Journal of Physics: Conference Series **766** (2016) 012015 doi:10.1088/1742-6596/766/1/012015

Quantum calculus of classical vortex images, integrable models and quantum states

Oktay K Pashaev

Department of Mathematics, Izmir Institute of Technology, Gulbahce Campus, Urla-Izmir, 35430, Turkey

E-mail: `oktaypashaev@iyte.edu.tr`

Abstract. From two circle theorem described in terms of q-periodic functions, in the limit $q \to 1$ we have derived the strip theorem and the stream function for N vortex problem. For regular N-vortex polygon we find compact expression for the velocity of uniform rotation and show that it represents a nonlinear oscillator. We describe q-dispersive extensions of the linear and nonlinear Schrödinger equations, as well as the q-semiclassical expansions in terms of Bernoulli and Euler polynomials. Different kind of q-analytic functions are introduced, including the pq-analytic and the golden analytic functions.

1. Introduction

The quantum or the so called q-calculus rises from classical works of Euler, Gauss, Fermat etc., but only recently, after discovering quantum integrable models and quantum groups, the subject has attracted much attention. In present paper we are going to study several applications of this calculus to problems of classical hydrodynamics, quantum theory and integrable systems.

1.1. Fermat partition

The Fermat partition plays the central role in q-calculus. Introduced first for calculation of integrals, it divides an interval in geometric progression and has many applications. For guitar frets it gives the scales $L, Lq, Lq^2, ..., Lq^{12}$, where $q = e^{-\ln 2/12} = 0.94387645$. Another example is given by intervals for vertical distance traveled by a bouncing ball if the height of each rebound is reduced by the factor $q < 1$. The total distance is $h+2hq+2hq^2+... = -h+2h(1+q+q^2+...) = -h + 2h[\infty]_q = h(1+q)/(1-q)$.

In general, any real analytic function $f(q) = a_0 + a_1 q + a_2 q^2 + ...$, convergent for $0 < q < 1$, has geometrical meaning as an area on Fermat partition $(1, q, q^2, ...)$,

$$A = a_0(1-q) + a_1(q-q^2) + a_2(q^2-q^3) + ... = (1-q)(a_0 + a_1 q + a_2 q^2 + ...) = (1-q)f(q).$$

In particular case, when $a_n = F(q^n)$, this area becomes just the Jackson integral

$$A = (1-q)\sum_{n=0}^{\infty} F(q^n)q^n = \int_0^1 F(x)d_q x.$$

One more example is given by problem of point vortex in concentric annular domain, where the set of vortex images is distributed according to the Fermat partition.

Content from this work may be used under the terms of the Creative Commons Attribution 3.0 licence. Any further distribution of this work must maintain attribution to the author(s) and the title of the work, journal citation and DOI.
Published under licence by IOP Publishing Ltd

2. Hydrodynamic images

As a first application we consider the hydrodynamic flow in annular domain and the two circle theorem [1]. For incompressible and irrotational planar flow in annular domain $r_1 < |z| < r_2$, between two concentric circles $C_1 : |z| = r_1^2$ and $C_2 : |z| = r_2^2$, the complex potential $F(z)$ is analytic function, q-periodically extended to the whole complex plane:$F(qz) = F(z)$ or $D_q^z F(z) = 0$. Explicit form of this function is given in terms of $f(z)$, as a complex potential of flow in the plain without boundaries:

$$F(z) = f_q(z) + \overline{f_q(z^*)} = \sum_{n=-\infty}^{\infty} f(q^n z) + \sum_{n=-\infty}^{\infty} \bar{f}\left(q^n \frac{r_1^2}{z}\right), \tag{1}$$

where $z^* = r_1^2/\bar{z}$ is inversion of the point z, $f_q(z) \equiv \sum_{n=-\infty}^{\infty} f(q^n z)$ - q-periodic extension of the flow to even annular, and $\overline{f_q(z^*)} = \bar{f}_q(r_1^2/z) = \sum_{n=-\infty}^{\infty} \bar{f}\left(q^n \frac{r_1^2}{z}\right)$ - q-periodic extension to the odd ones. This expression shows that restriction of the flow from whole plane to annular domain is equivalent to the q-periodic flow in the plane. It happens due to the boundary conditions $\Im F(z)|_{C_{1,2}} = 0$, identifying imaginary parts of the analytic function on the circles and as follows, identifying values of this function on the boundaries (up to irrelevant constant). In addition to q-periodicity, function $F(z)$ must be Q- periodic, with $Q = e^{2\pi i}$, under rotations on 2π around the origin. The real and imaginary parts of this function in polar coordinates $F(z) = \Phi(r,\theta) + i\Psi(r,\theta)$ are the velocity potential and the stream function respectively. These functions are q-periodic in r, and periodic in θ: $\Phi(qr,\theta) = \Phi(r,\theta)$, $\Phi(r,\theta+2\pi) = \Phi(r,\theta)$; $\Psi(qr,\theta) = \Psi(r,\theta)$, $\Psi(r,\theta+2\pi) = \Psi(r,\theta)$. In explicit form

