

# General thermostatistical properties of a $q$ -deformed fermion gas in two dimensions

Abdullah Algin<sup>1</sup> and Mustafa Senay<sup>2</sup>

<sup>1</sup>Department of Physics and Centre for Quantum Research and Applications,  
Eskisehir Osmangazi University, Meselik, 26480 Eskisehir, Turkey

<sup>2</sup>Graduate School of Sciences, Eskisehir Osmangazi University, Meselik,  
26480 Eskisehir, Turkey

E-mail: aalgin@ogu.edu.tr; mustafasenay86@hotmail.com

**Abstract.** Starting with a deformed fermionic grand partition function, we study the high and low temperature thermostatistical properties of a special  $q$ -deformed fermion gas in two spatial dimensions. Many of the deformed thermostatistical functions such as the specific heat and the entropy are derived in terms of the real deformation parameter  $q$  for the range  $q < 1$ . For high temperatures, we specifically focus on the behavior of both the entropy function and the deformed virial coefficients in the equation of state for the  $q$ -fermion gas in two dimensions. Possible physical applications of the present  $q$ -fermion gas are briefly discussed.

## 1. Introduction

In recent years, quantum algebras [1-3] have attracted a great deal of interest for applications in natural and applied sciences. The statistical and thermodynamical properties of such boson and fermion systems have been extensively investigated [4-16]. As a matter of fact, interacting quantum systems exhibit many interesting phenomena not only for three spatial dimensions, but also in low-dimensional spaces. When confined to two dimensions, fermionic systems show phenomenon such as the fractional quantum Hall effect [17] and the behavior of an anyon gas [18]. In the meantime, some approaches with deformed fermionic and bosonic oscillator algebras to analyze the thermostatistical properties of such physical systems in two dimensions have been carried out [19,20]. However, for two dimensions, both quantum mechanical and thermostatistical properties of deformed oscillator systems have been less studied and less applied to the up-to-date scientific researches of different fields.

Motivated by the above considerations, in the present work, we aim to study the high and low temperature thermodynamical and statistical properties of a special  $q$ -deformed fermion gas model in two spatial dimensions. The  $q$ -deformed fermionic oscillator algebra, which we are interested in here, has been first introduced by Viswanathan *et al.* [21] and some of its statistical properties were also discussed by Chaichian *et al.* [8]. Therefore, we will use these fermion oscillators to construct a deformed gas of  $q$ -fermions describing a  $q$ -fermion gas model. Due to the historical reasons, we will call this model as the VPJC-type  $q$ -fermion gas. We should mention that high and low temperature thermostatistical properties of the VPJC  $q$ -fermion gas for three spatial dimensions have been elaborately investigated in [22,23]. Here, we continue to study further on the same model, and consider its quantum statistical behavior in two dimensions. In this framework, we will derive many



of the important thermostistical functions of the model depending on a function of the deformation parameter  $q$ . We specifically focus on the behavior of both deformed entropy function and deformed virial expansion of the equation of state for the VPJC  $q$ -fermion gas confined in two spatial dimensions. The effect of fermionic  $q$ -deformation on the thermodynamical quantities of the model in two dimensions and possible physical applications of the results are finally discussed.

## 2. The VPJC-type $q$ -fermion gas model

The model containing the VPJC-type  $q$ -deformed fermion oscillators is based on the following deformed anti-commutation and commutation relations [21] for the deformed fermionic creation and annihilation operators  $c^*$ ,  $c$  and the number operator  $\hat{N}$ :

$$\begin{aligned} cc^* + qc^*c &= 1, \\ [\hat{N}, c^*] &= c^*, \quad [\hat{N}, c] = -c, \end{aligned} \quad (1)$$

where  $q$  is the real positive deformation parameter having the range  $q < 1$ . It has different properties in the sense of its quantum mechanical and thermostistical aspects from the other fermion algebras in the literature [24-26]. Also, the deformed fermion number operator for the VPJC-fermion model is defined as

