(t)\right|^2\right)^{\frac{1}{2}} \text{ for } \quad y:q^{\mathbb{N}}\to\mathbb{C}$$

by $l_2(q^{\mathbb{N}},\mathbb{C}^2)$.

Let us consider the system

$$\begin{cases}(y^{(2)}(t))^{\Delta}+p(t)y^{(1)}(t)=\dfrac{\lambda y^{(1)}(t)}{\mu(t)},\\ -(y^{(1)}(\dfrac{t}{q}))^{\Delta}+r(t)y^{(2)}(t)=\dfrac{\lambda y^{(2)}(t)}{\mu(t)},\end{cases}\quad t\in q^{\mathbb{N}} \tag{1.1}$$

where $\{p(t)\}_{t\in q^{\mathbb{N}}}$, $\{r(t)\}_{t\in q^{\mathbb{N}}}$ and $\{\mu(t)\}_{t\in q^{\mathbb{N}}}$ are real sequences for all $t\in q^{\mathbb{N}}$ and $\lambda$ is a spectral parameter. Note that, the system of equations (1.1) is quantum analogue of the well-known Dirac system

$$\begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}y_1'\\y_2'\end{pmatrix}+\begin{pmatrix}p(x)&0\\0&q(x)\end{pmatrix}\begin{pmatrix}y_1\\y_2\end{pmatrix}=\lambda\begin{pmatrix}y_1\\y_2\end{pmatrix}$$

([6], Chapter 2). For this reason, the system (1.1) is called a quantum Dirac system. It is worth to point out here that the construction of the quantum Dirac system (1.1) is completely different from other studies [7, 8] because of the quantum derivative of a function.

Let us define the operator $L$ generated in $l_2(q^{\mathbb{N}},\mathbb{C}^2)$ by the following system of quantum difference expression:

$$(ly)(t)=\begin{pmatrix}y^{(2)}(qt)-y^{(2)}(t)+p(t)\mu(t)y^{(1)}(t)\\ y^{(1)}(\dfrac{t}{q})-y^{(1)}(t)+r(t)\mu(t)y^{(2)}(t)\end{pmatrix}$$

where $\{p(t)\}_{t\in q^{\mathbb{N}}}$, $\{r(t)\}_{t\in q^{\mathbb{N}}}$ and $\{\mu(t)\}_{t\in q^{\mathbb{N}}}$ are real sequences. It is clear that the system of equations (1.1) can be rewritten as

$$(Ly)(t)=\lambda y(t),\quad t\in q^{\mathbb{N}}.$$

The set up of this paper organized as follows: Section 2 is concerned with the investigation of the polynomial type Jost solution of the system (1.1) with the boundary condition

$$y^{(1)}(1)=0 \tag{1.2}$$

and investigate analytic properties and asymptotic behavior of the Jost solution. Section 3 presents continuous and discrete spectrum and some properties of the eigenvalues of the boundary value problem BVP (1.1)-(1.2).

## 2. Jost solution and Jost function of (1.1)

Let the real sequences $\{p(t)\}_{t\in q^{\mathbb{N}}}$, $\{r(t)\}_{t\in q^{\mathbb{N}}}$ and $\{\mu(t)\}_{t\in q^{\mathbb{N}}}$ satisfy the condition

$$\sum_{t\in q^{\mathbb{N}}}\frac{\ln t}{\ln q}\left(\left|p(t)\mu(t)\right|+\left|r(t)\mu(t)\right|\right)<\infty. \tag{2.1}$$