$$\Phi(r,\theta) = \sum_{n=-\infty}^{\infty} \left(\varphi(q^n r,\theta) + \varphi(q^n \frac{r_1^2}{r},\theta)\right), \quad \Psi(r,\theta) = \sum_{n=-\infty}^{\infty} \left(\psi(q^n r,\theta) - \psi(q^n \frac{r_1^2}{r},\theta)\right), \tag{2}$$

where $f(z) = \varphi(r,\theta) + i\psi(r,\theta)$ is the flow in plane without boundaries, periodic in θ: $\varphi(r,\theta+2\pi) = \varphi(r,\theta)$, $\psi(r,\theta+2\pi) = \psi(r,\theta)$. From the above formulas, the stream function vanishes at boundary circles: $\Psi(r_1,\theta) = 0$ and $\Psi(r_2,\theta) = 0$. Moreover, if $\Gamma_k : |z| = r_1 q^k$, and $\gamma_k : |z| = r_2 q^k$, $k = 0, \pm1, \pm2, ...$ are the boundary circles images, then $\Psi|_{\Gamma_k} = 0$ and $\Psi|_{\gamma_k} = 0$ as well.

Several extensions of the two circle theorem, requiring different type of q-calculus were considered in [2] . The wedge theorem for the domain with angle $\alpha = 2\pi/N$ was given with the base $Q^N = 1$. Geometry of the double circular wedge theorem requires calculus with two bases, $q = r_2^2/r_1^2$ and $Q^N = 1$. By using these theorems, the Kummer kaleidoscope of vortices was described as well. In the limit $q \to \infty$ our two circles theorem reduces to the Milne-Thomson one circle theorem. However, more common in q-calculus limit $q \to 1$ was not studied. It happens due to the fact that for finite r_1 and r_2, at this limit the annular domain vanishes and problem has no meaning. But, as we describe below, for infinite radiuses $r_1 \to \infty$, $r_2 \to \infty$, the finite limit $q = r_2^2/r_1^2 \to 1$ of the domain exists and it is the strip domain.

2.1. The strip theorem

For incompressible and irrotational flow in the strip domain S: $\{z = x + iy; -h/2 < y < h/2\}$, the complex potential is

$$F(z) = \sum_{n=-\infty}^{\infty} f(z + (2n)ih) + \sum_{n=-\infty}^{\infty} \bar{f}(z + (2n-1)ih), \tag{3}$$

where $f(z)$ is flow in the whole plane. The proof is straightforward by checking the boundary conditions: $\Im F(z)|_{z=x+i\frac{h}{2}} = 0$ and $\Im F(z)|_{z=x-i\frac{h}{2}} = 0$. The flow (3) satisfies periodicity and the combined periodicity conditions, respectively

$$F(z+2ih)=F(z),\ \ \bar{F}(z+ih)=F(z). \tag{4}$$

2.1.1. Vortex in strip For the point vortex $f(z)=\frac{\Gamma}{2\pi i}\ln(z-z_0)$ in the strip, we have

$$F(z)=\frac{-i\Gamma}{2\pi}\left(\ln(z-z_0)+\sum_{n=1}^{\infty}\ln\frac{(z-z_0)^2+(2n)^2h^2}{(z-\bar{z}_0)^2+(2n-1)^2h^2}\right). \tag{5}$$

By using infinite product representation

$$\sinh z=z\prod_{n=1}^{\infty}\left(1+\frac{z^2}{\pi^2n^2}\right)$$

it can be rewritten as

$$F(z)=\frac{-i\Gamma}{2\pi}\left(\ln\prod_{n=1}^{\infty}\frac{4h}{\pi}\left(\frac{2n}{2n-1}\right)^2+\ln\frac{\sinh\frac{\pi}{2h}(z-z_0)\sinh\frac{\pi}{2h}(z-\bar{z}_0)}{\sinh\frac{\pi}{h}(z-\bar{z}_0)}\right)$$

and due to $\sinh\frac{\pi}{h}(z-\bar{z}_0)=2\sinh\frac{\pi}{2h}(z-\bar{z}_0)\cosh\frac{\pi}{2h}(z-\bar{z}_0)$, up to irrelevant constant we get

$$F(z)=\frac{-i\Gamma}{2\pi}\ln\frac{\sinh\frac{\pi}{2h}(z-z_0)}{\cosh\frac{\pi}{2h}(z-\bar{z}_0)}. \tag{6}$$

