

From q -analytic functions to double q -analytic Hermite binomials and q -traveling waves

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Abstract. We extend the concept of q -analytic function in two different directions. First we find expansion of q -binomial in terms of q -Hermite polynomials, analytic in two complex arguments. Based on this representation, we introduce a new class of complex functions of two complex arguments, which we call the double q -analytic functions. As another direction, by the hyperbolic version of q -analytic functions we describe the q -analogue of traveling waves, which is not preserving the shape during evolution. The IVP for corresponding q -wave equation we solved in the q -D'Alembert form.

1. Introduction

Recently, we have introduced a new class of complex valued functions $f(z)$ of one complex variable z according to equation [1],

$$D_q^{\bar{z}} f(z) = \frac{1}{2} \left(D_q^x + i D_q^y \right) f(z) = 0 \quad (1)$$

and we called it as the q -analytic function. In the limit $q \rightarrow 1$ it reduces to the standard analytic function satisfying the Cauchy-Riemann equations. The basic example of q -analytic function is given by complex q -binomial

$$(x + iy)_q^n \equiv (x + iy)(x + iqy)(x + iq^2y) \dots (x + iq^{n-1}y) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}} x^{n-k} (iy)^k, \quad (2)$$

which is not analytic function in the usual sense $\frac{\partial}{\partial \bar{z}}(x + iy)_q^n \neq 0$. But, as was shown in [1] it is a generalized analytic function [2], which satisfies the $\bar{\partial}$ - equation

$$\frac{\partial \Phi}{\partial \bar{z}} = A(z, \bar{z}) \Phi. \quad (3)$$

This can be solved in a closed form [2], [3], [4]:

$$\Phi(z, \bar{z}) = \omega(z) e^{\frac{1}{2\pi i} \int \int_D \frac{A(\zeta, \bar{\zeta})}{\zeta - z} d\zeta \wedge d\bar{\zeta}}, \quad (4)$$



providing following representation for q -binomial [1]:

$$(x + iy)_q^n = (x + iy)^n \left(\prod_{k=0}^{n-1} \frac{[2]_{q^k}}{2} \right) \exp \left[\frac{1}{2\pi i} \int \int_D \frac{1-q}{\zeta-z} \sum_{k=1}^{n-1} \frac{[k]_q}{(1+q^k)\zeta + (1-q^k)\bar{\zeta}} d\zeta \wedge d\bar{\zeta} \right]. \quad (5)$$

It was shown that every complex q -analytic function $\psi(z)$ determines a quantum state in the q -analytic Fock-Bargmann representation [1]. So that the orthonormal basis states $|n\rangle$ in this representation are given by complex q -binomials (2),

$$\psi_n(z) = \frac{(x + iy)_q^n}{\sqrt{[n]_q!}}. \quad (6)$$

In the present paper we are going to extend the concept of q -analytic function in two different directions. First we describe the double q -analytic functions and expansion of them in the q -Hermite binomials. Then, as a hyperbolic version of q -analyticity we introduce the q -traveling waves.

2. Double analytic and q -analytic functions

2.1. Double analytic function

A complex valued function $f(z, w)$ of two complex variables z and w , analytic in both variables $\frac{\partial}{\partial \bar{z}} f(z, w) = \frac{\partial}{\partial \bar{w}} f(z, w) = 0$ we call a double analytic if it satisfies the following equation

$$\frac{1}{2} \left(\frac{\partial}{\partial z} + i \frac{\partial}{\partial w} \right) f(z, w) = 0. \quad (7)$$

As an example, $f(z, w) = (z + iw)^n$ is double analytic function. In general, the double analytic function can be written as a power series in complex binomials $f(z, w) = \sum_{n=0}^{\infty} a_n (z + iw)^n$. For such double analytic binomials we can derive the following Hermite binomial formula

$$(z + iw)^n = \frac{1}{2^n} \sum_{k=0}^n \binom{n}{k} i^k H_{n-k}(z) H_k(w). \quad (8)$$

This formula is motivated by description of the Dirac type δ -function for quantum states in the Fock-Bargmann representation [6]. By generalization of this approach to two complex variables we find expansion (8). In the present paper we derive the q -analogue of this identity,

$$(z + iw)_q^n = \frac{1}{[2]_q^n} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q i^k q^{\frac{k(k-1)}{2}} H_{n-k}(z; q) H_k(qw, \frac{1}{q}), \quad (9)$$

from which (8) follows as a particular case at $q = 1$.