$$c^*c = [\hat{N}] = \frac{1 - (-1)^{\hat{N}} q^{\hat{N}}}{1 + q}, \quad (2)$$

whose spectrum is

$$[n] = \frac{1 - (-1)^n q^n}{1 + q}. \quad (3)$$

This is called the  $q$ -fermionic basic number for the VPJC-fermion model. By using the relations in Eqs. (1)-(3), it is possible to construct the representations of fermion operators in the Fock space, and therefore one can see that the VPJC-fermion model does not satisfy the Pauli exclusion principle for the interval  $0 < q < 1$ . However, it reduces to the standard fermion relations in the limit  $q = 1$ . Moreover, it has the following fermionic Jackson derivative (JD) operator  $\hat{D}_x^{(q)}$  [26]:

$$\hat{D}_x^{(q)} f(x) = \frac{1}{x} \left[ \frac{f(x) - f(-qx)}{1 + q} \right], \quad (4)$$

which is a fermionic extension of the usual JD operator [27,28]. We will see below that this fermionic JD operator plays an important role for examining thermostistics of the VPJC-type  $q$ -fermion gas even in two spatial dimensions.

## 3. Thermostistical properties of the VPJC $q$ -fermion gas

We consider a free deformed gas of  $q$ -fermions satisfying the relations in Eqs. (1)-(4) confined in a two-dimensional area ( $A$ ). We can assume the Hamiltonian  $\hat{H}$  for such a  $q$ -fermion gas model as  $\hat{H} = \sum_i (\varepsilon_i - \mu) \hat{N}_i$ , where  $\mu$  is the chemical potential and  $\varepsilon_i$  is the kinetic energy in the state  $i$  related to the number operator  $\hat{N}_i$ . Hence, we can determine the mean value of the occupation number  $n_{i,q}$  for the VPJC-type  $q$ -fermion gas using the relation  $[n_i] = \text{Tr}(e^{-\beta \hat{H}} c_i^* c_i) / Z$ , where  $Z = \text{Tr}(e^{-\beta \hat{H}})$  is the partition function and  $\beta = 1/kT$  with the Boltzmann constant  $k$  and the temperature  $T$ . From the above relations along with the use of the cyclic property of the trace [5,7], we obtain the following result for the mean occupation number  $n_{i,q}$  of the VPJC  $q$ -fermion gas:

$$n_{i,q} = \frac{1}{|\ln q|} \left| \ln \left( \frac{|z^{-1} e^{\beta \varepsilon_i} - 1|}{z^{-1} e^{\beta \varepsilon_i} + q} \right) \right|, \quad (5)$$

where  $z = e^{\beta \mu}$  is the fugacity and  $0 < q < 1$ . Using Eqs. (4) and (5), one can deduce the following logarithm of the grand partition function:

$$\ln Z^{(VPJC)} = \left( \frac{1+q}{|\ln q|} \right) \sum_i \left| \ln(1 - ze^{-\beta \epsilon_i}) \right|, \quad (6)$$

which satisfies the modified thermodynamic expression  $N = z \hat{D}_z^{(q)} \ln Z^{(VPJC)} \equiv \sum_i n_{i,q}$ , where  $n_{i,q}$  is given in

Eq. (5) and  $\hat{D}_z^{(q)}$  has the same form as in Eq. (4) with the variable  $z$  instead of  $x$ . In order to find the high temperature thermostistical functions of the VPJC  $q$ -fermion gas, we can replace the sums over states by integrals for a large area and a large number of particles as in the standard procedure [18,29]. From the thermodynamic relations  $PA/kT = \ln Z^{(VPJC)}$  and  $N = \sum_i n_{i,q}$ , we obtain the following results for a two-dimonsional system by using the fermionic JD operator in Eq. (4) :

$$\frac{P^{(q)}}{kT} = \frac{1}{\lambda^2} f_2(z, q), \quad \frac{N^{(q)}}{A} = \frac{1}{\lambda^2} f_1(z, q), \quad (7)$$

where  $\lambda = h/\sqrt{2\pi mkT}$  is the thermal wavelength and the  $q$ -deformed function  $f_n(z, q)$  is