For N vortices $\Gamma_1,...,\Gamma_N$ at positions $z_1,...,z_N$ this gives the complex potential

$$F(z)=\sum_{k=1}^{N}\frac{-i\Gamma_k}{2\pi}\ln\frac{\sinh\frac{\pi}{2h}(z-z_k)}{\cosh\frac{\pi}{2h}(z-\bar{z}_k)}, \tag{7}$$

and the stream function $\Psi=\Im F(z)$,

$$\Psi(x,y)=\sum_{k=1}^{N}\frac{-\Gamma_k}{4\pi}\ln\frac{\sinh^2\frac{\pi}{2h}(x-x_k)+\sin^2\frac{\pi}{2h}(y-y_k)}{\sinh^2\frac{\pi}{2h}(x-x_k)+\cos^2\frac{\pi}{2h}(y+y_k)}. \tag{8}$$

For $N=2$ this function was obtained in [3], though formula in the paper has some typos.

2.1.2. Gamma function To express (5) in terms of the Gamma function we split

$$F(z)=\frac{-i\Gamma}{2\pi}\left(\ln(z-z_0)+\ln\prod_{n=1}^{\infty}\frac{(z-z_0)+2nih}{(z-\bar{z}_0)+(2n-1)ih}\frac{(z-z_0)-2nih}{(z-\bar{z}_0)-(2n-1)ih}\right).$$

Rewriting the second term as

$$C_\infty+\ln\prod_{n=1}^{\infty}\frac{(1+\frac{z-z_0}{2ihn})\,(1+\frac{z-\bar{z}_0}{2ihn})}{(1+\frac{z-\bar{z}_0}{ihn})}\frac{(1+\frac{z_0-z}{2ihn})\,(1+\frac{\bar{z}_0-z}{2ihn})}{(1+\frac{\bar{z}_0-z}{ihn})} \tag{9}$$

we find

$$F(z)=\frac{-i\Gamma}{2\pi}\left(\ln(z-z_0)+\ln\frac{\Gamma(1+\frac{z-\bar{z}_0}{ih})\Gamma(1-\frac{z-\bar{z}_0}{ih})}{\Gamma(1+\frac{z-z_0}{2ih})\Gamma(1-\frac{z-z_0}{2ih})\Gamma(1+\frac{z-\bar{z}_0}{2ih})\Gamma(1-\frac{z-\bar{z}_0}{2ih})}\right),$$

where we have used definition of the Gamma function as an infinite product

$$\frac{1}{\Gamma(z)}=ze^{\gamma z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}.$$

2.2. Strip theorem as the limit $q \to 1$

Now we are going to show that our two circle theorem (1) in the limit $q \to 1$ reduces to the strip theorem (3). Let $r_1 \to \infty$, $r_2 \to \infty$, so that $r_2 = r_1 + h$. Then $q = \frac{r_2^2}{r_1^2} = (1 + \frac{h}{r_1})^2$ and $q^n = (1 + \frac{h}{r_1})^{2n} = 1 + 2n\frac{h}{r_1} + O(\frac{h^2}{r_1^2})$. To get the strip domain we need to shift our coordinate system z with origin z=0, to the new one z' with origin at $i(r_1 + \frac{h}{2})$. These coordinates are related by equation $z = z' + i(r_1 + \frac{h}{2})$. Then $q^n z = (1 + 2n\frac{h}{r_1} + O(\frac{h^2}{r_1^2})(i(r_1 + \frac{h}{2}) + z') = i(r_1 + \frac{h}{2}) + z' + 2nhi + O(\frac{h}{r_1})$. For analytic in plane function $f(z)$ we have Taylor expansion $f(z) = f(i(r_1 + \frac{h}{2}) + z') = \sum_{n=0}^{\infty} \frac{f^{(n)}(i(r_1+\frac{h}{2}))}{n!}(z')^n \equiv g(z')$, so that

$$f(q^n z) = f((1 + \frac{h}{r_1})^n (i(r_1 + \frac{h}{2}) + z')) = f(i(r_1 + \frac{h}{2}) + z' + 2nih + O(\frac{h}{r_1})) = g(z' + 2nih).$$