2.2. Double q -analytic function

A complex valued function $f(z, w)$ of two complex variables z and w , analytic in both variables and satisfying equation

$$\bar{D}_{z,w} f(z, w) = \frac{1}{2} \left(D_q^z + i D_{\frac{1}{q}}^w \right) f(z, w) = 0 \quad (10)$$

we call as a *double q -analytic* function, where $D_q^z f(z) = \frac{f(qz) - f(z)}{(q-1)z}$, $D_{\frac{1}{q}}^w f(w) = \frac{f(\frac{w}{q}) - f(w)}{(\frac{1}{q}-1)w}$.

As an example, complex q -binomial $(z + iw)_q^2 = z^2 + [2]_q iwz - qw^2$, is analytic in z and w , and double q -analytic due to $\bar{D}_{z,w}(z + iw)_q^2 = 0$.

The complex q -binomial $(z + iw)_q^n$, for n -positive integer, is double q -analytic. This is why, any convergent power series

$$f(z, w)_q \equiv \sum_{n=0}^{\infty} a_n (z + iw)_q^n$$

represents a double q -analytic function. For example, analytic function of two complex variables

$$e_q(z + iw)_q = \sum_{n=0}^{\infty} \frac{(z + iw)_q^n}{[n]_q!} \tag{11}$$

is double q -analytic in the disk $|w| < \frac{q}{q-1}$, for $q > 1$, and admits factorization

$$e_q(z + iw)_q = e_q(z) e_{\frac{1}{q}}(iw) = e_q(z) (\cos_{\frac{1}{q}} w + i \sin_{\frac{1}{q}} w) \tag{12}$$

in terms of Jackson's exponential functions

$$e_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!},$$

where q -numbers and q -factorials are defined as follows:

$$[n]_q = \frac{q^n - 1}{q - 1}, \quad [n]_q! = [1]_q [2]_q \dots [n]_q. \tag{13}$$

As a central result of the present paper, we prove an expansion (9) of the q -binomial in terms of q -Hermite polynomials. Here, the q -Hermite polynomials are defined according to generating function [7]

$$e_q(-t^2) e_q([2]_q tx) = \sum_{n=0}^{\infty} H_n(x; q) \frac{t^n}{[n]_q!}. \tag{14}$$

In the limit $q \rightarrow 1$, (9) reduces to the Hermite binomial formula (8).

2.2.1. q -binomial and q -translation operator Our proof is based on representation of q -binomial as a q -translation:

$$e_{\frac{1}{q}}^{aD_q^x} x^n = (x + a)_q^n. \tag{15}$$

This formula can be proved by expanding q -exponential function as follows

$$\begin{aligned} e_{\frac{1}{q}}^{aD_q^x} x^n &= \sum_{k=0}^{\infty} \frac{a^k (D_q^x)^k}{[k]_{\frac{1}{q}}!} x^n = \sum_{k=0}^n \frac{a^k (D_q^x)^k}{[k]_{\frac{1}{q}}!} x^n = \sum_{k=0}^n a^k \frac{[n]_q [n-1]_q \dots [n-k+1]_q}{[k]_q!} q^{\frac{k(k-1)}{2}} x^{n-k} \\ &= \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}} x^{n-k} a^k = (x + a)_q^n, \end{aligned} \tag{16}$$

where $(D_q^x)^k = [n]_q [n-1]_q \dots [n-k+1]_q x^{n-k}$, $[k]_{\frac{1}{q}}! = \frac{1}{q^{\frac{k(k-1)}{2}}} [k]_q!$.

3. q -binomial expansion in terms of q -Hermite polynomials

Complex q -binomial expansion in terms of q -Hermite polynomials we find in the next form:

$$e_{\frac{1}{q}}^{iyD_q^x} x^n = (x + iy)_q^n = \frac{1}{[2]_q^n} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q i^k q^{\frac{k(k-1)}{2}} H_{n-k}(x; q) H_k(qy; \frac{1}{q}). \quad (17)$$

In order to prove this formula, from the generating function for q -Hermite polynomials (14) we get

$$H_n(x; q) = e_q \left(-\frac{1}{[2]_q^2} (D_q^x)^2 \right) [2]_q^n x^n. \quad (18)$$