$$\begin{aligned} f_n(z, q) &= \frac{1}{\Gamma(n)} \int_0^\infty \frac{x^{n-1}}{|\ln q|} \left| \ln \left( \frac{1 - ze^{-x}}{1 + qze^{-x}} \right) \right| dx \\ &= \frac{1}{|\ln q|} \left[ \sum_{l=1}^\infty (-1)^{(l-1)} \frac{(zq)^l}{l^{n+1}} - \sum_{l=1}^\infty \frac{z^l}{l^{n+1}} \right], \end{aligned} \quad (8)$$

where  $x = \beta \epsilon$ . These functions are different from both the  $h_n^*(z, q)$  functions of Ref. [20] and the standard  $f_n(z)$  functions for an undeformed fermion gas in two dimensions [18]. Also, the internal energy  $U^{(q)}$  is obtained as

$$U = - \left( \frac{\partial \ln Z^{(VPJC)}}{\partial \beta} \right)_{z,A} \Rightarrow \frac{U^{(q)}}{A} = \frac{1}{\lambda^2} kT f_2(z, q). \quad (9)$$

It is noteworthy that from Eqs. (7) and (9), the relation  $P^{(q)} = U^{(q)} / A$  is satisfied for the VPJC  $q$ -fermion gas. The specific heat of our model can be derived from both the thermodynamic definition  $C_A = (\partial U / \partial T)_{N,A}$  and Eq. (4) as

$$\frac{C_A^{(q)} \lambda^2}{kA} = 2(z \hat{D}_z^{(q)} f_3(z, q)) - \frac{(z \hat{D}_z^{(q)} f_2(z, q))^2}{(z \hat{D}_z^{(q)} f_1(z, q))}, \quad (10)$$

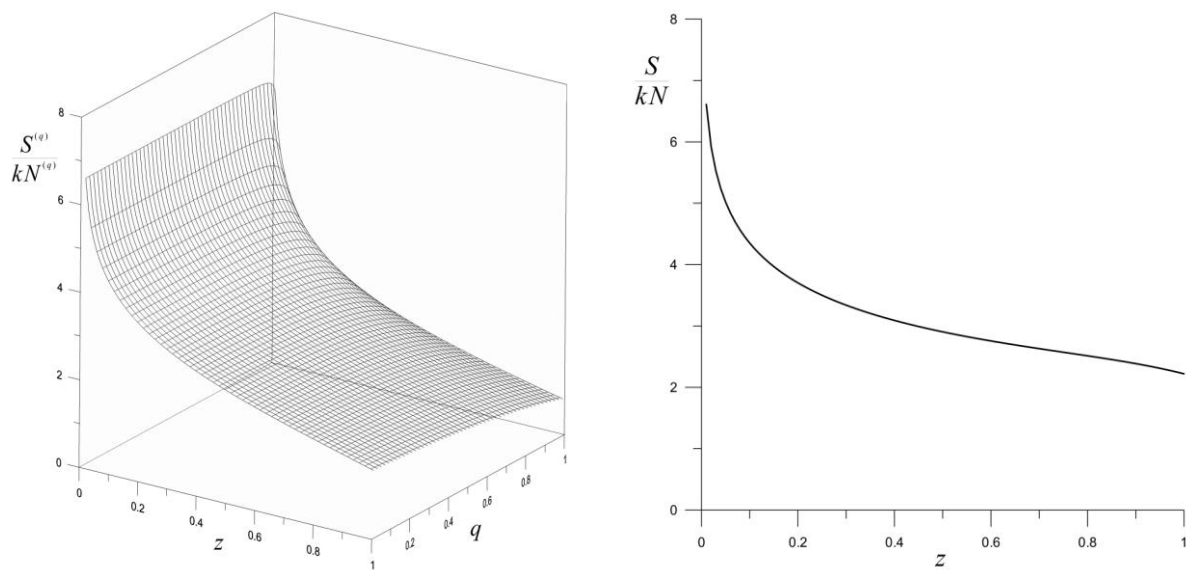
where the function  $f_n(z, q)$  is given in Eq. (8). The Helmholtz free energy  $F = \mu N - PA$  can be found from Eqs. (7) and (9) as

$$F^{(q)} = N^{(q)} kT \left[ \ln z - \left( \frac{f_2(z, q)}{f_1(z, q)} \right) \right], \quad (11)$$

and the  $q$ -deformed entropy function  $S^{(q)} / kN^{(q)}$  for the VPJC  $q$ -fermion gas model for high temperatures is derived from the relation  $S = (U - F) / T$  as

$$\frac{S^{(q)}}{kN^{(q)}} = \left[ 2 \left( \frac{f_2(z, q)}{f_1(z, q)} \right) - \ln z \right]. \quad (12)$$