This gives the first part of formula (1): $f_q(z) = \sum_{-\infty}^{\infty} g(z' + 2nih)$. To get the second part we expand $r_1^2/z = r_1^2/(z' + i(r_1 + \frac{h}{2})) = r_1/i(1 + z'/ir_1 + h/2r_1) = -i(r_1 + \frac{h}{2}) + ih + z' + O(h/r_1)$, and $q^n r_1^2/z = -i(r_1 + \frac{h}{2}) + ih + z' + O(h/r_1)$. Then $\bar{f}(r_1^2/z) = \bar{g}(z' + ih)$ and $\bar{f}(q^n r_1^2/z) = \bar{g}(z' - (2n-1)ih)$. For the second part of (1) it gives: $\bar{f}_q(r_1^2/z) = \sum_{n=-\infty}^{\infty} \bar{f}(q^n r_1^2/z) = \sum_{n=-\infty}^{\infty} \bar{g}(z' - (2n-1)ih)$. Combining both terms together we find that in the limit $q \to 1$ the two circle theorem (1) reduces to the strip theorem (3), $(n \to -n,\ n \to n-1)$,

$$G(z') = \sum_{-\infty}^{\infty} g(z' + 2nih) + \sum_{n=-\infty}^{\infty} \bar{g}(z' + (2n-1)ih).$$

3. N vortex polygon as nonlinear oscillator

From two circle theorem it is easy to get equations of motion for N point vortices [4]. When vortices with equal strength Γ are located at vertices of the regular polygon in annular domain, the problem admits exact solution $z_k(t) = r\exp(i\omega t + i2\pi k/N)$, $k = 1, ..., N$, with uniform rotation frequency

$$\omega(r) = \frac{\Gamma}{2\pi r^2}\frac{N-1}{2} + \frac{\Gamma}{2\pi r^2}\frac{1}{q-1}\sum_{j=1}^{N}\left[Ln_q\left(1 - \frac{r_2^2}{r^2}e^{i\frac{2\pi}{N}j}\right) - Ln_q\left(1 - \frac{r^2}{r_1^2}e^{i\frac{2\pi}{N}j}\right)\right]. \quad (10)$$

Summation in this formula can be performed explicitly if we notice that $\zeta = e^{i\frac{2\pi}{N}}$ is the primitive root of unity and $\zeta^N = 1$ implies $\zeta^N - 1 = (\zeta - 1)(1 + \zeta + \zeta^2 + ... + \zeta^{N-1}) = 0$. Then, $1 + \zeta + \zeta^2 + ... + \zeta^{N-1} = 0$ and for the sum of q-logarithm functions we have

$$\sum_{n=1}^{N} Ln_q(1 - x\zeta^n) = -\sum_{k=1}^{\infty}\frac{x^k}{[k]_q}(1 + \zeta^k + \zeta^{2k} + ... + \zeta^{k(N-1)}). \quad (11)$$

The sum $1 + \zeta^k + \zeta^{2k} + ... + \zeta^{k(N-1)} = N$, if $k = Nl$ and vanishes for $k \neq Nl$, where $l = 1, 2, ...$ It can be easily undersood if we notice that ζ^2, ζ^3,...ζ^{N-1} are primitive roots and $(\zeta^k)^N - 1 = (\zeta^k - 1)(1 + \zeta^k + \zeta^{2k} + ... + \zeta^{(N-1)k}) = 0$ for $k \neq Nl$. For $k = Nl$, $1 + \zeta^k + \zeta^{2k} + ... + \zeta^{k(N-1)} = 1 + 1 + ... + 1 = N$. Thus, in (11) nonvanishing terms are

$$\sum_{n=1}^{N} Ln_q(1 - x\zeta^n) = -\sum_{l=1}^{\infty}\frac{x^{Nl}}{[Nl]_q}N = -N(q-1)\sum_{l=1}^{\infty}\frac{x^{Nl}}{q^{Nl}-1} = -N(q-1)\sum_{l=1}^{\infty}\frac{(q^N-1)x^{Nl}}{(q^{Nl}-1)(q^N-1)}$$

and

$$\sum_{n=1}^{N} Ln_q(1 - x\zeta^n) = -\frac{N}{[N]_q}\sum_{l=1}^{\infty}\frac{(x^N)^l}{[l]_{q^N}} = \frac{N}{[N]_q}Ln_{q^N}(1 - x^N). \quad (12)$$

By applying formula (12) finally we rewrite frequency (10) in a simple form

$$\omega(r) = \frac{\Gamma(N-1)}{4\pi r^2} + \frac{\Gamma N}{2\pi r^2 (q^N - 1)} \left[Ln_{q^N} \left(1 - \frac{r_2^{2N}}{r^{2N}} \right) - Ln_{q^N} \left(1 - \frac{r^{2N}}{r_1^{2N}} \right) \right]. \tag{13}$$

The result shows that frequency $\omega(r)$ is function of r, and uniformly rotating polygon of vortices represents nonlinear oscillator. Quantization of this polygon as a nonlinear oscillator can be done in a similar way as for the one vortex case [2].

