

# Clifford Algebras and Their Decomposition into Conjugate Fermionic Heisenberg Algebras

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## Abstract.

We discuss a construction scheme for Clifford numbers of arbitrary dimension. The scheme is based upon performing direct products of the Pauli spin and identity matrices. Conjugate fermionic algebras can then be formed by considering linear combinations of the Clifford numbers and the Hermitian conjugates of such combinations. Fermionic algebras are important in investigating systems that follow Fermi-Dirac statistics. We will further comment on the applications of Clifford algebras to Fueter analyticity, twistors, color algebras, M-theory and Leech lattice as well as unification of ancient and modern geometries through them.

## 1. Clifford Algebras

These are associative hermitian matrix algebras<sup>[1]</sup> with basis  $\gamma_\alpha$  ( $\alpha = 1, \dots, 2n + 1$ ) that obey

$$\gamma_\alpha \gamma_\beta = \frac{1}{2} \{ \gamma_\alpha, \gamma_\beta \} = \frac{1}{2} (\gamma_\alpha \gamma_\beta + \gamma_\beta \gamma_\alpha) = \delta_{\alpha\beta} I \quad (1)$$

where  $I$  is a unit matrix of the same dimension as  $\gamma_\alpha$ . We have

$$\gamma_\alpha = \gamma_\alpha^\dagger, \quad \text{and} \quad \gamma_\alpha (\gamma_\beta \gamma_\rho) = (\gamma_\alpha \gamma_\beta) \gamma_\rho \quad (2)$$



The basis elements  $\gamma_\alpha$  are linearly independent.

The Clifford relation means that the square of each  $\gamma_\alpha$  is unity and two different  $\gamma_\alpha$  anticommute.

(a) Suppose there are  $2n$  such elements  $\gamma_\alpha$ . Define

$$\gamma_{2n+1} = \epsilon \gamma_1 \gamma_2 \cdots \gamma_{2n-1} \gamma_{2n} \quad (3)$$

Then

$$\{\gamma_{2n+1}, \gamma_i\} = 0 \quad (i = 1, \dots, 2n) \quad (4)$$

Also

$$\gamma_{2n+1}^2 = \epsilon^2 (-1)^n \quad (5)$$

so that if  $\epsilon = (-1)^{\frac{n}{2}}$  we have  $\gamma_{2n+1}^2 = 1$ . Thus the basis is generated by  $2n$  elements  $\gamma_1, \dots, \gamma_{2n}$ .

(b) Construction of the algebra (Clifford, Dirac, Jordan and Wigner):

Let  $n = 1$ . The answer is given by the hermitian traceless  $2 \times 2$  Pauli matrices  $\tau_1, \tau_2, \tau_3$ . In this case  $\tau_3 = -i\tau_1\tau_2$ . we have

$$\tau_i \tau_j + \tau_j \tau_i = 2\delta_{ij} I \quad (6)$$

where  $I$  is a  $2 \times 2$  unit matrix. The  $\tau_i$  act on complex 2-dimensional vector space  $U$ . We can also write  $\tau_j = ie_j$  where  $e_j$  are the 3 quaternion units.

Let  $n = 2$ . We take two commuting sets of Pauli operators  $\sigma_i^{(1)}$  and  $\sigma_j^{(2)}$  acting in the vector space  $U \otimes U$ . We have seen that

$$\sigma_i^{(1)} = \tau_i \times I \quad \text{and} \quad \sigma_j^{(2)} = I \times \tau_j \quad (7)$$

are  $4 \times 4$  matrices with the properties

$$[\sigma_i^{(1)}, \sigma_j^{(2)}] = 0 \quad (8)$$

We define

$$\begin{aligned} \gamma_1 &= \sigma_1^{(2)} & \gamma_3 &= \sigma_2^{(2)} \\ \gamma_2 &= \sigma_3^{(2)} \sigma_1^{(1)} & \gamma_4 &= \sigma_3^{(2)} \sigma_2^{(1)} \end{aligned} \quad (9)$$

Then we find

$$\gamma_5 = \sigma_3^{(1)} \sigma_3^{(2)} \quad \text{and} \quad \{\gamma_\alpha, \gamma_\beta\} = 2\delta_{\alpha\beta} \quad (10)$$

Dirac's notation:  $\sigma_j^{(1)} = \sigma_j$ ,  $\sigma_i^{(2)} = \rho_i$ .