By replacing $n \rightarrow k$, $x \rightarrow qy$, $q \rightarrow \frac{1}{q}$ we obtain

$$H_k(qy; \frac{1}{q}) = e_{\frac{1}{q}} \left(-\frac{1}{[2]_{\frac{1}{q}}^2} (D_{\frac{1}{q}}^{qy})^2 \right) [2]_{\frac{1}{q}}^k (qy)^k. \quad (19)$$

As the next step, to both sides of Gauss' binomial formula (2) we apply the operator

$$\frac{1}{[2]_q^n} e_q \left(-\frac{1}{[2]_q^2} (D_q^x)^2 \right) [2]_q^{n-k} e_{\frac{1}{q}} \left(-\frac{1}{[2]_{\frac{1}{q}}^2} (D_{\frac{1}{q}}^{qy})^2 \right) [2]_{\frac{1}{q}}^k q^k. \quad (20)$$

Then, by using (18) and (19), from the RHS of (2) we obtain

$$\frac{1}{[2]_q^n} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q i^k q^{\frac{k(k-1)}{2}} H_{n-k}(x; q) H_k(qy; \frac{1}{q}). \quad (21)$$

From another side the LHS of (2) becomes

$$e_q \left(-\frac{1}{[2]_q^2} (D_q^x)^2 \right) e_{\frac{1}{q}} \left(-\frac{1}{[2]_{\frac{1}{q}}^2} (D_{\frac{1}{q}}^{qy})^2 \right) e_{\frac{1}{q}}^{iyD_q^x} x^n,$$

and due to $(D_{\frac{1}{q}}^{qy})^2 e_{\frac{1}{q}}^{iyD_q^x} x^n = (\frac{i}{q})^2 (D_q^x)^2 e_{\frac{1}{q}}^{iyD_q^x} x^n$ and as follows

$$e_{\frac{1}{q}} \left(-\frac{1}{[2]_{\frac{1}{q}}^2} (D_{\frac{1}{q}}^{qy})^2 \right) e_{\frac{1}{q}}^{iyD_q^x} x^n = e_{\frac{1}{q}} \left(\frac{1}{[2]_q^2} (D_q^x)^2 \right) e_{\frac{1}{q}}^{iyD_q^x} x^n,$$

finally we find

$$e_{\frac{1}{q}}^{iyD_q^x} e_q \left(-\frac{1}{[2]_q^2} (D_q^x)^2 \right) + \frac{1}{[2]_q^2} (D_q^x)^2 e_{\frac{1}{q}}^{iyD_q^x} x^n = e_{\frac{1}{q}}^{iyD_q^x} x^n = (x + iy)_q^n. \quad (22)$$

By changing real variables to the complex ones $x \rightarrow z$ and $y \rightarrow w$, we obtain the expansion of double q -analytic q -binomial in terms of q -Hermite polynomials (9).

4. q -traveling waves

Here as another, hyperbolic extension of q -analytic functions, we consider the q -analogue of traveling waves as a solution of the q -wave equation. Direct extension of traveling waves to q -traveling waves is not possible. This happens due to the lack of the chain rule in q -calculus and as follows, impossibility to use moving frame as an argument of the wave function. Moreover, if in the Fourier harmonics $f(x, t) = e^{i(kx - \omega t)}$, we try naively to replace the exponential function by Jackson's q -exponential, $f_q(x, t) = e_q(i(kx - \omega t))$, then we find that it doesn't work due to the absence of factorization for q -exponential function $e_q(i(kx - \omega t)) \neq e_q(ikx) e_q(i\omega t)$.

4.1. q -wave equation

This is why, here we propose another way. First we observe that q -binomials

$$(x \pm ct)_q^n = (x \pm ct)(x \pm qct)\dots(x \pm q^{n-1}ct) \quad (23)$$

for $n = 0, \pm 1, \pm 2, \dots$, satisfy the first order one-directional q -wave equations

$$\left(D_{\frac{1}{q}}^t \mp cD_q^x \right) (x \pm ct)_q^n = 0, \quad (24)$$

which are hyperbolic analogs of q -analyticity (and anti-analyticity). Then, the Laurent series expansion in terms of these q -binomials determines the q -analog of traveling waves

$$f(x \pm ct)_q = \sum_{n=-\infty}^{\infty} a_n (x \pm ct)_q^n.$$

Due to (24) the q -binomials (23) satisfy the q -wave equation

$$\left(\left(D_{\frac{1}{q}}^t \right)^2 - c^2 \left(D_q^x \right)^2 \right) u(x, t) = 0 \quad (25)$$

and the general solution of this equation is expressed in the form of q -traveling waves

$$u(x, t) = F(x + ct)_q + G(x - ct)_q, \quad (26)$$

where

$$F(x + ct)_q = \sum_{n=-\infty}^{\infty} a_n (x + ct)_q^n, \quad G(x - ct)_q = \sum_{n=-\infty}^{\infty} b_n (x - ct)_q^n$$

are arbitrary functions.