4. q-semiclassical expansion

Quantization of nonlinear oscillators as q-dispersive equations has been studied in [2]. Here we develop the q-semiclassical expansion for the q-deformed theories. Expansion formulas in powers of $\lambda = \ln q$ are based on the Bernoulli polynomials and the Euler polynomials with generating functions correspondingly

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \quad |t| < 2\pi; \qquad \frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \quad |t| < \pi. \tag{14}$$

4.1. Non-symmetric q-calculus

We have expansion of q-number

$$[n]_q = \frac{q^n - 1}{q - 1} = \sum_{m=0}^{\infty} (B_m(n) - B_m) \frac{(\ln q)^{m-1}}{m!}, \tag{15}$$

for $e^{-2\pi} < q < e^{2\pi}$, where $B_m = B_m(0)$ are the Bernoulli numbers. By using $B_0(x) = 1, B_1(x) = x - 1/2$ for any real or complex number, or even an arbitrary operator A we get

$$[A]_q = A + \sum_{k=1}^{\infty} \left(B_{k+1}(A) - B_{k+1} \right) \frac{\lambda^k}{(k+1)!}. \tag{16}$$

4.1.1. Nonsymmetric q-dispersive Schrödinger equation The q-dispersive linear Schrödinger equation is derived as

$$i\hbar \frac{\partial \Psi}{\partial t} = \left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \right]_q \Psi,$$

and it admits the q-deformed Galilean boost operator

$$K = x + t \frac{\ln q}{q - 1} \frac{i\hbar}{m} \frac{d}{dx} q^{-\frac{\hbar^2}{2m} \frac{d^2}{dx^2}},$$

generating dynamical symmetry and producing solutions of the equation. By expansion (16) we get higher derivative corrections to the Schrödinger equation

$$i\hbar \frac{\partial \Psi}{\partial t} + \frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \Psi = \sum_{k=1}^{\infty} \frac{(\ln q)^k}{(k+1)!} \left(B_{k+1} \left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \right) - B_{k+1} \right)$$

and to the Galilean boost

$$K = \left(x + \frac{t}{m} i\hbar \frac{d}{dx} \right) + \frac{t}{m} i\hbar \frac{d}{dx} \sum_{k=1}^{\infty} \frac{(\ln q)^k}{k!} B_k \left(-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \right).$$

4.2. Symmetric q-calculus

For symmetric q-numbers, in a similar way we get

$$[n]_{\tilde{q}} = \frac{q^n - q^{-n}}{q - q^{-1}} = n + \sum_{m=1}^{\infty} \left(B_{m+1}\left(\frac{1+n}{2}\right) - B_{m+1}\left(\frac{1-n}{2}\right)\right) \frac{2^m (\ln q)^m}{(m+1)!}. \tag{17}$$

Due to symmetry $B_m(1-x) = (-1)^m B_m(x), m = 0, 1, 2...$ this formula can be simplified and extended to any operator

$$[A]_{\tilde{q}} = \frac{\sinh(A \ln q)}{\sinh(\ln q)} = A + \sum_{k=1}^{\infty} B_{2k+1}\left(\frac{I+A}{2}\right) \frac{2^{2k+1}(\ln q)^{2k}}{(2k+1)!}. \tag{18}$$

4.2.1. q-symmetric quantum oscillator For the spectrum of q-oscillator [5], [6], [7], ($\hbar = 1, \omega = 1$), we have q-semiclassical expansion

$$E_n = \frac{1}{2}([n]_{\tilde{q}} + [n+1]_{\tilde{q}}) = \frac{1}{2}[2n+1]_{\widetilde{\sqrt{q}}} = (n + \frac{1}{2}) + \sum_{k=1}^{\infty} B_{2k+1}(n+1) \frac{(\ln q)^{2k}}{(2k+1)!}.$$