For comparison, in figure 1, we show the plots of the  $q$ -deformed entropy function  $S^{(q)} / kN^{(q)}$  and the entropy function  $S / kN$  of an undeformed fermion gas as a function of  $z$  for values of the deformation parameter  $q$  for the cases  $q < 1$  and  $q = 1$ , respectively. For  $q < 1$  and at the same fugacity, the entropy values of the VPJC  $q$ -fermion gas in two dimensions decrease with the values of  $q$ . At the same fugacity, it is also lower than the result of an undeformed fermion gas in two dimensions as shown in figure 1.



**Figure 1.** The entropy functions  $S^{(q)}/kN^{(q)}$  and  $S/kN$  as a function of  $z$  for a two-dimensional system for the cases  $q < 1$  (left) and  $q = 1$  (right).

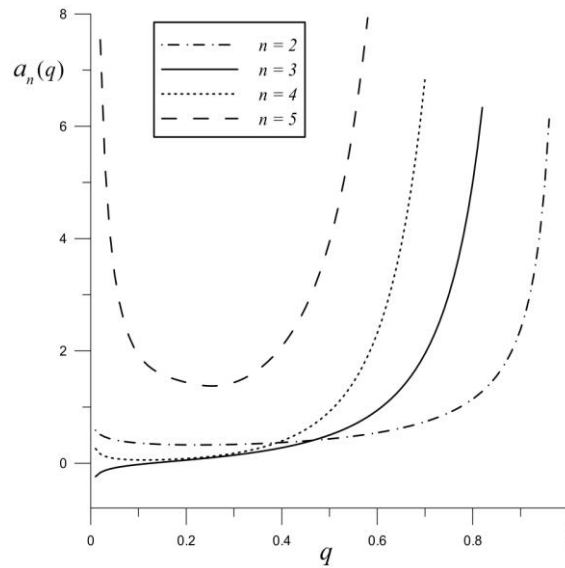
Furthermore, for high temperatures, the equation of state for the VPJC  $q$ -fermion gas model can be calculated from Eqs. (7) as a deformed virial expansion in the two-dimensional space as follows:

$$\frac{P^{(q)}A}{N^{(q)}kT} = a_1(q) + a_2(q)\left(\frac{N^{(q)}\lambda^2}{A}\right) + a_3(q)\left(\frac{N^{(q)}\lambda^2}{A}\right)^2 + a_4(q)\left(\frac{N^{(q)}\lambda^2}{A}\right)^3 + a_5(q)\left(\frac{N^{(q)}\lambda^2}{A}\right)^4 + \dots, \quad (13)$$

where the first five deformed virial coefficients are

$$\begin{aligned} a_1(q) &= 1, \\ a_2(q) &= \frac{1}{2^3} \frac{|\ln q|}{(q-1)^2} (q^2 + 1), \\ a_3(q) &= \frac{1}{2^4} \frac{|\ln q|^2}{(q-1)^4} (q^2 + 1)^2 - \frac{2}{3^3} \frac{|\ln q|^2}{(q-1)^3} (q^3 - 1), \\ a_4(q) &= \frac{5}{2^7} \frac{|\ln q|^3}{(q-1)^6} (q^2 + 1)^3 - \frac{1}{3 \cdot 2^2} \frac{|\ln q|^3}{(q-1)^5} (q^2 + 1)(q^3 - 1) + \frac{3}{2^6} \frac{|\ln q|^3}{(q-1)^4} (q^4 + 1), \\ a_5(q) &= \frac{7}{2^8} \frac{|\ln q|^4}{(q-1)^8} (q^2 + 1)^4 - \frac{1}{12} \frac{|\ln q|^4}{(q-1)^7} (q^2 + 1)^2 (q^3 - 1) + \frac{1}{2^4} \frac{|\ln q|^4}{(q-1)^6} (q^2 + 1)(q^4 + 1) + \frac{2}{3^4} \frac{|\ln q|^4}{(q-1)^6} (q^3 - 1)^2 \\ &\quad - \frac{4}{5^3} \frac{|\ln q|^4}{(q-1)^5} (q^5 - 1), \end{aligned} \quad (14)$$