**Theorem 2.1.** Under the condition (2.1), the system of equations (1.1) has unique solutions

$$f(t,z)=\begin{pmatrix} f^{(1)}(t,z)\\ f^{(2)}(t,z)\end{pmatrix}=\left[\begin{pmatrix}1&0\\0&1\end{pmatrix}+\sum_{r\in q^{\mathbb{N}_0}}A_{tr}z^{2\log_q r}\right]\begin{pmatrix}z\\-i\end{pmatrix}z^{2\log_q t}, \tag{2.2}$$

$$f^{(1)}(1,z)=z+\sum_{r\in q^{\mathbb{N}_0}}A_{1r}{}^{11}z^{2\log_q r+1}-i\sum_{r\in q^{\mathbb{N}_0}}A_{1r}{}^{12}z^{2\log_q r}, \tag{2.3}$$

for $\lambda=-iz-(iz)^{-1}$, $t\in q^{\mathbb{N}}$ and $|z|=1$ where $A_{tr}=\begin{pmatrix}A_{tr}^{11}&A_{tr}^{12}\\A_{tr}^{21}&A_{tr}^{22}\end{pmatrix}$.

**Proof.** If we substitute the equation (2.2) in the system of equations (1.1) and take $\lambda=-iz-(iz)^{-1}$ and $|z|=1$, then we obtain

$$A_{tq}{}^{12}=-\sum_{s\in[qt,\infty)\cap q^{\mathbb{N}}}p(s)\mu(s)+r(s)\mu(s),$$

$$A_{tq}{}^{11}=\sum_{s\in[qt,\infty)\cap q^{\mathbb{N}}}p(s)\mu(s)A_{sq}{}^{12},$$

$$A_{tq}{}^{22}=\sum_{s\in[t,\infty)\cap q^{\mathbb{N}}}p(s)\mu(s)A_{sq}{}^{12},$$

$$A_{tq}{}^{21}=A_{tq}{}^{12}+p(t)\mu(t)A_{tq}{}^{11}+\sum_{s\in[qt,\infty)\cap q^{\mathbb{N}}}\left[r(s)\mu(s)A_{sq}{}^{22}+p(s)\mu(s)A_{sq}{}^{11}\right],$$

$$A_{tq^2}{}^{12}=-\sum_{s\in[qt,\infty)\cap q^{\mathbb{N}}}\left[p(s)\mu(s)A_{sq}{}^{11}+r(s)\mu(s)A_{sq}{}^{22}\right],$$

$$A_{tq^2}{}^{11}=-A_{qt,q}{}^{22}+\sum_{s\in[qt,\infty)\cap q^{\mathbb{N}}}\left[p(s)\mu(s)A_{s,q^2}{}^{12}-r(s)\mu(s)A_{sq}{}^{21}\right],$$

$$A_{tq^2}{}^{22}=-A_{tq}{}^{11}+\sum_{s\in[t,\infty)\cap q^{\mathbb{N}}}\left[p(s)\mu(s)A_{s,q^2}{}^{12}-r(sq)\mu(sq)A_{sq,q}{}^{21}\right],$$

$$A_{tq^2}{}^{21}=A_{t,q^2}{}^{12}+\sum_{s\in[t,\infty)\cap q^{\mathbb{N}}}\left[p(s)\mu(s)A_{s,q^2}{}^{11}+r(sq)\mu(sq)A_{sq,q^2}{}^{22}\right],$$

for $r\geq q^3$

$$A_{tr}{}^{12}=A_{tq,\frac{r}{q^2}}{}^{21}-\sum_{s\in[tq,\infty)\cap q^{\mathbb{N}}}r(s)\mu(s)A_{s,\frac{r}{q}}{}^{22}+p(s)\mu(s)A_{s,\frac{r}{q}}{}^{11},$$

$$A_{tr}{}^{11}=-A_{tq,\frac{t}{q^2}}{}^{22}+\sum_{s\in[tq,\infty)\cap q^{\mathbb{N}}}p(s)\mu(s)A_{sr}{}^{12}-r(s)\mu(s)A_{s,\frac{r}{q}}{}^{21},$$

$$A_{tr}{}^{22}=-A_{t,\frac{t}{q}}{}^{11}+\sum_{s\in[t,\infty)\cap q^{\mathbb{N}}}\left(p(s)\mu(s)A_{sr}{}^{12}-r(sq)\mu(sq)A_{sq,\frac{r}{q}}{}^{21}\right),$$

$$A_{tr}{}^{21}=A_{tr}{}^{12}+p(t)\mu(t)A_{tr}{}^{11}+\sum_{s\in[tq,\infty)\cap q^{\mathbb{N}}}\left(r(s)\mu(s)A_{sr}{}^{22}+p(s)\mu(s)A_{sr}{}^{11}\right).$$