4.2.2. Symmetric q-dispersive Schrödinger equation For the linear Schrödinger equation with symmetric q-dispersion [2] we find following expansion with higher order derivatives

$$i\hbar \frac{\partial \Psi}{\partial t} + \frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} = \sum_{k=1}^{\infty} \frac{2^{2k+1}(\ln q)^{2k}}{(2k+1)!} B_{2k+1}\left(\frac{1}{2} - \frac{\hbar^2}{4m}\frac{\partial^2}{\partial x^2}\right)\Psi. \tag{19}$$

4.2.3. q-symmetric dispersive NLS Nonlinearization of model (19) consists in replacement of momentum operator by the recursion operator of NLS hierarchy [2], so that

$$i\sigma_3 \left(\begin{array}{c} \psi \\ \bar{\psi} \end{array}\right)_t = \frac{\sinh \lambda \frac{\hbar^2}{2m}\mathcal{R}^2}{\sinh \lambda} \left(\begin{array}{c} \psi \\ \bar{\psi} \end{array}\right) = \left[\frac{\hbar^2}{2m}\mathcal{R}^2\right]_{\tilde{q}} \left(\begin{array}{c} \psi \\ \bar{\psi} \end{array}\right).$$

Then, we have expansion

$$i\sigma_3 \left(\begin{array}{c} \psi \\ \bar{\psi} \end{array}\right)_t - \frac{\hbar^2}{2m}\mathcal{R}^2 \left(\begin{array}{c} \psi \\ \bar{\psi} \end{array}\right) = \sum_{k=1}^{\infty} \frac{2^{2k+1}(\ln q)^{2k}}{(2k+1)!} B_{2k+1}\left(\frac{I}{2} + \frac{\hbar^2}{4m}\mathcal{R}^2\right) \left(\begin{array}{c} \psi \\ \bar{\psi} \end{array}\right),$$

which gives q-corrections to the NLS equation ($\hbar = 1$, $m = 1/2$),

$$\left(\begin{array}{c} i\psi_t + \psi_{xx} + 2\kappa^2 |\psi|^2 \psi \\ -i\bar{\psi}_t + \bar{\psi}_{xx} + 2\kappa^2 |\psi|^2 \bar{\psi} \end{array}\right) = \sum_{k=1}^{\infty} \frac{2^{2k+1}(\ln q)^{2k}}{(2k+1)!} B_{2k+1}\left(\frac{I}{2} + \frac{1}{2}\mathcal{R}^2\right) \left(\begin{array}{c} \psi \\ \bar{\psi} \end{array}\right),$$

preserving integrability at any order of $\ln q$. The Lax pair can be expanded in a similar way.

4.3. Fibonacci polynomials for symmetric q-fermions

For $p = -1/q$ in pq calculus we have

$$[n]_{q,-q^{-1}} = \frac{q^n - (-q)^{-n}}{q - (-q)^{-1}} = \frac{q^n - (-1)^n q^{-n}}{q + q^{-1}} = \frac{e^{(1+n)\ln q} - (-1)^n e^{(1-n)\ln q}}{e^{2\ln q} + 1}$$

Journal of Physics: Conference Series **766** (2016) 012015 doi:10.1088/1742-6596/766/1/012015

and expanding in terms of Euler polynomials

$$[n]_{q,-q^{-1}} = \frac{1+(-1)^{n+1}}{2} + \frac{1}{2}\sum_{m=1}^{\infty}\left(E_m\left(\frac{1+n}{2}\right) - (-1)^n E_m\left(\frac{1-n}{2}\right)\right)\frac{2^m(\ln q)^m}{m!}. \quad (20)$$

By the relation $E_m(1-x) = (-1)^m E_m(x)$ for even and odd numbers we get

$$[2k]_{q,-q^{-1}} = \sum_{s=0}^{\infty} E_{2s+1}\left(k+\frac{1}{2}\right)\frac{2^{2s+1}(\ln q)^{2s+1}}{(2s+1)!},$$

$$[2k+1]_{q,-q^{-1}} = 1 + \sum_{s=1}^{\infty} E_{2s}\left(k+1\right)\frac{2^{2s}(\ln q)^{2s}}{(2s)!}.$$

From (20) for $q=1$ we find that the "classical" fermion number $[n]_{1,-1} = \frac{1+(-1)^{n+1}}{2}$ is the parity number, vanishing if $n = 2s$ is an even number, and equal to 1, if $n = 2s+1$ is an odd number. In terms of it we find following compact expression

$$[n]_{q,-q^{-1}} = [n]_{1,-1} + \sum_{m=1}^{\infty} E_m\left(\frac{n+1}{2}\right)[n+m]_{1,-1}\frac{2^m(\ln q)^m}{(m)!}. \quad (21)$$