Then  $\gamma_1 = \rho_1$ ,  $\gamma_3 = \rho_2$ ,  $\gamma_2 = \rho_3 \gamma_1$ ,  $\gamma_4 = \rho_3 \sigma_2$ ,  $\gamma_5 = \rho_3 \sigma_3$ .

Next we look at its generalization.

For arbitrary  $n$  we construct  $2n + 1$  different  $2^n \times 2^n$  matrices that act on the vector space

$$\underbrace{U \otimes U \otimes \cdots \otimes U}_{n \text{ factors}} \quad (11)$$

of dimension  $2^n$  in terms of  $n$  commuting sets of Pauli matrices

$$\begin{aligned} \sigma_j^{(n)} &= \tau_j \otimes I \otimes I \otimes \cdots \otimes I \\ \sigma_j^{(n-1)} &= I \otimes \tau_j \otimes I \cdots \otimes I \\ &\dots\dots\dots \\ \sigma_j^{(1)} &= I \otimes I \otimes \cdots \otimes \tau_j \end{aligned} \quad (12)$$

Then we construct the following  $2n + 1$  hermitian matrices:

$$\begin{aligned} \gamma_1 &= \sigma_1^{(n)} & , & & \gamma_{n+1} &= \sigma_2^{(n)} \\ \gamma_2 &= \sigma_3^{(n)} \sigma_1^{(n-1)} & , & & \gamma_{n+2} &= \sigma_3^{(n)} \sigma_2^{(n-1)} \\ \gamma_3 &= \sigma_3^{(n)} \sigma_3^{(n-1)} \sigma_1^{(n-2)} & , & & \gamma_{n+3} &= \sigma_3^{(n)} \sigma_3^{(n-1)} \sigma_2^{(n-2)} \\ &\dots\dots\dots \\ \gamma_{2n+1} &= \sigma_3^{(n)} \sigma_3^{(n-1)} \cdots \sigma_3^{(2)} \sigma_3^{(1)} \end{aligned} \quad (13)$$

They satisfy

$$\gamma_\alpha \gamma_\beta + \gamma_\beta \gamma_\alpha = 2\delta_{\alpha\beta} I \quad (\alpha, \beta = 1, \dots, 2n + 1) \quad (14)$$

### (c) General Representation of the Clifford Algebra:

The automorphism group of the Clifford algebra is the unitary group  $U(2^n)$ . Indeed, if

$$\gamma'_\alpha = U \gamma_\alpha U^\dagger, \quad (U U^\dagger = I) \quad (15)$$

then

$$\begin{aligned} \gamma'_\alpha \gamma'_\beta + \gamma'_\beta \gamma'_\alpha &= U \gamma_\alpha U^\dagger U \gamma_\beta U^\dagger + U \gamma_\beta U^\dagger U \gamma_\alpha U^\dagger \\ &= 2U(\delta_{\alpha\beta} I)U^\dagger = \delta_{\alpha\beta} I \end{aligned} \quad (16)$$

For example the following different representations of the Dirac algebra:

$$\begin{aligned} \text{I :} \quad \gamma_1 &= \rho_2 \sigma_1, \quad \gamma_2 = \rho_2 \sigma_2, \\ \gamma_3 &= \rho_2 \sigma_3, \quad \gamma_4 = \rho_1, \quad \gamma_5 = \rho_3 \end{aligned} \quad \text{Weyl rep.} \quad (17)$$

$$\begin{aligned} \text{II :} \quad \gamma_1 &= \rho_2 \sigma_1, \quad \gamma_2 = \rho_2 \sigma_2, \\ \gamma_3 &= \rho_2 \sigma_3, \quad \gamma_4 = -\rho_3, \quad \gamma_5 = \rho_1 \end{aligned} \quad \text{Dirac rep.} \quad (18)$$

$$\text{III :} \quad \begin{aligned} \gamma_1 &= \sigma_1, & \gamma_2 &= \rho_2 \sigma_2, \\ \gamma_3 &= \sigma_3, & \gamma_4 &= \sigma_2 \rho_1, & \gamma_5 &= \sigma_2 \rho_3 \end{aligned} \quad \text{Majorana rep.} \quad (19)$$

are unitarily equivalent to the standard representation given above.