Based on the condition (2.1), the series in the definition of $A_{tr}^{ij}$ $(i,j=1,2)$ are absolutely convergent. Hence, $A_{tr}^{ij}$ $(i,j=1,2)$ can be uniquely determined by $p(t)$, $r(t)$ and $\mu(t)$ $(t\in q^{\mathbb{N}})$, i.e., the system (1.1) has the solution given by the equations (2.2) and (2.3). □

The solution $f$ is called Jost solution of the system of equations (1.1). Using the inequalities for $A_{tr}^{ij}$ $(i,j=1,2)$ given in Theorem (2.1), we find

$$\left|A_{tr}^{\ ij}\right| \le C \sum_{s\in[tq^{\left\lceil \frac{\ln r}{2\ln q}\right\rceil},\infty)\cap q^{\mathbb{N}}}^{\infty} \left(\left|p(s)\mu(s)\right|+\left|r(s)\mu(s)\right|\right), \quad i,j=1,2, \tag{2.4}$$

by induction, where $\left[\frac{\ln r}{2\ln q}\right]$ is the integer part of $\frac{\ln r}{2\ln q}$, $C>0$ is a constant and $t,r\in q^{\mathbb{N}}$.

It is clear from the equations (2.2), (2.3) and (2.4) that $\begin{pmatrix} f^{(1)}(t,z) \\ f^{(2)}(t,z) \end{pmatrix}$ has analytic continuation from $\{z:|z|=1\}$ to $D:=\{z:|z|<1\}\setminus\{0\}$.

**Theorem 2.2.** If the condition (2.1) holds then the Jost solution $f$ satisfies the following asymptotic:

$$\begin{pmatrix} f^{(1)}(t,z) \\ f^{(2)}(t,z) \end{pmatrix} = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + o(1)\right]\begin{pmatrix} z \\ -i \end{pmatrix} z^{2\log_q t},\ z\in\overline{D}:=\{z:|z|\le 1\}\setminus\{0\},\ t\to\infty. \tag{2.5}$$

**Proof.** It can be written from the equation (2.2) that

$$f^{(1)}(t,z)z^{-2\log_q t-1} = 1+\sum_{r\in q^{\mathbb{N}_0}} A_{tr}^{\ 11} z^{2\log_q r} - i\sum_{r\in q^{\mathbb{N}_0}} A_{tr}^{\ 12} z^{2\log_q r-1}. \tag{2.6}$$

From the equations (2.4) and (2.6), it is seen that

$$\begin{aligned}
\left|f^{(1)}(t,z)z^{-2\log_q t-1}\right| &\le 1+\sum_{r\in q^{\mathbb{N}_0}}\left|A_{tr}^{\ 11}\right|+\sum_{r\in q^{\mathbb{N}_0}}\left|A_{tr}^{\ 12}\right| \\
&\le 1+2C\sum_{r\in q^{\mathbb{N}_0}}\sum_{s\in\left[tq^{\left\lceil\frac{\ln r}{2\ln q}\right\rceil},\infty\right)\cap q^{\mathbb{N}}}\left(\left|p(s)\mu(s)\right|+\left|r(s)\mu(s)\right|\right) \\
&\le 1+2C\sum_{r\in q^{\mathbb{N}_0}}\sum_{s\in[tq,\infty)\cap q^{\mathbb{N}}}\left(\frac{\ln s}{\ln q}-\frac{\ln t}{\ln q}\right)\left(\left|p(s)\mu(s)\right|+\left|r(s)\mu(s)\right|\right) \\
&\le 1+2C\sum_{s\in[tq,\infty)\cap q^{\mathbb{N}}}\frac{\ln s}{\ln q}\left(\left|p(s)\mu(s)\right|+\left|r(s)\mu(s)\right|\right)
\end{aligned} \tag{2.7}$$

where $C>0$ is a constant. Then, we get from the equation (2.7) that

$$f^{(1)}(t,z) = z^{2\log_q t+1}\left[1+o(1)\right],\ z\in\overline{D},\ t\to\infty. \tag{2.8}$$