4.4. Fibonacci numbers and Euler polynomials

From Binet formula for Fibonacci numbers

$$F_n = \frac{\varphi^n - \varphi'^n}{\varphi - \varphi'} = [n]_{\varphi,\varphi'} \equiv [n]_F \quad (22)$$

we find

$$F_n = [n]_{1,-1} + \sum_{m=1}^{\infty} E_m\left(\frac{n+1}{2}\right)[n+m]_{1,-1}\frac{2^m(\ln\varphi)^m}{(m)!}. \quad (23)$$

This provides an expansion of Fibonacci numbers in powers of $\ln\varphi$ and Euler polynomials

$$F_{2k} = \sum_{s=0}^{\infty} E_{2s+1}\left(k+\frac{1}{2}\right)\frac{2^{2s+1}(\ln\varphi)^{2s+1}}{(2s+1)!},$$

$$F_{2k+1} = 1 + \sum_{s=1}^{\infty} E_{2s}\left(k+1\right)\frac{2^{2s}(\ln\varphi)^{2s}}{(2s)!}.$$

4.4.1. Golden oscillator By these formulas spectrum of the golden oscillator [8], $E_n = \frac{\hbar\omega}{2}(F_{n+1}+F_n) = \frac{\hbar\omega}{2}F_{n+2}$, can be expanded in golden ratio as

$$E_{2k} = \frac{\hbar\omega}{2}\sum_{s=0}^{\infty} E_{2s+1}\left(k+\frac{3}{2}\right)\frac{2^{2s+1}}{(2s+1)!}\left(\ln\frac{1+\sqrt{5}}{2}\right)^{2s+1},$$

$$E_{2k+1} = \frac{\hbar\omega}{2} + \frac{\hbar\omega}{2}\sum_{s=1}^{\infty} E_{2s}\left(k+2\right)\frac{2^{2s}}{(2s)!}\left(\ln\frac{1+\sqrt{5}}{2}\right)^{2s}.$$

5. Diversity of q-analytic functions

Here we introduce several types of q-analytic functions. These functions are determined by the q-binomials and represent quantum states in the q-analytic Fock-Bargman representation. The q-binomials itself correspond to states $|n>$ and are the generalized analytic functions.

5.1. Nonsymmetric q-analytic functions
By q-translation

$$e_{1/q}^{iyD_q^x} x^n = (x+iy)_q^n$$

from any real analytic function $f(x)$ we get the q-analytic function [9],

$$f(x+iy)_q = e_{1/q}^{iyD_q^x} f(x) = \sum_{n=0}^{\infty} a_n (x+iy)_q^n,$$

satisfying $\bar{\partial}_q$ equation, $(D_q^x + iD_{1/q}^y) f(x+iy)_q = 0$. The real part $u(x,y) = \cos_{1/q}(yD_q^x) f(x)$ and imaginary part $v(x,y) = \sin_{1/q}(yD_q^x) f(x)$ of this function are q-harmonic and satisfy the q-Cauchy-Riemann equations

$$D_q^x u(x,y) = D_{1/q}^y v(x,y), \quad D_{1/q}^y u(x,y) = -D_q^x v(x,y).$$

5.2. Symmetric q-analytic functions
For symmetric q-binomials we have q-translation

$$e_{\tilde{q}}^{iyD_{\tilde{q}}^x} x^n = (x+iy)_{\tilde{q}}^n,$$

which determines symmetric q-analytic function

$$f(x+iy)_{\tilde{q}} = e_{\tilde{q}}^{iyD_{\tilde{q}}^x} f(x) = \sum_{n=0}^{\infty} a_n (x+iy)_{\tilde{q}}^n,$$

satisfying $\bar{\partial}_{\tilde{q}}$ equation

$$\frac{1}{2}(D_{\tilde{q}}^x + iD_{\tilde{q}}^y) f(x+iy)_{\tilde{q}} = 0.$$

As an example,

$$e(z;\tilde{q}) = \sum_{n=0}^{\infty} \frac{(x+iy)_{\tilde{q}}^n}{[n]_{\tilde{q}}!}$$

is an entire symmetric q-analytic function. The real part $u(x,y) = \cos_{\tilde{q}}(yD_{\tilde{q}}^x) f(x)$ and imaginary part $v(x,y) = \sin_{\tilde{q}}(yD_{\tilde{q}}^x) f(x)$ of the function satisfy the q-Cauchy-Riemann equations

$$D_{\tilde{q}}^x u(x,y) = D_{\tilde{q}}^y v(x,y), \quad D_{\tilde{q}}^y u(x,y) = -D_{\tilde{q}}^x v(x,y),$$

and are q-harmonic: $(D_{\tilde{q}}^x)^2 u(x,y) + (D_{\tilde{q}}^y)^2 u(x,y) = 0$.