(d) Reality properties in the standard representation:

From the construction given above  $n + 1$  elements of the basis ( $\gamma_1, \dots, \gamma_n$  and  $\gamma_{2n+1}$ ) are real and symmetric while the remaining  $n$  elements ( $\gamma_{n+1}, \dots, \gamma_{2n}$ ) are imaginary and antisymmetric.

(e) Generation of the  $2^n \times 2^n$  matrix algebra from the Clifford algebra.

$\gamma_\alpha$  and  $I$  close under the symmetric Jordan product. They do not close under the matrix product since for instance

$$\begin{aligned} \gamma_\alpha \gamma_\beta &= \frac{1}{2} \{ \gamma_\alpha \gamma_\beta \} + \frac{1}{2} [ \gamma_\alpha, \gamma_\beta ] \\ &= \frac{1}{2} \delta_{\alpha\beta} I + i \sigma_{\alpha\beta} \end{aligned} \quad (20)$$

where

$$\sigma_{\alpha\beta} = \frac{1}{2i} [ \gamma_\alpha, \gamma_\beta ] \quad (21)$$

(called the spin matrices) are hermitian and linearly independent of  $\gamma_\alpha$ . Similarly  $\gamma_\alpha \gamma_\beta \gamma_\rho$  will contain a part independent of  $\gamma_\alpha$  and  $\sigma_{\beta\rho}$  up to products of  $\gamma_\alpha$  of degree  $n$ . The product of degree  $2n + 1$ , namely

$$\gamma_1 \gamma_2 \cdots \gamma_{2n} \gamma_{2n+1} \quad (22)$$

is proportional to the unit matrix as we have already seen. Hence a linear combination of all the powers of  $\gamma_\alpha$  from 0 to  $n$  span the whole  $2^n \times 2^n$  matrix algebra. In the case of the Dirac algebra we have  $I$ ,  $\gamma_\alpha$  and  $\sigma_{\alpha\beta}$  respectively 1, 5, and  $\frac{5 \times 4}{2} = 10$  in number spanning the 16 independent elements of the  $4 \times 4$  matrix algebra.

(f) Let  $a = a_\alpha \gamma_\alpha$ . We have  $a^2 = a_\alpha a_\alpha I$ .

Let us now show an application: Take 4 of the  $\gamma_\alpha$  for  $n = 2$ :  $\gamma_\mu$  ( $\mu = 1, 2, 3, 4$ ). Let  $\partial = \gamma_\mu \frac{\partial}{\partial x_\mu}$ . Let  $\partial\psi = m\psi$  where  $\psi$  is a 4-column. Then

$$\partial^2 \psi = \partial_\mu \partial_\mu \psi = \square \psi = m^2 \psi \quad (23)$$

This is the relativistic wave equation for  $\psi$ .  $\partial\psi = m\psi$  is the Dirac equation.

(g) Rotation of  $2n + 1$  dimensional vectors.

Consider an infinitesimal transformation of the vector  $a_\alpha$

$$a'_\alpha = a_\alpha + \delta a_\alpha \quad (24)$$

This transformation is a rotation if

$$a'_\alpha a'_\alpha = (a_\alpha + \delta a_\alpha)(a_\alpha + \delta a_\alpha) = a_\alpha a_\alpha \quad (25)$$

This entails  $a_\alpha \delta a_\alpha = 0$ . Hence if

$$a = \gamma_\rho a_\rho, \quad \delta a = \gamma_\rho \delta a_\rho \quad (26)$$

we must have

$$\{a, \delta a\} = 0 \quad (27)$$

Now let

$$\delta a = \left[ \frac{i}{4} \sigma_{\alpha\beta} \omega_{\alpha\beta}, a \right] = \left[ \frac{i}{2} \omega, a \right] \quad (28)$$

where  $\omega = \frac{1}{2} \sigma_{\alpha\beta} \omega_{\alpha\beta}$ . We have

$$\begin{aligned} \{a, \delta a\} &= \frac{i}{2} (a[\omega, a] + [\omega, a]a) \\ &= \frac{i}{2} (a\omega a - a^2\omega + \omega a^2 - a\omega a) = 0 \end{aligned} \quad (29)$$