In a similar way to equation (2.8), we have

$$f^{(2)}(t,z) = -iz^{2\log_q t}\left[1+o(1)\right],\ z\in\overline{D},\ t\to\infty. \tag{2.9}$$

From equations (2.8) and (2.9), we obtain the equation (2.5).

□

## 3. Main Results

Now, we shall give some theorems and definitions to prove our main results. We begin with investigating continuous spectrum of the operator $L$.

**Theorem 3.1.** Assume the condition (2.1) satisfies. Then $\sigma_c(L)=[-2,2]$, where $\sigma_c(L)$ represents the continuous spectrum of the operator $L$.

**Proof.** Let $L_0$ be the operator generated in $l_2(q^{\mathbb{N}},\mathbb{C}^2)$ by the following system of quantum difference expression:

$$\left(l_0 y\right)(t)=\begin{pmatrix} y^{(2)}(qt)-y^{(2)}(t) \\ y^{(1)}(\frac{t}{q})-y^{(1)}(t) \end{pmatrix}$$

with the boundary condition $y^{(1)}(1)=0$. We also define the operator $L_1$ in $l_2(q^{\mathbb{N}},\mathbb{C}^2)$ by the following

$$\left(l_1 y\right)(t)=\begin{pmatrix} p(t)\mu(t)y^{(1)}(t) \\ r(t)\mu(t)y^{(2)}(t) \end{pmatrix}.$$

One can easily observe that the operator $L_0$ is self-adjoint and $L_1$ is compact operator [16]. Also, $L=L_0+L_1$ holds. From the Weyl Theorem [17] of a compact perturbation, we find

$$\sigma_c(L)=\sigma_c(L_0)=\left[-2,2\right]. \qquad \square$$

**Definition 3.1.** The Wronskian of two solutions $y=\left\{y(t,z)\right\}_{t\in q^{\mathbb{N}}}$ and $u=\left\{u(t,z)\right\}_{t\in q^{\mathbb{N}}}$ of (1.1) is defined by $W[y,u](t)=y(t,z)u(qt,z)-y(qt,z)u(t,z)$ for $t\in q^{\mathbb{N}}$.

Let $\varphi(z)=\varphi(\lambda)=\left\{\varphi(t,\lambda)\right\}$, $t\in q^{\mathbb{N}}$ be the solution of the system of equations (1.1) subject to the initial conditions

$$\varphi^{(1)}(1,z)=0, \quad \varphi^{(2)}(q,z)=1.$$

Then from definition (3.1)

$$\begin{aligned} W\left[f(z),\varphi(z)\right] &= f^{(1)}(t,z)\varphi^{(2)}(qt,z)-f^{(2)}(qt,z)\varphi^{(1)}(t,z) \\ &= f^{(1)}(1,z). \end{aligned} \tag{3.1}$$

Since the operator $L$ is self-adjoint, the eigenvalues of $L$ is real valued. By using equation (3.1) and the definition of eigenvalues, we get

$$\sigma_d(L)=\left\{\lambda:\lambda=-iz-(iz)^{-1}, iz\in(-1,0)\cup(0,1), f^{(1)}(1,z)=0\right\}, \tag{3.2}$$

where $\sigma_d(L)$ sembolizes the eigenvalues of the operator $L$.

