Construction of two qutrit entanglement by using magnetic resonance selective pulse sequences

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Abstract: Quantum entanglement is essential for some applications of quantum information processing such as quantum cryptography, quantum teleportation and superdence coding. A qubit is a two level quantum system and four two−qubit entangled states called Bell states can be easily obtained for two−qubit states. A qutrit is a three level quantum system and Zeeman levels of spin−1 electron or nucleus can be referred as qutrit. For SI (S=1, I=1) spin system there exist nine two−qutrit states. So nine two−qutrit entangled states can be obtained by using the Hadamard and CNOT logic gates. In this study by considering $N^{+}@C_{60}$ molecule as SI (S=1, I=1) spin system, two−qutrit entangled states are also obtained by using the magnetic resonance selective pulse sequences of Hadamard and CNOT logic gates. Then it is shown that these entangled states can be transformed into each other by the suggested transformation operators.

1. Introduction

Quantum information has a great interest in different areas such as physics, mathematics and computer science and engineering as the new technologies are expected in computing, cryptography and communication [1−3]. In these technologies qubits or qutrits (in general qudits) will be used instead of bits [3,4]. Quantum entanglement is essential for some applications of Quantum Information Processing (QIP) such as superdence coding, quantum cryptography and quantum teleportation [5]. In quantum entanglement there is a correlation between the quantum states of entangled particles. For SI (S=1/2, I=1/2) spin system two−qubit entangled states are obtained and they are called Bell states [6].

In this study, first two−qutrit entangled states are introduced in section 2. Then, in section 3, two qutrit entangled states are also obtained by using magnetic resonance selective pulse sequences for SI (S=1, I=1) spin system. In section 3, transformations of two qutrit entangled states into each other are also achieved by using suggested transformation operators.

2. Theory

A qutrit is a three level quantum system. As shown in Table 1, Zeeman levels of spin−1 electron or nucleus are referred as qutrit [4]. The matrix representation of one qutrit Hadamard gate can be written as [7]

$$\mathrm{H}=\frac{1}{\sqrt{3}}\begin{pmatrix}1 & 1 & 1\\ 1 & c & c^{2}\\ 1 & c^{2} & c\end{pmatrix}. \qquad (1)$$

Where $c=e^{i\frac{2\pi}{3}}$, $c^{2}=e^{-i\frac{2\pi}{3}}=c^{*}$. By applying this Hadamard gate, superpositions of single qutrit states are found as given in Table 2.

Table 1. Single qutrit states for spin−1.

m_I	qutrit
1	$\|0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$
0	$\|1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$
−1	$\|2\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Table 2. The superpositions of single qutrit states.

qutrit, $\|a\rangle$	$H\|a\rangle$
$\|0\rangle$	$(\|0\rangle + \|1\rangle + \|2\rangle)/\sqrt{3}$
$\|1\rangle$	$(\|0\rangle + c\|1\rangle + c^2\|2\rangle)/\sqrt{3}$
$\|2\rangle$	$(\|0\rangle + c^2\|1\rangle + c\|2\rangle)/\sqrt{3}$

For two spin−1 system such as SI (S=1, I=1) spin system, nine two-qutrit states of $|00\rangle,|01\rangle,|02\rangle,|10\rangle,|11\rangle,|12\rangle,|20\rangle,|21\rangle$ and $|22\rangle$ are obtained by direct products of single qutrit states. Two qutrit CNOT gates can be found by using the ternary addition of qutrit states:

$$CNOT_a(T)|a,b\rangle = |a,b \oplus a\rangle, \tag{2a}$$

$$CNOT_b(T)|a,b\rangle = |a \oplus b,b\rangle. \tag{2b}$$

These two−qutrit CNOT gates are 9x9 matrices and they can be written in Dirac notation as following:

$$\begin{aligned} CNOT_a(T) = {} & |00\rangle\langle 00| + |01\rangle\langle 01| + |02\rangle\langle 02| + |10\rangle\langle 12| + |11\rangle\langle 10| \\ & + |12\rangle\langle 11| + |20\rangle\langle 21| + |21\rangle\langle 22| + |22\rangle\langle 20|, \end{aligned} \tag{3a}$$

$$\begin{aligned} CNOT_b(T) = {} & |00\rangle\langle 00| + |01\rangle\langle 21| + |02\rangle\langle 12| + |10\rangle\langle 10| + |11\rangle\langle 01| \\ & + |12\rangle\langle 22| + |20\rangle\langle 20| + |21\rangle\langle 11| + |22\rangle\langle 02|. \end{aligned} \tag{3b}$$

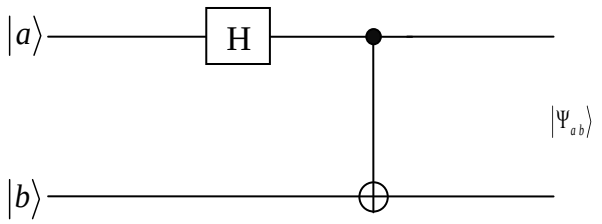

Figure 1. Quantum circuit for generating entangled states.