Hence

$$a \rightarrow a + \frac{i}{2} [\omega, a] \quad (30)$$

represents a rotation and  $\frac{1}{2} \sigma_{\alpha\beta}$  are called spin generators. In component form we write

$$a'_\alpha = a_\alpha + \omega_{\alpha\beta} a_\beta \quad (31)$$

In the case of the Dirac algebra ( $n = 2$ )

$$\frac{1}{2} \sigma_{\alpha\beta} \quad (\alpha, \beta = 1, \dots, 5) \quad (32)$$

generate 5 dimensional rotations, while

$$\frac{1}{2} \sigma_{\mu\nu} \quad (\mu, \nu = 1, \dots, 4) \quad (33)$$

generate 4 dimensional rotations. When  $\omega_{23}, \omega_{31}, \omega_{12}$  are real and  $\omega_{41}, \omega_{42}, \omega_{43}$  are imaginary

$$\delta a = \left[ \frac{i}{2} \sigma_{\mu\nu} \omega_{\mu\nu}, a \right] \quad (34)$$

for the vector  $a$  with  $a_1, a_2, a_3$  real and  $a_4$  imaginary represents a Lorentz transformation in 3 space and 1 time directions (the Minkowski space-time). Under such a Lorentz transformation

$$\partial \rightarrow \partial + \left[ \frac{i}{2} \omega, \partial \right] \quad (35)$$

while

$$\psi \rightarrow \psi + \frac{i}{2}\omega\psi \quad (36)$$

Thus  $\partial$  transforms like a vector while  $\psi$  transforms like a spinor. Under such a Lorentz transformation the bilinear forms

$$\bar{\psi}\psi \quad \text{and} \quad i\bar{\psi}\gamma_5\psi \quad (37)$$

(where  $\bar{\psi} = \psi^\dagger\gamma_4$ ) remains invariant, while

$$\bar{\psi}\gamma_\mu\psi \quad \text{and} \quad i\bar{\psi}\gamma_5\gamma_\mu\psi \quad (38)$$

transform like vectors.

## 2. Grassmann Algebra as Subalgebra of Clifford Algebra

### Grassmann algebra and its conjugate Grassmann algebra

The Grassmann algebra is the algebra of associative anticommuting numbers. It is of order  $n$  if we can choose a basis  $b_j$  ( $j = 1, \dots, n$ ) such that

$$b_i(b_j b_k) = (b_i b_j)b_k \quad (39)$$

and

$$\{b_i b_j\} = b_i b_j + b_j b_i = 0, \quad (\text{or} \quad b_i \cdot b_j = 0) \quad (40)$$

The associativity relation is automatically satisfied if  $b_i$  is chosen to be matrices and they are multiplied with the rules of matrix multiplication.

A standard set satisfying the anticommutativity relation can be obtained from the Clifford algebra of order  $2n + 1$  by choosing

$$b_j = \frac{1}{2}(\gamma_j + i\gamma_{n+j}), \quad (j = 1, \dots, n) \quad (41)$$

We then verify that  $b_j^2 = 0$ , and  $b_j b_k = -b_k b_j$  for  $j \neq k$ . In the standard representation  $\gamma_j$  ( $j \leq n$ ) are real while  $\gamma_{n+j}$  are imaginary so that  $b_j$  are real non-hermitian  $2^n \times 2^n$  matrices. Another set of Grassmann numbers  $b'_j$  of the same order can be obtained from  $b_j$  by a unitary transformation

$$b'_j = U b_j U^\dagger \quad (42)$$

with  $U U^\dagger = I$ , where  $I$  is the  $2^n \times 2^n$  unit matrix, or by the transformation to the conjugate set

$$b_j \rightarrow b_j^\dagger = \frac{1}{2}(\gamma_j - i\gamma_{n+j}) \quad (43)$$