**Definition 3.2.** The multiplicity of a zero of $f^{(1)}(1,z)$ is called the multiplicity of the corresponding eigenvalue or spectral singularity of BVP (1.1), (1.2).

**Theorem 3.2.** Under the condition (2.1), the operator $L$ has a finite number of real eigenvalues in $D$.

**Proof.** In order to prove the theorem, it is necessary and sufficient to show that the function $f^{(1)}(1,z)$ has finite number of real zeros in $D$. The accumulation points of the zeros of the analytic function $f^{(1)}(1,z)$ can take real values $-i$, 0 and $i$. It is known that $L$ is self-adjoint bounded operator. So its eigenvalues is different from infinity. Assume $z_0=0$ is a zero of the function $f^{(1)}(1,z)$. But in this case the eigenvalue $\lambda$ is infinite. Therefore $z_0=0$ is not a zero of the function $f^{(1)}(1,z)$. Now, assume $z=\pm i$. In this case, $\lambda=\pm 2$ and $D$ is bounded. From Theorem (3.1), $\pm 2$ are elements of continuous spectrum of the operator $L$. From operator theory, discrete spectrum of a self-adjoint operator and continuous spectrum of this operator are distinct. Hence, the set of zeros of the function $f^{(1)}(1,z)$ in $D$ is finite from the Bolzano Weierstrass Theorem. It is obvious from (3.2) that the eigenvalues of the operator $L$ are real. This completes the proof. $\square$

## References

[1] Naimark M A 1960 Investigation of the spectrum and expansion of eigenfunctions of a non-selfadjoint operator of second order on a semi-axis *AMS Transl.* **2** pp 103-193
[2] Lyance V E 1967 A differential opeator with spectral singularities I, II *AMS Transl.* **60** pp 185-225
[3] Agarwal R P and Wong P J Y 1997 *Advanced Topics in Difference Euations* (Dordrecht: Kluwer Academic Publishers Group)
[4] Bairamov E, Cakar O and Krall A M 2001 Nonselfadjoint difference operators and Jacobi matrices with spectral singularities *Math. Nachr.* **229** pp 5-14
[5] Kelley W G and Peterson A C 2001 *Difference Equations: An Introduction with Applications* (New York: Academic Press)
[6] Levitan B M and Sargsjan I S 1991 *Sturm-Liouville and Dirac Operators* (Dordrecht: Kluwer Academic Publishers Group)
[7] Bairamov E and Celebi A O 1999 Spectrum and spectral expansion for the non-selfadjoint discrete Dirac operators *Q. J. Math.* **50** pp 371-384
[8] Aygar Y and Olgun M 2014 Investigation of spectrum and the Jost solutions of discrete Dirac system on the whole axis *Journal of Inequalities and Applications* **73**
[9] Kac V and Cheung P 2002 *Quantum Calculus* (New York: Springer)
[10] Bohner M and Peterson A 2003 *Dynamic equations on time scales* (Boston: Birkhauser)
[11] Trijtzinsky W J 1933 Analytic theory of linear q-difference equations *Acta Math.* **61** pp 1-38
[12] Berg C and Ruffing A 2001 Generalized q-Hermite polynomials *Comm. Math. Phys.* **223** pp 29-46
[13] Adıvar M and Bohner M 2006 Spectral analysis of q-difference equations with spectral singularities *Math. Comput. Modelling* **43** pp 695-703
[14] Adıvar M and Bohner M 2006 Spectrum and principal vectors of second order q-difference equations *Indian J. Math.* **48** pp 17-33
[15] Adıvar M 2010 Quadratic Pencil of Difference Equations: Jost Solutions, Spectrum and Principal vectors *Quaestiones Mathematicae* **33** pp 305-323
[16] Lusternik L A and Sobolev V I 1974 *Elements of Functional Analysis* (New York: Halsted)
[17] Glazman I M 1965 *Direct Methods of Qualitative Spectral Analysis of Singular Differential Operators* (Jerusalem: Israel Program for Scientific Translations)