By using the quantum circuit given in Figure 1, two−qutrit entangled states can be generated. Obtained two−qutrit entangled states are presented in Table 3.

Table 3. Two−qutrit entangled states.

Input qutrit, $\left\|ab\right\rangle$	Output, $\left\|\Psi_{ab}\right\rangle$
$\left\|00\right\rangle$	$(\left\|00\right\rangle+\left\|11\right\rangle+\left\|22\right\rangle)/\sqrt{3}$
$\left\|01\right\rangle$	$(\left\|01\right\rangle+\left\|12\right\rangle+\left\|20\right\rangle)/\sqrt{3}$
$\left\|02\right\rangle$	$(\left\|02\right\rangle+\left\|10\right\rangle+\left\|21\right\rangle)/\sqrt{3}$
$\left\|10\right\rangle$	$(\left\|00\right\rangle+c\left\|11\right\rangle+c^2\left\|22\right\rangle)/\sqrt{3}$
$\left\|11\right\rangle$	$(\left\|01\right\rangle+c\left\|12\right\rangle+c^2\left\|20\right\rangle)/\sqrt{3}$
$\left\|12\right\rangle$	$(\left\|02\right\rangle+c\left\|10\right\rangle+c^2\left\|21\right\rangle)/\sqrt{3}$
$\left\|20\right\rangle$	$(\left\|00\right\rangle+c^2\left\|11\right\rangle+c\left\|22\right\rangle)/\sqrt{3}$
$\left\|21\right\rangle$	$(\left\|01\right\rangle+c^2\left\|12\right\rangle+c\left\|20\right\rangle)/\sqrt{3}$
$\left\|22\right\rangle$	$(\left\|02\right\rangle+c^2\left\|10\right\rangle+c\left\|21\right\rangle)/\sqrt{3}$

3. Results and Discussion

For spin−1 electron and nucleus, magnetic quantum numbers and corresponding one qutrit states are given in Figure 2. Selective transitions are also shown in this figure. Selective magnetic resonance pulses can be used to prepare some quantum logic gates. Modified matrix representation of some selective pulse operators for spin−1 is as follows [8]:

$$R_x^{12}(\theta)=\begin{pmatrix}\cos\left(\frac{\theta}{2}\right) & \sin\left(\frac{\theta}{2}\right) & 0\\ \sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right) & 0\\ 0 & 0 & 1\end{pmatrix},\quad R_x^{23}(\theta)=\begin{pmatrix}1 & 0 & 0\\ 0 & e^{i\frac{2\pi}{3}}\cos\left(\frac{\theta}{2}\right) & e^{-i\frac{2\pi}{3}}\sin\left(\frac{\theta}{2}\right)\\ 0 & e^{-i\frac{2\pi}{3}}\sin\left(\frac{\theta}{2}\right) & e^{i\frac{2\pi}{3}}\cos\left(\frac{\theta}{2}\right)\end{pmatrix}. \tag{4}$$

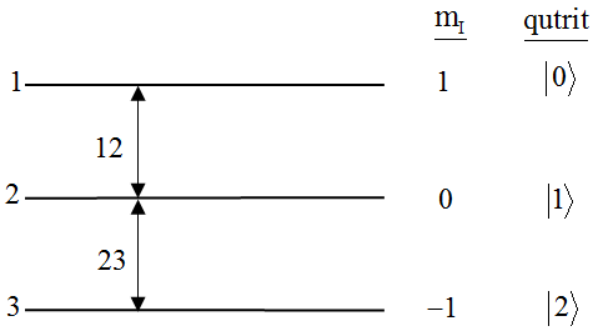

Figure 2. Magnetic quantum numbers and corresponding one qutrit states with selective transition for spin-1.

$N^+@C_{60}$ molecule can be considered as SI (S=1, I=1) spin system [9−10]. In this molecule total electron spin of N^+ in the ground state is 1 and nuclear spin of ^{14}N is also 1 with the abundance of 99,63%. For this spin system total Hamiltonian in an external magnetic field is

$$\hat{H}=\Omega_S\hat{S}_Z-\Omega_I\hat{I}_Z+A\hat{S}_Z\hat{I}_Z. \tag{5}$$

Where $\Omega_S = g\mu_B B$, $\Omega_I = \gamma_I B$. Energies of the spin Hamiltonian and corresponding two-qutrit states for SI (S=1, I=1) spin system are presented in Table 4.

Table 4. Energies of the spin Hamiltonian for SI (S=1, I=1) spin system and their two-qutrit states.