which are also real and obey

$$b_j^\dagger b_k^\dagger + b_k^\dagger b_j^\dagger = 0 \quad (44)$$

Let us now give examples: For  $n = 1$  we have

$$b = \frac{1}{2}(\sigma_1 + i\sigma_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (45)$$

$$b^\dagger = \frac{1}{2}(\sigma_1 - i\sigma_2) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad (46)$$

$$b^2 = 0, \quad b^{\dagger 2} = 0 \quad (47)$$

$$b' = UbU^\dagger, \quad \text{and} \quad b'^\dagger = Ub^\dagger U^\dagger \quad (48)$$

are not real in general.

For  $n = 2$  (Dirac algebra) we can construct a Grassmann algebra of order 2 and its conjugate, briefly:

$$b_1 = \frac{\rho_1 + i\rho_2}{2} \quad (49)$$

$$b_2 = \rho_3 \frac{\sigma_1 + i\sigma_2}{2} \quad (50)$$

$$b_1^2 = b_2^2 = 0, \quad b_1 b_2 = -b_2 b_1 \quad (51)$$

$$b_1^\dagger = \frac{\rho_1 - i\rho_2}{2} \quad (52)$$

$$b_2^\dagger = \rho_3 \frac{\sigma_1 - i\sigma_2}{2} \quad (53)$$

$$b_1^{\dagger 2} = b_2^{\dagger 2} = 0, \quad b_1^\dagger b_2^\dagger = -b_2^\dagger b_1^\dagger \quad (54)$$

### 3. Fermionic Heisenberg Algebra

If the Grassmann algebra  $b_j$  and its conjugate  $b_j^\dagger$  are taken together we obtain

$$b_j b_k^\dagger + b_k^\dagger b_j = \frac{1}{4} \{ \gamma_j + i\gamma_{n+j}, \gamma_k - i\gamma_{n+k} \} = \delta_{jk} I \quad (55)$$

These relations are also invariant under the unitary transformation  $b_j \rightarrow Ub_j U^\dagger$ . Thus from  $b_j$  and  $b_k^\dagger$  we obtain the Fermionic Heisenberg algebra

$$\{b_j, b_k\} = 0, \quad \{b_j^\dagger, b_k^\dagger\} = 0, \quad \{b_j, b_k^\dagger\} = \delta_{jk} I \quad (56)$$

These have the same structure as Poisson bracket relations among canonical variables (coordinates and momenta) in classical mechanics. The difference is that the Poisson bracket is anticommutative while the bracket used for the Fermionic Heisenberg algebra is commutative since it is an anticommutator. In the bosonic Heisenberg algebra commutator brackets with same formal properties as Poisson brackets are used.

The  $2n$  generators  $\gamma_1, \dots, \gamma_n$  of the Clifford algebra of order  $2n + 1$  can be recovered from the two conjugate Grassmann algebras through

$$\gamma_j = b_j + b_j^\dagger, \quad \gamma_{n+j} = -i(b_j - b_j^\dagger) \quad (57)$$

The last element  $\gamma_{2n+1}$  of the Clifford basis can be expressed by means of  $b_j, b_j^\dagger$  through the relation

$$\gamma_1 \gamma_{n+1} \gamma_2 \gamma_{n+2} \cdots \gamma_n \gamma_{2n} \gamma_{2n+1} = i^n I \quad (58)$$

Now

$$\gamma_j \gamma_{n+j} = i(I - 2b_j^\dagger b_j) \quad (59)$$

so that

$$\gamma_{2n+1} = \prod_{j=1}^n (I - 2b_j^\dagger b_j) \quad (60)$$

#### 4. Further Remarks

The present authors earlier developed octonion and split octonion<sup>[2],[3]</sup> algebras that led into algebraic foundations of color symmetries in nature, into quark-antiquark symmetries<sup>[4],[5]</sup>, exotics and pentaquark formations. The formulations we put forth in this paper also have deeper applications in hadronic physics and in astrophysics. Further applications into Fueter analyticity, quaternion analyticity, twistor formalism, hyperbolic extensions of exceptional groups, M-theory and Leech lattices will be published in our forthcoming papers.

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