where all the virial coefficients are  $q$ -dependent with the interval  $0 < q < 1$ , except that the first one. In order to visualize the behavior of these virial coefficients, we present the plots of  $a_2(q)$ ,  $a_3(q)$ ,  $a_4(q)$ ,  $a_5(q)$  as a function of  $q$  for a two-dimensional system for the case  $q < 1$  in figure 2. As is shown in figure 2, we have numerically analyzed the behavior of these virial coefficients and have found that only the third virial coefficient  $a_3(q)$  has both positive and negative values. The other virial coefficients have always positive values in the interval  $0 < q < 1$ . Thus, the high-temperature behavior of the VPJC  $q$ -fermion gas depends on the parameter  $q$  even in two dimensions.



**Figure 2.** The virial coefficients  $a_n(q)$  as a function of  $q$  for a two-dimensional system for the case  $q < 1$ .

We now consider some of the low-temperature thermostistical properties of our model in two dimensions. From the relations  $N^{(q)}(T) = \int_0^\infty g(\varepsilon) n(\varepsilon, T, q) d\varepsilon$  and  $U^{(q)}(T) = \int_0^\infty \varepsilon g(\varepsilon) n(\varepsilon, T, q) d\varepsilon$ , where  $g(\varepsilon) = 2\pi mA/h^2$ , we derive the following relations for the chemical potential and the total energy of the VPJC-fermion model by using both Eq. (5) and the Sommerfeld expansion method [30,31] for low temperatures:

$$\mu = \frac{\hbar^2}{2m} \left( \frac{4\pi N^{(q)}(0)}{A} \right) \equiv \varepsilon_F, \quad U^{(q)}(T) = \frac{(N^{(q)}(0)) \varepsilon_F}{2} \left[ 1 - \frac{I^{(VPJC)}(q)}{4} \left( \frac{kT}{\varepsilon_F} \right)^2 \right], \quad (15)$$

where  $\varepsilon_F$  is the Fermi energy and  $I^{(VPJC)}(q)$  is defined as

$$I^{(VPJC)}(q) = \int_{-\infty}^{\infty} dx \frac{x^2}{\ln q} \left[ \frac{1}{(1-e^{-x})} - \frac{1}{(1+q e^{-x})} \right], \quad (16)$$

where  $x = \beta(\varepsilon - \mu)$ . This function has always negative values for the interval  $0 < q < 1$ . The Helmholtz free energy and the entropy function for our model in two spatial dimensions can be calculated for low temperatures as

$$\frac{F^{(q)}(T)}{N^{(q)}(0)} = \frac{\varepsilon_F}{2} \left[ 1 + \frac{I^{(VPJC)}(q)}{4} \left( \frac{kT}{\varepsilon_F} \right)^2 \right], \quad \frac{S^{(q)}(T)}{kN^{(q)}(0)} = -\frac{I^{(VPJC)}(q)}{4} \left( \frac{kT}{\varepsilon_F} \right), \quad (17)$$

where  $I^{(VPJC)}(q)$  is given in Eq. (16).

#### 4. Conclusions

Following our previous works in [22,23], we further investigated the high and low temperature thermostistical properties of the VPJC-type  $q$ -fermion gas model in two dimensions. According to figure 1, it should be emphasized that an enhancement of the values of  $q$  results in a reduction of the values of entropy function  $S^{(q)}/kN^{(q)}$  for the case of two-dimensional system. This leads to a less chaoticity in the VPJC  $q$ -fermion gas. This model also describes an intermediate-statistics behavior via Eq. (5), and from the results obtained above, the deformation parameter  $q$  with the range  $q < 1$  is responsible for the behavior of the VPJC  $q$ -fermion gas as a whole. We hope that our results peculiar to two spatial dimensions could be useful for studies on several quantum systems such as strongly interacting electron gases and semiconductor quantum wells.

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