Level Number	$\|M_S\rangle$	$\|M_I\rangle$	Energy	Two−qutrit state, $\|ab\rangle$
1	$\|1\rangle$	$\|1\rangle$	$\Omega_S - \Omega_I + A$	$\|00\rangle$
2	$\|1\rangle$	$\|0\rangle$	Ω_S	$\|01\rangle$
3	$\|1\rangle$	$\|-1\rangle$	$\Omega_S + \Omega_I - A$	$\|02\rangle$
4	$\|0\rangle$	$\|1\rangle$	$-\Omega_I$	$\|10\rangle$
5	$\|0\rangle$	$\|0\rangle$	0	$\|11\rangle$
6	$\|0\rangle$	$\|-1\rangle$	Ω_I	$\|12\rangle$
7	$\|-1\rangle$	$\|1\rangle$	$-\Omega_S - \Omega_I - A$	$\|20\rangle$
8	$\|-1\rangle$	$\|0\rangle$	$-\Omega_S$	$\|21\rangle$
9	$\|-1\rangle$	$\|-1\rangle$	$-\Omega_S + \Omega_I + A$	$\|22\rangle$

Two−qutrit entangled states can be also obtained by using magnetic resonance pulse sequences of Hadamard and CNOT logic gates. For the construction of entanglement for two−qutrit state of $|00\rangle$, microwave pulse sequence of the Hadamard gate

$$\mathrm{H} = \left(\frac{\pi}{2}\right)_x^{\boxed{1}-\boxed{7}} - \left(70.52^\circ\right)_x^{\boxed{1}-\boxed{4}} \tag{6}$$

should be applied. Where the numbers such as $\boxed{1}$, $\boxed{4}$ and $\boxed{7}$ represent the level number as given in Table 4. Matrix representation of selective microwave pulses are as following:

$$\left(70.52^\circ\right)_x^{\boxed{1}-\boxed{4}} = R_x^{12}\left(70.52\right)\otimes I^1 + S_E \otimes \left(I_E - I^1\right), \tag{7a}$$

$$\left(\frac{\pi}{2}\right)_x^{\boxed{1}-\boxed{7}} = R_x^{13}\left(\frac{\pi}{2}\right)\otimes I^1 + S_E \otimes \left(I_E - I^1\right). \tag{7b}$$

Matrices used in the microwave and radio frequency pulses for spin-1 are given as following:

$$S_E = I_E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad S^1 = I^1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$S^0 = I^0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad S^{-1} = I^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{8}$$

When the Hadamard pulse sequence is applied to $|00\rangle$ state,

$$\mathrm{H}|00\rangle = \left(|00\rangle + |10\rangle + |20\rangle\right)/\sqrt{3} \tag{9}$$

is obtained. Then as a $\mathrm{CNOT_a(T)}$ logic gate, radio frequency pulse sequence

$$CNOT_a\left(T\right) = \left(\pi\right)_x^{\boxed{7}-\boxed{9}} - \left(\pi\right)_x^{\boxed{4}-\boxed{5}} \tag{10}$$

can be applied. Where the matrix representation of selective radio frequency pulses are

$$(\pi)_x^{\boxed{4}-\boxed{5}} = S^0 \otimes R_x^{12}(\pi) + (S_E - S^0) \otimes I_E, \tag{11a}$$

$$(\pi)_x^{\boxed{7}-\boxed{9}} = S^{-1} \otimes R_x^{13}(\pi) + (S_E - S^{-1}) \otimes I_E. \tag{11b}$$

After the application of radio frequency pulse sequence of $CNOT_a(T)$ to equation (9), entanglement of $|00\rangle$ state is found:

$$CNOT_a(T)(\mathrm{H}|00\rangle) = (|00\rangle + |11\rangle + |22\rangle)/\sqrt{3}. \tag{12}$$

The entanglements of the other two-qutrit states can be also constructed by using suitable Hadamard and $CNOT_a(T)$ pulse sequences. For the construction of two−qutrit entangled states, all suitable Hadamard microwave selective pulse sequences and $CNOT_a(T)$ radio frequency pulse sequences are presented in Table 5. It can be shown that after the application of these pulse sequences to two qutrit states, the same entangled states (given in Table 3) are found.

Table 5. Suitable Hadamard and $CNOT_a(T)$ pulse sequences for the entanglement of two−qutrit states.