4.2. q -D'Alembert solution

This general solution can be fixed by the IVP for the q -wave equation

$$\left[\left(D_{\frac{1}{q}}^t \right)^2 - c^2 \left(D_q^x \right)^2 \right] u(x, t) = 0, \quad (27)$$

$$u(x, 0) = f(x), \quad (28)$$

$$D_{\frac{1}{q}}^t u(x, 0) = g(x), \quad (29)$$

where $-\infty < x < \infty$. In the D'Alembert form the solution is

$$u(x, t) = \frac{f(x + ct)_q + f(x - ct)_q}{2} + \frac{1}{2c} \int_{(x-ct)_q}^{(x+ct)_q} g(x') d_q x', \quad (30)$$

where the Jackson integral is defined as

$$\int_{(x-ct)_q}^{(x+ct)_q} g(x') d_q x' = (1-q)(x+ct) \sum_{j=0}^{\infty} q^j g(q^j(x+ct))_q - (1-q)(x-ct) \sum_{j=0}^{\infty} q^j g(q^j(x-ct))_q.$$

If the initial velocity is zero, $g(x) = 0$, the formula reduces to

$$u(x, t) = \frac{1}{2} (f(x + ct)_q + f(x - ct)_q). \quad (31)$$

It should be noted here that the q -traveling wave is not traveling wave in the standard sense, since it is not preserving shape during evolution. It can be seen from simple observation. The traveling wave polynomial $(x - ct)_q^n = (x - ct)(x - qct)(x - q^2ct)\dots(x - q^{n-1}ct)$ includes the set of moving frames (as zeros of this polynomial) with geometrically re-scaled set of speeds $(c, qc, q^2c, \dots, q^{n-1}c)$. It means that zeros of this polynomial are moving with different speeds and therefore the shape of polynomial wave is not preserving. Only in the linear case and in the case $q = 1$, when speeds of all frames coincide, we are getting standard traveling wave.

4.3. Example: q -Gaussian traveling wave

For the initial function in the Gaussian form: $u(x, 0) = e^{-x^2}$ in the standard case $q = 1$ we have the Gaussian traveling wave with the same shape $u(x, t) = e^{-(x-ct)^2}$ (see Figure 1). For the q -traveling wave with the same initial condition $u(x, 0) = e^{-x^2}$, we have solution

$$u(x, t) = \left(e^{-(x-ct)^2} \right)_q \equiv \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (x-ct)_q^{2n}$$

(see Figure 2). As we can see, during evolution the shape of the wave is changing as an amplitude is growing.

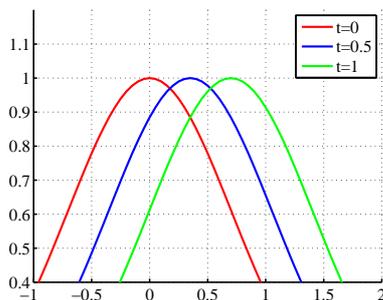


Figure 1. Gaussian traveling wave at time intervals $t=0$, $t=0.5$ and $t=1$.

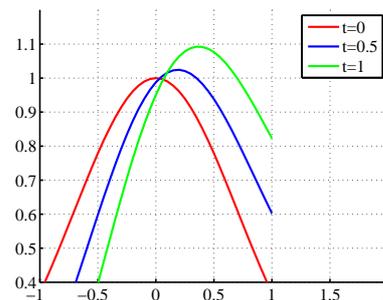


Figure 2. q -Gaussian traveling wave at time intervals $t=0$, $t=0.5$ and $t=1$.

5. Conclusions

As a potential application of our results we should mention that an analytic function of two complex variables can be related to the tensor product of Glauber coherent states. Then, the double analytic functions, as well as the q -double analytic functions correspond to some symmetry restrictions on these states. Expansion of these states in binomial and Hermite binomials form would reflect some entanglement properties of these states. These questions are under the study now. Interesting problem also is to find the symmetry group of the q -wave equation as a q -deformation of the Lorentz group.

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