5.3. pq-analytic function
For pq-calculus we have binomials

$$e_{\frac{1}{p}\frac{1}{q}}^{iyD_{pq}^x} x^n = (x+iy)_{pq}^n,$$

where

$$(x+iy)_{pq}^n = (x+ip^{n-1}y)(x+ip^{n-2}qy)...(x+ipq^{n-2}y)(x+iq^{n-1}y) = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_{pq} (pq)^{\frac{k(k-1)}{2}} x^{n-k} i^k y^k$$

and
$$e_{pq}(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]_{pq}!}.$$
Then, the pq-analytic function generated by pq-translation
$$f(x+iy)_{pq} = e_{\frac{1}{p}\frac{1}{q}}^{iyD^x_{pq}} f(x) = \sum_{n=0}^{\infty} a_n (x+iy)^n_{pq},$$
satisfies $\bar{\partial}_{pq}$ equation
$$\frac{1}{2}(D^x_{pq} + iD^y_{\frac{1}{p}\frac{1}{q}}) f(x+iy)_{pq} = 0.$$
For $u(x,y) = \cos_{\frac{1}{p}\frac{1}{q}}(yD^x_{pq}) f(x)$ and $v(x,y) = \sin_{\frac{1}{p}\frac{1}{q}}(yD^x_{pq}) f(x)$ we have the pq-Cauchy-Riemann equations
$$D^x_{pq} u(x,y) = D^y_{\frac{1}{p}\frac{1}{q}} v(x,y), \quad D^y_{\frac{1}{p}\frac{1}{q}} u(x,y) = -D^x_{pq} v(x,y)$$
and the pq-Laplace equation
$$(D^x_{pq})^2 u(x,y) + (D^y_{\frac{1}{p}\frac{1}{q}})^2 u(x,y) = 0.$$

5.4. Golden analytic function
Complex golden binomials, defined as
$$(x+iy)^n_F = (x+i\varphi^{n-1}y)(x-i\varphi^{n-3}y)...(x+i(-1)^{n-1}\varphi^{1-n}y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_F (-1)^{\frac{k(k-1)}{2}} x^{n-k} i^k y^k,$$
can be generated by the golden translation
$$E_F^{iyD^x_F} x^n = (x+iy)^n_F,$$
where
$$E^x_F = \sum_{n=0}^{\infty} (-1)^{\frac{n(n-1)}{2}} \frac{x^n}{F_n!}.$$
They determine the golden analytic function
$$f(z,F) = E_F^{iyD^x_F} f(x) = \sum_{n=0}^{\infty} a_n \frac{(x+iy)^n_F}{F_n!},$$
satisfying the golden $\bar{\partial}_F$ equation $\frac{1}{2}(D^x_F + iD^y_{-F}) f(z;F) = 0$, where $D_{-F} = (-1)^{x\frac{d}{dx}} D_F$. For $u(x,y) = Cos_F(yD^x_F) f(x)$ and $v(x,y) = Sin_F(yD^x_F) f(x)$, the golden Cauchy-Riemann equations are
$$D^x_F u(x,y) = D^y_{-F} v(x,y), \quad D^y_{-F} u(x,y) = -D^x_F v(x,y),$$
and the golden-Laplace equation is $(D^x_F)^2 u(x,y) + (D^y_{-F})^2 u(x,y) = 0$.

References
[1] Pashaev O K 2014 *J. Phys.: Conf. Series* **482** 012033
[2] Pashaev O K 2015 *Phys. Scr.*. **90** 074010
[3] Erdakova N N and Mamaev I S 2014 *Fluid Dyn. Res.*. **46** 031420
[4] Pashaev O K and Yilmaz O 2008 *J. Phys. A: Math. Theor.*. **41** 135207
[5] Biedenharn L C 1989 *J. Phys. A.* **22** L873
[6] Macfarlane A J 1989 *J. Phys. A.* **22** 4581
[7] Sun C P and Fu H C 1989 *J. Phys. A.* **22** L983
[8] Pashaev O K and Nalci S 2012 *J. Phys. A: Math. Theor.*. **45** 015303
[9] Pashaev O K and Nalci S 2014 *J. Phys. A: Math. Theor.*. **47** 045204