Two−qutrit state, $\|ab\rangle$	Suitable H microwave pulse sequence	Suitable $CNOT_a(T)$ radio frequency pulse sequence
$\|00\rangle$	$\left(\frac{\pi}{2}\right)_x^{\boxed{1}-\boxed{7}} - (70.52^\circ)_x^{\boxed{1}-\boxed{4}}$	$(\pi)_x^{\boxed{7}-\boxed{9}} - (\pi)_x^{\boxed{4}-\boxed{5}}$
$\|01\rangle$	$\left(\frac{\pi}{2}\right)_x^{\boxed{2}-\boxed{8}} - (70.52^\circ)_x^{\boxed{2}-\boxed{5}}$	$(\pi)_x^{\boxed{7}-\boxed{8}} - (\pi)_x^{\boxed{5}-\boxed{6}}$
$\|02\rangle$	$\left(\frac{\pi}{2}\right)_x^{\boxed{3}-\boxed{9}} - (70.52^\circ)_x^{\boxed{3}-\boxed{6}}$	$(\pi)_x^{\boxed{8}-\boxed{9}} - (\pi)_x^{\boxed{4}-\boxed{6}}$
$\|10\rangle$	$\left(\frac{\pi}{2}\right)_x^{\boxed{4}-\boxed{7}} - (70.52^\circ)_x^{\boxed{1}-\boxed{4}}$	$(\pi)_x^{\boxed{7}-\boxed{9}} - (\pi)_x^{\boxed{4}-\boxed{5}}$
$\|11\rangle$	$\left(\frac{\pi}{2}\right)_x^{\boxed{5}-\boxed{8}} - (70.52^\circ)_x^{\boxed{2}-\boxed{5}}$	$(\pi)_x^{\boxed{8}-\boxed{9}} - (\pi)_x^{\boxed{5}-\boxed{6}}$
$\|12\rangle$	$\left(\frac{\pi}{2}\right)_x^{\boxed{6}-\boxed{9}} - (70.52^\circ)_x^{\boxed{3}-\boxed{6}}$	$(\pi)_x^{\boxed{8}-\boxed{9}} - (\pi)_x^{\boxed{4}-\boxed{6}}$
$\|20\rangle$	$\left(\frac{\pi}{2}\right)_x^{\boxed{4}-\boxed{7}} - (70.52^\circ)_x^{\boxed{4}-\boxed{7}}$	$(\pi)_x^{\boxed{7}-\boxed{9}} - (\pi)_x^{\boxed{4}-\boxed{5}}$
$\|21\rangle$	$\left(\frac{\pi}{2}\right)_x^{\boxed{5}-\boxed{8}} - (70.52^\circ)_x^{\boxed{2}-\boxed{8}}$	$(\pi)_x^{\boxed{7}-\boxed{8}} - (\pi)_x^{\boxed{5}-\boxed{6}}$
$\|22\rangle$	$\left(\frac{\pi}{2}\right)_x^{\boxed{6}-\boxed{9}} - (70.52^\circ)_x^{\boxed{3}-\boxed{9}}$	$(\pi)_x^{\boxed{8}-\boxed{9}} - (\pi)_x^{\boxed{4}-\boxed{6}}$

Weyl operators (also called d^2 operators) can be obtained for dimension d as following [11]:

$$U_{nm} = \sum_{k=0}^{d-1} e^{i\frac{2\pi}{d}km} |k\rangle\langle(k+m) \bmod d|. \tag{13}$$

Where n, m=0, 1,…, d−1. For qutrits (d=3) these operators can be generated as given in Table 6. These operators can be called as Pauli bases for qutrits. In order to transform the two−qutrit entangled states between each other, transformation operators can be defined as $U_{nm} \otimes I_E$. For example $U_{01} \otimes I_E$

transformation operator can be used to transform the $|\psi_{01}\rangle$ entangled state to $|\psi_{02}\rangle$ entangled state. Applications of all the transformation operators to all two−qutrit entangled states can be also achieved.

Table 6. Weyl operators for qutrits [11]. Where $c = e^{i\frac{2\pi}{3}}$, $c^2 = e^{-i\frac{2\pi}{3}} = c^*$.

$U_{00} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$U_{01} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$	$U_{02} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$
$U_{10} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c^2 \end{pmatrix}$	$U_{11} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & c \\ c^2 & 0 & 0 \end{pmatrix}$	$U_{12} = \begin{pmatrix} 0 & 0 & 1 \\ c & 0 & 0 \\ 0 & c^2 & 0 \end{pmatrix}$
$U_{20} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c^2 & 0 \\ 0 & 0 & c \end{pmatrix}$	$U_{21} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & c^2 \\ c & 0 & 0 \end{pmatrix}$	$U_{22} = \begin{pmatrix} 0 & 0 & 1 \\ c^2 & 0 & 0 \\ 0 & c & 0 \end{pmatrix}$

4. Conclusion

For SI (S=1, I=1) spin system, there exist 9 two−qutrit states. By using the matrix representation of one qutrit Hadamard and two−qutrit CNOT(T) gates, two−qutrit entangled states are acquired for this spin system. Two−qutrit entangled states are also obtained by using the magnetic resonance selective pulse sequences of Hadamard and CNOT(T) logic gates. In order to transform two−qutrit entangled states between each other, transformation operators are suggested by using the Weyl operators. Then, the applications of these operators are presented.

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